East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.2025-092.100825

Vol. **xx**, No. x, pp. 1-22
xxx 2025

Quasi-Periodic Breathers of the Hirota-Maccari System

Mao-Wei Yin and Jian-Qing Sun*

School of Mathematical Sciences, Ocean University of China,
Qingdao 266100, China.

Received 13 May 2025; Accepted (in revised version) 10 August 2025.

Abstract. The Hirota-Maccari (HM) system generalizing the Hirota equation, serves as a generalized (2+1)-dimensional model in fluid dynamics, plasma physics, and optical fiber communication. In this paper, we obtain the quasi-periodic breathers to HM system by use of Hirota's bilinear method and the theta function. Asymptotic analysis demonstrates that the quasi-periodic breathers can be reduced to regular breathers under small amplitude limits. Moreover, we also classified solutions based on their asymptotic behavior. Numerical examples are given to confirm the theoretical analysis.

AMS subject classifications: 65M10, 78A48

Key words: Quasi-periodic breather, Hirota-Maccari system, Hirota's bilinear method, theta function.

1. Introduction

Integrable systems and soliton equations have garnered significant attention in both physics and mathematics. These nonlinear partial differential equations serve as powerful tools to model various nonlinear phenomena in physical sciences, including fluid mechanics, nonlinear optics, magnetic fluids, and many other areas. The solutions of soliton equations are of significant interest due to their diverse types, such as soliton solutions, breather solutions, lump solutions, and rogue wave solutions. These solutions provide deep insights into the behavior of nonlinear systems and have broad applications in both theoretical and applied research.

It is well known that the Kadomtsev-Petviashvili (KP) equation is regarded as the most fundamental integrable nonlinear dispersive wave equations in (2+1)-dimensions — cf. [1,19,20] and references therein. This is because many integrable systems can be derived as special reductions of the KP hierarchy, which comprises the KP equation along with its infinitely many symmetries. Starting from the KP equations, Maccari introduced a new type of coupled nonlinear evolution equation in (2+1)-dimensions, called the Hirota-Maccari

*Corresponding author. *Email address:* `sunjq@lsec.cc.ac.cn` (J.Q. Sun)

system

$$\begin{aligned}&\mathrm{i}u_t+u_{xy}+\mathrm{i}\beta u_{xxx}+uv-\mathrm{i}\beta|u|^2u_x=0,\\&3v_x+(|u|^2)_y=0,\end{aligned}\tag{1.1}$$

which was derived using an asymptotically exact reduction method based on Fourier expansion and spatiotemporal rescaling in [26]. The Hirota-Maccari system (1.1) is integrable due to the existence of the corresponding Lax pairs, and it could be reduced to the (1+1)-dimensional integrable Hirota equation [13,27,28,30,38] when $x=y$. Its exact solutions including soliton solutions, periodic waves solutions, travelling wave solutions, have been obtained using a variety of methods, such as the unified algebraic method [9], the Weierstrass elliptic function expansion method [5], the complex hyperbolic function method [4], the direct algebraic method [39], the solitary wave ansatz method [33], the extended trial equation method [7] and the bilinear method [37]. Through the coordinate transformation $x\to \mathrm{i}x, y\to -y, t\to -\mathrm{i}t$, the HM system (1.1) can be rewritten as

$$\begin{aligned}&u_t-\mathrm{i}u_{xy}+\beta u_{xxx}+uv+\beta|u|^2u_x=0,\\&3\mathrm{i}v_x-\left(|u|^2\right)_y=0,\end{aligned}\tag{1.2}$$

which can be further transformed into the bilinear form

$$\begin{aligned}&\left(D_t-\mathrm{i}D_xD_y+\beta D_x^3+\beta D_x\right)g\cdot f=0,\\&\left(3D_x^2+1\right)f\cdot f=gg^*\end{aligned}\tag{1.3}$$

under the dependent variable transformation

$$u=\frac{g}{f},\quad v=-2\mathrm{i}(\ln f)_{xy},$$

where f is real, g is complex, $*$ means the complex conjugation, and D is Hirota operator [14] defined as

$$\begin{aligned}&D_t^mD_x^na(t,x)\cdot b(t,x)=\frac{\partial^m}{\partial s^m}\frac{\partial^n}{\partial y^n}a(t+s,x+y)b(t-s,x-y)|_{s=0,y=0},\\&m,n=0,1,2,\ldots.\end{aligned}$$

The bilinear HM equation (1.3) is a special case of the coupled DS-KP equation proposed by Hietarinta in [12]. The rational and semi-rational solutions to (1.3) were studied in [31,34] by use of the KP hierarchy reduction method.

