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Piecewise Smooth N-Dimensional Nonlinear Singular Singularly Perturbed Boundary Value Problems

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Abstract. Internal layer phenomena are often appear in various piecewise smooth nonlinear singular singularly perturbed problems of natural sciences. To characterize these special structures, we study piecewise smooth n-dimensional nonlinear singular singularly perturbed boundary value problems. In particular, we show the existence of solutions with an internal layer and construct their asymptotic expansions. The remainder estimations of the approximate solutions are also given. Finally, an example aimed to verify the correctness of the developed theory is presented.

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Key words: Asymptotic method, singularly perturbed equation, discontinuous equation.

1. Introduction

Many practical problems in science can be described as piecewise smooth singularly perturbed problems associated with various types of differential equations [1, 4, 6]. The significant feature of this problems is that their solutions will vary rapidly in the thin internal layer [12]. In order to characterize the solutions, many experts and scholars have studied the singular perturbation problems with discontinuous right-hand side [2, 7, 8, 13]. Ding *et al.* [2] gave an asymptotic solution for a class of singularly perturbed semi-linear boundary value problems with discontinuous functions. Nefedov and Ni [7] developed asymptotic methods for a one-dimensional stationary reaction-diffusion equation in which the term describing reaction undergoes a discontinuous at some point. Ni *et al.* [8] considered a boundary value problem for a piecewise smooth singularly perturbed second order

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differential equation. However, researches of asymptotic methods for the piecewise smooth n-dimensional nonlinear singularly perturbed problems are not yet fully developed. The difficulties lies in the vector problems are more complex than that in the scalar problems.

This paper investigates the following piecewise smooth n-dimensional nonlinear singular singularly perturbed boundary value problem:

$$\begin{aligned} &\mu\frac{\mathrm{d}x}{\mathrm{d}t}=f(x,t,\mu), \quad 0<t<1,\\ &Ax(0,\mu)=Ax^0, \quad Bx(1,\mu)=Bx^1, \end{aligned} \tag{1.1}$$

where $0<\mu\ll 1$ is a small parameter, $x\in\mathbb{R}^n$, the discontinuous function $f(x,t,\mu)$ is defined as

$$\begin{aligned} &f(x,t,\mu)\triangleq\begin{cases} f^{(-)}(x,t,\mu), & (t,x)\in D^{(-)},\\ f^{(+)}(x,t,\mu), & (t,x)\in D^{(+)}, \end{cases}\\ &f^{(-)}(x,t,\mu)\neq f^{(+)}(x,t,\mu), \ \ (t,x)\in D^{(0)} \end{aligned}$$

with

$$\begin{aligned} &f^{(\mp)}:\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}^n,\\ &D^{(-)}\triangleq\big\{(t,x)\,|\,0\leqslant t\leqslant 1, H(x,t)<0\big\},\\ &D^{(0)}\triangleq\big\{(t,x)\,|\,0\leqslant t\leqslant 1, H(x,t)=0\big\},\\ &D^{(+)}\triangleq\big\{(t,x)\,|\,0\leqslant t\leqslant 1, H(x,t)>0\big\}. \end{aligned}$$

To construct an asymptotic solution the problem (1.1), we assume the functions $f^{(\mp)}(x,t,\mu)$ and the surface $H(x,t)=0$ be sufficiently smooth on the domain $D\triangleq D^{(-)}\cup D^{(0)}\cup D^{(+)}$. Besides,

$$A=\mathrm{diag}(E_s,0,E_c), \quad B=\mathrm{diag}(0,E_u,0), \quad s+u+c=n \tag{1.2}$$

are $(n\times n)$-order matrices, E_a is the a-dimensional identity matrix. If $\mu=0$ in (1.1), the degenerate problem

$$f^{(\mp)}(x,t,0)=0 \tag{1.3}$$

can be obtained.

Assumption 1.1. The degenerate problem (1.3) has a family of solution of the form

$$\bar{x}=\begin{cases} \varphi^{(-)}\big(\alpha^{(-)}(t),t\big)=\varphi^{(-)}\Big(\alpha_1^{(-)}(t),\cdots,\alpha_c^{(-)}(t),t\Big), & (t,x)\in D^{(-)},\\ \varphi^{(+)}\big(\alpha^{(+)}(t),t\big)=\varphi^{(+)}\Big(\alpha_1^{(+)}(t),\cdots,\alpha_c^{(+)}(t),t\Big), & (t,x)\in D^{(+)}, \end{cases}$$

where $\varphi^{(\mp)}(\alpha^{(\mp)}(t),t)$ satisfy the conditions:

(1) $\varphi^{(\mp)}(\alpha^{(\mp)}(t),t)$ are sufficiently smooth in the domain D.

(2) $\varphi_\alpha^{(\mp)}(\alpha^{(\mp)}(t),t)$ have constant rank c.