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A Fast Cascadic Multigrid Method for Exponential Compact FD Discretization of Singularly Perturbed Convection-Diffusion Equations

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Abstract. Solving singularly perturbed convection-diffusion equations, especially for 3D problems, is a challenging problem. In this paper, we extend our work on the extrapolation cascadic multigrid (EXCMG) method for solving the 3D Poisson equation — cf. [Pan *et al.*, J. Sci. Comput. 2017], to 3D convection-diffusion equations with singularly perturbed parameters. First, we introduce an exponential higher order compact finite difference scheme to discretize the 3D convection-diffusion equation with variable convection coefficients, resulting in a larger-scale nonsymmetric linear system. Then, we propose an EXCMG method combined with the biconjugate gradient stabilized smoother to solve the larger-scale nonsymmetric system efficiently. Numerical experiments demonstrate that the EXCMG method is a highly effective solver for convection-dominated problems, which outperforms the existing multigrid methods such as aggregation-based algebraic multigrid method.

AMS subject classifications: 65M10, 78A48

Key words: Singular perturbation, convection-diffusion problem, boundary layer, high efficiency, exponential finite difference method, multigrid, high-order compact scheme.

1. Introduction

Let $\Omega=(0,1)\times(0,1)\times(0,1)$. We consider the convection-diffusion equations

$$-\varepsilon(u_{xx}+u_{yy}+u_{zz})+pu_x+qu_y+ru_z=f \tag{1.1}$$

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with a positive diffusive coefficient $\varepsilon > 0$. The convection coefficients p, q, r, the forcing term f and the unknown function u depend on three spatial variables x, y and z. All these functions are supposed sufficiently smooth on the domain Ω. This problem is well-known as a crucial aspect of computational fluid dynamics. If ε is sufficiently small, then the Eq. (1.1) becomes a singularly perturbed problem, which presents significant challenges in simulations because of the presence of nonphysical oscillations [28].

To overcome this problem, various methods have been proposed, including the widely used upwind scheme [14, 16, 21]. However, the standard upwind method often struggles to accurately capture the characteristics of boundary layer due to pollution effects. On the other hand, in recent decades, high-order compact (HOC) FD schemes have been developed to handle singularly perturbed problems. A prominent example of such methods is the HOC polynomial scheme, known for its robust numerical stability and high-order accuracy [9, 35–37]. Despite its advantages, this method may not be suitable for certain physical problems, particularly those involving sharp boundary layers in convection-dominated problems, unless a layer-adapted mesh — e.g. Shishkin mesh, is used. Besides, increased computational cost associated with high-order schemes, can become a limiting factor in large-scale or real-time simulations. As the result, high-order methods are most effective if the solution is smooth and the problem is well-conditioned. Therefore, non-uniform meshes and local mesh refinement have been also intensively studied [3, 7, 8, 10, 15, 17, 18, 38]. Ge and Cao [10] developed a multigrid method based on HOC scheme proposed by Kalita *et al.* [15] on non-uniform grids to solve the 2D convection with boundary layers. Later on the method has been extended to 3D problems [3]. Zhang *et al.* [38] proposed a HOC scheme to solve the 3D convection-diffusion equation without source terms on nonuniform grids. They also conducted direct numerical simulations of problems involving boundary layers. However, the methods mentioned require prior knowledge of the exact location of boundary layer or singularity region. Another high-order FD technique is the EHOC FD scheme. Pillai [27] introduced a fourth-order exponential FD method specifically designed for convection-diffusion problems. Similar work by Tian and Dai [33] produced fourth-order schemes for convection-diffusion equations with both constant and variable coefficients. Later on the method was extended to 3D problems by Mohamed *et al.* [20]. The exponential FD schemes exhibit remarkable properties such as the inherent integration of upwind effects through exponential functions and the unconditional diagonal dominance of the coefficient matrix, making them well-suited for solving singular perturbation problems associated with boundary and interior layers. Tian and Ge [31] theoretically analyzed the constraint on the grid step size for achieving fourth-order accuracy with the proposed scheme for large Péclet numbers. However, the HOC formulation for discretizing 3D convection-diffusion equations leads to large, sparse matrix systems [12], which is the most computationally expensive component of the method. Therefore, developing efficient algorithms to speed up the solution process is crucial.

High-quality simulations demand accurate solutions of 3D equations. This is computationally expensive operation requiring high memory and large processing time. A practical way to address these problems is the employment of high-order methods along with efficient iterative solvers. High-order methods can achieve comparable accuracy on coarser