Recently, the quasi-periodic solution (also called algebra-geometric solutions or finite-gap solutions) to the (2+1)-dimensional integrable systems have been investigated by different methods [6, 10, 11, 17, 18, 22, 24]. This kind of solutions is expressed by the theta function [8], a summation of an infinite number of exponential functions. Since breathers characterized by theta functions reduce to regular breathers described by elementary functions (e.g., trigonometric functions), they are termed quasi-periodic breathers. Based on Hirota's bilinear method and theta functions, one of the authors and her collaborators introduced a direct approach for efficiently and directly computing quasi-periodic breathers of the Davey-Stewartson (DS) equation in [29]. Subsequently, Zhao *et al.* [41] and Xin *et al.* [35] successfully applied this method to quasi-periodic breathers for the KP-based system and the Fokas system, respectively. In this paper, we apply this direct method to construct quasi-periodic breathers for the HM system (1.2) and analyze their asymptotic behavior in the small-amplitude limit. To the best of our knowledge, no previous work has addressed this particular aspect.

The paper is organized as follows. In Section 2, we provide a brief overview of Hirota's bilinear method and theta functions, which serve as the foundational tools for our study. We then derive regular breather solutions for the HM system and conduct a detailed analysis and classification of these solutions. Section 3 focuses on constructing quasi-periodic breather solutions by formulating an over-determined nonlinear algebraic system. Through asymptotic analysis, we demonstrate that quasi-periodic breathers can reduce to regular breathers under certain limiting conditions, allowing us to categorize them into three distinct types. In Section 4, we present numerical examples of quasi-periodic breathers for the HM system. Section 5 contains discussion and conclusions.

2. Breathers of HM System and Theta Functions

In this section, we review regular breathers to the (2+1)-dimensional HM system (1.2) and introduce theta functions of even dimension to construct the quasi-periodic breathers.

2.1. Bilinear method and breathers

In order to construct breather solutions to the HM system (1.2), we introduce the following dependent variable transformation:

$$u = a\exp\left\{\mathrm{i}(\tilde{k}x + \tilde{l}y + \tilde{w}t + \tilde{\eta}^0)\right\}\frac{g}{f}, \quad v = -2\mathrm{i}(\ln f)_{xy}, \tag{2.1}$$

where $a, \tilde{k}, \tilde{l}, \tilde{w}, \tilde{\eta}^0$ are real numbers. Then the HM system (1.2) can be transformed to

$$\left(3D_x^2 - c\right) f \cdot f - a^2 g g^* = 0, \tag{2.2}$$

$$\Big(D_t + \beta D_x^3 + 3\mathrm{i}\beta\tilde{k}D_x^2 - \left(3\beta\tilde{k}^2 - \tilde{l} + c\beta\right)D_x + \tilde{k}D_y - \mathrm{i}D_xD_y + \mathrm{i}\left(\tilde{w} + \tilde{k}\tilde{l} - \beta\tilde{k}^3 - c\beta\tilde{k}\right)\Big) g \cdot f = 0. \tag{2.3}$$

