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Schwartz Duality for Singularly Perturbed Differential Equations with Chebyshev Spectral Methods

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Abstract. Singularly perturbed differential equations with the Dirac delta function usually yield discontinuous solutions. Therefore, careful consideration is required when using numerical methods to solve these equations because of the Gibbs phenomenon. A remedy based on the Schwartz duality has been previously proposed, yielding superior results without oscillations. However, this approach has primarily been applied to linear problems and still exhibits the Gibbs phenomenon when extended to nonlinear or higher-dimensional problems. In this paper, we propose a consistent yet simple approach based on Schwartz duality that can handle such problems. Our proposed approach utilizes a modified direct projection method with a consistent discrete derivative of the Heaviside function, which directly approximates the Dirac delta function. As numerical examples, we consider several problems, including the Burgers' equation and the two-dimensional time-dependent advection equation. The proposed method effectively eliminates Gibbs oscillations without the need for traditional regularization and demonstrates uniform error reduction for the problems considered.

AMS subject classifications: 65M70

Key words: Differential equations with singular sources, Dirac delta function, Gibbs phenomenon, Schwartz duality, spectral collocation method.

1. Introduction

Singularly perturbed partial differential equations (PDEs) frequently appear in various applications, especially for modeling phenomena where small changes in certain parameters can lead to significant variations in the solution, such as in boundary layers or reaction-diffusion processes. However, solving these PDEs numerically with guaranteed stability and non-oscillatory solution behavior is challenging due to the presence of singular or steep solution gradients. Various numerical methods have been developed to address this problem,

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including [1, 4, 6, 9, 22], to name a few. In particular, singular differential equations that contain Dirac delta function-type source terms are challenging to solve numerically. The most common approach relies on regularization methods to approximate the Dirac delta function, which includes methods such as the immersed boundary method [20], the volume of fluid method [12], and the level set method [16]. However, the regularization of the Dirac delta function could prevent instability but often degrades accuracy. When high-order methods such as spectral methods are used, the challenges increase, as spectral decomposition with orthogonal polynomials such as Chebyshev and Legendre polynomials is highly sensitive to solution regularity. Consequently, applying spectral approximation to differential equations with the Dirac delta function can lead to spurious oscillations, degrading accuracy and sometimes stability for nonlinear problems. Although these spectral methods are susceptible to oscillation, they offer the advantage of superior convergence accuracy compared to low-order numerical methods if used successfully. In [10], Schwartz duality was proposed to achieve exact cancellation on the grid points and obtain numerically stable solutions with spectral accuracy for the differential equations with the Dirac delta function. Further in [19], fractional differential equations with the Dirac delta function were successfully solved with the Schwartz duality approach. Nevertheless, this approach is limited to linear problems, and for the non-conservative nonlinear problems the oscillation persists due to the lack of consistency between the term with the differential operator and the source term. The main idea behind the Schwartz duality approach consists in using the cancellation of oscillations only at the collocation points. That is, the interpolation does not necessarily eliminate the oscillations, as the approximated solution at these points may be discontinuous. The observed cancellation arises from the consistent formulation of the Dirac delta function over the collocation points, which is evident in linear and one-dimensional (1D) problems. However, in nonlinear problems, exact cancellation is not straightforward. Additionally, for high-dimensional problems, the presence of multiple derivatives across dimensions makes it challenging to maintain a consistent formulation. It was observed in [17] that regularization methods—commonly employed to approximate the high-dimensional Dirac delta function—are easy to fail to accurately reconstruct the exact solution.

In this paper, we first consider the following perturbed conservation law for $u(x,t)$: $[-1,1]\times[0,\infty)\to\mathbb{R}$

$$u_t + f_x(u) = \delta(x-c), \tag{1.1}$$

where the flux $f(u)$ is, in general, a nonlinear function of u and the singular source term, represented by the Dirac delta function $\delta(x-c)$, is located at $x=c$ with $c\in(-1,1)$. To demonstrate the effectiveness of the proposed method, we considered several problems, including the 1D Burgers' equation and two-dimensional (2D) advection and non-linear equations. For example, in order to demonstrate the effectiveness of the proposed method, we considered the Burgers' equation in its non-conservative form — e.g. $f_x(u)=uu_x$.

In this work, we solve Eq. (1.1) numerically by the Chebyshev collocation method. The spectral collocation method provides highly accurate solutions for PDEs, exhibiting so-called spectral convergence when the solution is sufficiently smooth. However, this desirable convergence behavior deteriorates when the solution is non-smooth or contains