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A Greedy Randomized Average Block Projection Method for Linear Feasibility Problems

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Abstract. The randomized projection (RP) method is a simple iterative scheme for solving linear feasibility problems and has gained popularity due to its speed and low memory requirement. This paper develops an accelerated variant of the standard RP method by using two ingredients: the greedy probability criterion and the average block approach, and obtains a greedy randomized average block projection (GRABP) method for solving large-scale systems of linear inequalities. We demonstrate that the GRABP method achieves deterministic linear convergence with various extrapolated step sizes. Numerical experiments on both randomly generated and real-world data show the advantage of GRABP over several state-of-the-art solvers, such as the RP method, the sampling Kaczmarz Motzkin (SKM) method, the generalized SKM method, and the Nesterov acceleration of SKM method.

AMS subject classifications: 65K05, 90C15, 65F10

Key words: Linear feasibility, randomized projection, average block, greedy probability criterion, convergence property.

1. Introduction

1.1. Model and notation

We consider the problem of solving large-scale systems of linear inequalities

$$Ax \le b, \tag{1.1}$$

where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. We confine the scope of this work to the regime of $m \gg n$, where iterative methods are more competitive for such problems. We denote the feasible region of (1.1) by $S = \{x \in \mathbb{R}^n \mid Ax \le b\}$. Throughout this paper, we assume that the coefficient matrix A has no zero rows and $S \neq \emptyset$.

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For a given matrix G, we use $\|G\|_2$, $\|G\|_F$, and $G^\dagger$ to denote the spectral norm, the Frobenius norm, and the Moore-Penrose pseudoinverse, respectively. We use $\sigma_{\min}(G)$ to denote the smallest nonzero singular value of the matrix G. For an integer $m \geq 1$, let $[m] := \{1, \ldots, m\}$. For any vector $x \in \mathbb{R}^n$, we use $x_i, x^\top$, $\|x\|_2$, and $\|x\|_p$ to denote the i-th entry, the transpose, the Euclidean norm and the p-norm of x, respectively. For any $u \in \mathbb{R}$ and $v \in \mathbb{R}^n$, we define $(u)_+ = \max\{0, u\}$ and $(v)_+ = ((v_1)_+, \ldots, (v_n)_+)^\top$. We refer to $\{\mathscr{I}_1, \mathscr{I}_2, \cdots, \mathscr{I}_t\}$ as a partition of $[m]$ if $\mathscr{I}_i \bigcap \mathscr{I}_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^t \mathscr{I}_i = [m]$. For a given index set $\mathscr{I}_i$, we use $G_{\mathscr{I}_i,:}$ to denote the row submatrix of the matrix G indexed by $\mathscr{I}_i$ and $u_{\mathscr{I}_i}$ denote the subvector of the vector u with components listed in $\mathscr{I}_i$. We use $P_S(u)$ to represent the orthogonal projection of u onto the feasible region S. For any random variables ξ, we use $\mathbb{E}[\xi]$ to denote the expectation of ξ.

1.2. The randomized Kaczmarz method

The Kaczmarz method [17], also known as the algebraic reconstruction technique (ART) [8,12], is a widely used algorithm for solving linear systems $Ax = b$. Starting from $x^0 \in \mathbb{R}^n$, the canonical Kaczmarz method constructs x^{k+1} by

$$x^{k+1} = x^k - \frac{A_{i,:}x^k - b_i}{\|A_{i,:}\|_2^2} A_{i,:}^\top,$$

where i is selected from $[m]$ cyclically. In fact, the current iterate is projected orthogonally onto the selected hyperplane $\{x \mid A_{i,:}x = b_i\}$ at each iteration. The iteration sequence $\{x^k\}_{k\geq 0}$ converges to $x_*^0 := A^\dagger b + (I - A^\dagger A)x^0$. However, the rate of convergence is hard to obtain. In the seminal paper [31], Strohmer and Vershynin analyzed the randomized variant of the Kaczmarz method. In particular, they proved that if the i-th row of A is selected with probability proportional to $\|A_{i,:}\|_2^2$, then the method converges linearly in expectation.

Leventhal and Lewis [19] extended the randomized Kaczmarz (RK) method to solve the linear feasibility problem (1.1). At each iteration k, if the inequality is already satisfied for the selected row i, then set $x_{k+1} = x_k$. If the inequality is not satisfied, the previous iterate only projects onto the solution hyperplane $\{x \mid A_{i,:}x = b_i\}$. The update rule for this algorithm is thus

$$x^{k+1} = x^k - \frac{(A_{i,:}x^k - b_i)_+}{\|A_{i,:}\|_2^2} (A_{i,:})^\top. \tag{1.2}$$

One can see that x^{k+1} in (1.2) is indeed the projection of x^k onto the set $\{x \mid A_{i,:}x \leq b_i\}$. Leventhal and Lewis proved that such RP method converges to a feasibility solution linearly in expectation — cf. [19, Theorem 4.3].

Combining the ideas of Kaczmarz and Motzkin methods [1,24], Loera *et al.* [6] recently proposed an SKM method for solving the linear feasibility problem (1.1). Later, Morshed *et al.* [23] developed a generalized framework, namely the generalized SKM (GSKM) method that extends the SKM algorithm and proves the existence of a family of SKM-type methods. In addition, they also proposed a Nesterov-type acceleration scheme in the SKM method