We still call Eqs. (2.2)-(2.3) the HM system without confusions. Consider the functions

$$f = 1 + \exp(\eta) + \exp(\eta^*) + A\exp(\eta + \eta^*), \tag{2.4}$$

$$g = 1 + \exp(\eta + \mathrm{i}\phi) + \exp(\eta^* + \mathrm{i}\phi^*) + A\exp\left(\eta + \eta^* + \mathrm{i}(\phi + \phi^*)\right), \tag{2.5}$$

where $\eta = kx + ly + wt + \eta^0$ and k, l, w, ϕ are complex numbers, η^0 is an arbitrary phase, and A is a real number. Substituting (2.4)-(2.5) into the HM system (2.2)-(2.3), we have the following relations:

$$c = -a^2, \tag{2.6}$$

$$\tilde{w} = -\tilde{k}\tilde{l} + \beta\tilde{k}^3 + c\beta\tilde{k}, \tag{2.7}$$

$$k^2 = -\frac{2}{3}a^2\sin^2\left(\frac{\phi}{2}\right), \tag{2.8}$$

$$w = -\beta k^3 + \left(3\beta\tilde{k}^2 - \tilde{l} + c\beta\right)k - \tilde{k}l + \left(kl - 3\beta\tilde{k}k^2\right)\cot\left(\frac{\phi}{2}\right), \tag{2.9}$$

$$\begin{aligned} A &= \frac{-3(k-k^*)^2 - 2a^2\sin^2((\phi-\phi^*)/2)}{3(k+k^*)^2 + 2a^2\sin^2((\phi+\phi^*)/2)} \\ &= 1 + \frac{8a^2\sin^2(\phi/2)\sin^2(\phi^*/2)}{3(k+k^*)^2 + 2a^2\sin^2((\phi+\phi^*)/2)}. \end{aligned} \tag{2.10}$$

Non-singular solutions to Eqs. (2.1) with (2.4)-(2.5) are obtained when the inequality $A > 1$ is satisfied. These solutions correspond to 1-breathers of the HM system. Under this constraint and owing to the arbitrariness of parameters k, l, ϕ, three different types of breather solutions emerge. Notably, the analysis of the following three cases remains unchanged regardless of the value of β:

(i) In the first scenario, the parameters k, l and ϕ take complex values (neither pure real nor pure imaginary). The solution in this case corresponds to general breather solutions. The condition $A > 1$ is inherently satisfied.

(ii) The second situation is characterized by real values of k and l, while ϕ is pure imaginary. At this point, the parameter w is generally complex, and the resulting solution remains a general breather. However, if an additional constraint $\mathscr{R}(w) = 0$ holds, the corresponding solutions represent stationary breathers, manifesting periodic behavior in time and spatial localization which are the same as Kuznetsov-Ma breathers [21,25]. The condition $A > 1$ remains inherently satisfied.

(iii) The third case arises when both k and l are pure imaginary, and ϕ is real. Under these conditions, these solutions represent approximate homoclinic orbits. Although the condition $A > 1$ still holds, (2.8) implies that k must satisfy the constraint $-2a^2/3 < k^2 < 0$. To further investigate the structure of homoclinic orbits, we set $k = \mathrm{i}\alpha$, $l = \mathrm{i}\gamma$, $w = \lambda_1 + \mathrm{i}\lambda_2$, $\eta^0 = \sigma_1 + \mathrm{i}\sigma_2$. Here α, γ, λ_j, σ_j $(j = 1,2)$ are real parameters. With these values substituted into (2.8)-(2.9), the resulting expressions are as follows:

$$\lambda_1 = \pm\left(-\alpha\gamma + 3\beta\tilde{k}\alpha^2\right)\sqrt{\frac{2a^2 - 3\alpha^2}{3\alpha^2}},$$
$$\lambda_2 = \beta\alpha^3 + \left(3\beta\tilde{k}^2 - \tilde{l} + c\beta\right)\alpha - \tilde{k}\gamma.$$

Additionally, the solution u can be represented as

$$u = a\exp\{\mathrm{i}(\tilde{\eta} + \phi)\} \cdot \frac{\sqrt{A}\cosh(\lambda_1 t + \sigma_1 + \ln(A)/2 + \mathrm{i}\phi) + \cos(\alpha x + \gamma y + \lambda_2 t + \sigma_2)}{\sqrt{A}\cosh(\lambda_1 t + \sigma_1 + \ln(A)/2) + \cos(\alpha x + \gamma y + \lambda_2 t + \sigma_2)}.$$

Here, $\tilde{\eta} = \tilde{k}x + \tilde{l}y + \tilde{w}t + \tilde{\eta}^0$ is pure real. Furthermore, depending on the value of λ_1, we consider two cases: $\lambda_1 > 0$ and $\lambda_1 < 0$. In the first case, $u \to a\exp\{\mathrm{i}(\tilde{\eta} + 2\phi)\}$ as $t \to +\infty$, and $u \to a\exp(\mathrm{i}\tilde{\eta})$ when $t \to -\infty$. In the second case, $u \to a\exp(\mathrm{i}\tilde{\eta})$ as $t \to +\infty$, and $u \to a\exp\{\mathrm{i}(\tilde{\eta} + 2\phi)\}$ when $t \to -\infty$. The asymptotic behavior indicates that the solution first grows exponentially, reaches a point of maximum modulation, and eventually decays back to where it started. Moreover, this type of solution, characterized by temporal localization and spatial periodicity, is also known as an Akhmediev breather [3].

2.2. Theta function

Quasi-periodic breathers can be effectively constructed using the theta function. In this context, we consider the following $2N$-dimensional theta function defined by its Fourier series of the form

$$\theta(\boldsymbol{\eta}, \mathbf{0}, \boldsymbol{\phi} \,|\boldsymbol{\tau}) = \sum_{\boldsymbol{n} \in \mathbb{Z}^{2N}} \exp\left\{ \langle \boldsymbol{n}, \boldsymbol{\eta} + \mathrm{i}\boldsymbol{\phi} \rangle - \frac{1}{2}\langle \boldsymbol{\tau n}, \boldsymbol{n} \rangle \right\} \tag{2.11}$$

with $\langle \cdot, \cdot \rangle$ denoting the Euclidean scalar product

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \sum_{j=0}^{j=2N} a_j b_j, \quad \langle \boldsymbol{\tau a}, \boldsymbol{b} \rangle = \sum_{i,j=1}^{2N} \tau_{ij} a_i b_j.$$

Here $\boldsymbol{\phi} \in \mathbb{C}^{2N}$ are characteristics, and $\boldsymbol{\tau} \in \mathbb{C}^{2N \times 2N}$ (which is called Riemann matrix) is a symmetric matrix whose real part is positive definite. Besides $\boldsymbol{\eta}$, $\boldsymbol{\phi}$ and $\boldsymbol{\tau}$ satisfy the following relation:

$$\eta_{j+N} = \eta_j^*, \quad \phi_{j+N} = \phi_j^*, \quad j = 1, 2, \ldots, N, \tag{2.12}$$

$$\tau_{j+N, l+N} = \tau_{jl}^*, \qquad j, l = 1, 2, \ldots, N, \tag{2.13}$$

$$\tau_{j, l+N} = \tau_{l, j+N}^*, \qquad j, l = 1, 2, \ldots, N, \tag{2.14}$$

where

$$\eta_j = k_j x + l_j y + w_j t + \eta_j^0, \quad j = 1, 2, \ldots, N.$$

The elements k_j and l_j correspond to the wave numbers, w_j corresponds to the frequencies, η_j^0 corresponds to the phase positions, the diagonal elements τ_{jj} correspond to the amplitudes, while the off-diagonal elements τ_{jl} $(j \neq l)$ correspond to the interactions of waves.

The theta function (2.11) has the following quasi-periodic property :

$$\theta\big(\boldsymbol{\eta} + 2\pi\mathrm{i}\boldsymbol{\xi} + \boldsymbol{\tau\zeta}, \mathbf{0}, \boldsymbol{\phi} | \boldsymbol{\tau}\big) = \exp\left\{ 2\pi\mathrm{i}\langle \mathbf{0}, \boldsymbol{\xi} \rangle + \langle \boldsymbol{\zeta}, \boldsymbol{\eta} + \mathrm{i}\boldsymbol{\phi} \rangle + \frac{1}{2}\langle \boldsymbol{\tau\zeta}, \boldsymbol{\zeta} \rangle \right\} \theta\big(\boldsymbol{\eta}, \mathbf{0}, \boldsymbol{\phi} | \boldsymbol{\tau}\big),$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ are integer vectors [8]. The terms $2\pi\mathrm{i}\boldsymbol{\xi}$ and $\tau\boldsymbol{\zeta}$ represent the periods along the imaginary axis and the real axis, respectively.

For convenience, we rewrite the vectors and the Riemann matrix in the following block form:

$$\boldsymbol{n}=\begin{pmatrix}\boldsymbol{n}_A\\ \boldsymbol{n}_B\end{pmatrix},\quad \boldsymbol{\tau}=\begin{pmatrix}\boldsymbol{\tau}_A & \boldsymbol{\tau}_B^*\\ \boldsymbol{\tau}_B & \boldsymbol{\tau}_A^*\end{pmatrix},\quad \boldsymbol{\eta}=\begin{pmatrix}\boldsymbol{\eta}_A\\ \boldsymbol{\eta}_A^*\end{pmatrix},\quad \boldsymbol{\phi}=\begin{pmatrix}\boldsymbol{\phi}_A\\ \boldsymbol{\phi}_A^*\end{pmatrix},$$

where the subscript A and B represent the upper and lower half of the vectors respectively, $\boldsymbol{n}_A,\boldsymbol{n}_B,\boldsymbol{\eta}_A,\boldsymbol{\phi}_A$ are vectors of dimension N, $\boldsymbol{\tau}_A$ is a symmetric matrix of $N\times N$ and $\boldsymbol{\tau}_B$ is a Hermitian matrix of $N\times N$. As a result, theta function (2.11) and its complex conjugation can be respectively written as

$$\begin{aligned}\theta(\boldsymbol{\eta},\boldsymbol{0},\boldsymbol{\phi}|\boldsymbol{\tau})=\sum_{\boldsymbol{n}_A,\boldsymbol{n}_B\in\mathbb{Z}^N}\exp\bigg\{&\langle\boldsymbol{n}_A,\boldsymbol{\eta}_A+\mathbf{i}\boldsymbol{\phi}_A\rangle+\langle\boldsymbol{n}_B,\boldsymbol{\eta}_A^*+\mathbf{i}\boldsymbol{\phi}_A^*\rangle-\frac{1}{2}\langle\boldsymbol{\tau}_A\boldsymbol{n}_A,\boldsymbol{n}_A\rangle\\ &-\frac{1}{2}\langle\boldsymbol{\tau}_A^*\boldsymbol{n}_B,\boldsymbol{n}_B\rangle-\frac{1}{2}\langle\boldsymbol{\tau}_B\boldsymbol{n}_A,\boldsymbol{n}_B\rangle-\frac{1}{2}\langle\boldsymbol{\tau}_B^*\boldsymbol{n}_B,\boldsymbol{n}_A\rangle\bigg\},\end{aligned}\tag{2.15}$$

and

$$\begin{aligned}\theta^*(\boldsymbol{\eta},\boldsymbol{0},\boldsymbol{\phi}|\boldsymbol{\tau})=\sum_{\boldsymbol{n}_A,\boldsymbol{n}_B\in\mathbb{Z}^N}\exp\bigg\{&\langle\boldsymbol{n}_A,\boldsymbol{\eta}_A^*-\mathbf{i}\boldsymbol{\phi}_A^*\rangle+\langle\boldsymbol{n}_B,\boldsymbol{\eta}_A-\mathbf{i}\boldsymbol{\phi}_A\rangle-\frac{1}{2}\langle\boldsymbol{\tau}_A^*\boldsymbol{n}_A,\boldsymbol{n}_A\rangle\\ &-\frac{1}{2}\langle\boldsymbol{\tau}_A\boldsymbol{n}_B,\boldsymbol{n}_B\rangle-\frac{1}{2}\langle\boldsymbol{\tau}_B^*\boldsymbol{n}_A,\boldsymbol{n}_B\rangle-\frac{1}{2}\langle\boldsymbol{\tau}_B\boldsymbol{n}_B,\boldsymbol{n}_A\rangle\bigg\}.\end{aligned}\tag{2.16}$$

Substituting the transformations $\boldsymbol{n}_A\mapsto\boldsymbol{n}_B'$, $\boldsymbol{n}_B\mapsto\boldsymbol{n}_A'$ in Eq. (2.16) and comparing it to Eq. (2.15), we have the following proposition.

Proposition 2.1. *If $f=\theta(\boldsymbol{\eta},\boldsymbol{0},\boldsymbol{\phi}|\boldsymbol{\tau})$ is the theta function of the form* (2.11)-(2.14)*, then its conjugation is $f^*=\theta(\boldsymbol{\eta},\boldsymbol{0},-\boldsymbol{\phi}|\boldsymbol{\tau})$.*

It follows from Proposition 2.1 that the conjugation of the theta function is only related to $\boldsymbol{\phi}$. Therefore, $\boldsymbol{\theta(\eta,0,\phi|\tau)}$ is real when $\boldsymbol{\phi}=0$.

3. Quasi-Periodic Waves in Form of Theta Functions

In this section, we conduct a detailed study of the quasi-periodic breathers constructed by the theta function and Hirota's bilinear method. We further show that, through asymptotic analysis, these quasi-periodic breathers can be reduced to regular breathers.

Firstly, we show how to construct quasi-periodic breathers by theta functions. Based on these expressions of breathers, an over-determined complex algebraic system with complex and real unknown parameters can be obtained. To simplify the problem, we separate the real and imaginary parts of the unknown complex parameters and treat each complex equation in the system as a set of independent real equations. Thereby, the complex algebraic system is transformed into a purely real system with real unknowns, which can be formulated as a nonlinear least squares problem. Secondly, we verify the existence of an asymptotic relation between the quasi-periodic breathers and regular breathers in the case of $N=1$.

3.1. Conditions for quasi-periodic breathers and numerical approach

Set f and g as

$$f = \theta(\boldsymbol{\eta}, \mathbf{0}, \mathbf{0}|\boldsymbol{\tau}), \tag{3.1}$$

$$g = \theta(\boldsymbol{\eta}, \mathbf{0}, \boldsymbol{\phi}|\boldsymbol{\tau}). \tag{3.2}$$

According to Proposition 2.1, f is real and

$$g^* = \theta(\boldsymbol{\eta}, \mathbf{0}, -\boldsymbol{\phi}|\boldsymbol{\tau}). \tag{3.3}$$

Note that according to [14], the operator D has the following property:

$$F(D_x)e^{k_1 x} \cdot e^{k_2 x} = F(k_1 - k_2)e^{(k_1+k_2)x}, \tag{3.4}$$

where F is an arbitrary polynomial. Substituting the expressions (3.1)-(3.3) into the bilinear equations (2.2)-(2.3) with the identity (3.4), leads to the following theorem.

Theorem 3.1. *Functions f, g defined by* (3.1)-(3.2) *are quasi-periodic breathers of the bilinear equations* (2.2)-(2.3) *if*

$$F_1(\boldsymbol{m}) = 0, \tag{3.5}$$

$$F_2(\boldsymbol{m}) = 0 \tag{3.6}$$

for all $\boldsymbol{m} \in \mathbb{Z}^{2N}$ with elements $m_j = 0, 1,\ j = 1, 2, \ldots, 2N$, where

$$\begin{aligned}
F_1(m) = \sum_{\boldsymbol{n}\in\mathbb{Z}^{2N}} & \left\{3\langle \mathbf{2}\boldsymbol{n}-\boldsymbol{m},\boldsymbol{k}\rangle^2 - c - a^2\exp\big(\mathrm{i}\langle \mathbf{2}\boldsymbol{n}-\boldsymbol{m},\boldsymbol{\phi}\rangle\big)\right\} \\
& \times \exp\left\{-\frac{1}{2}\langle \boldsymbol{\tau}\boldsymbol{n},\boldsymbol{n}\rangle - \frac{1}{2}\langle \boldsymbol{\tau}(\boldsymbol{m}-\boldsymbol{n}),\boldsymbol{m}-\boldsymbol{n}\rangle\right\}, \\
F_2(m) = \sum_{\boldsymbol{n}\in\mathbb{Z}^{2N}} & \Big\{\langle \boldsymbol{m}-\mathbf{2}\boldsymbol{n},\boldsymbol{w}\rangle + \beta\langle \boldsymbol{m}-\mathbf{2}\boldsymbol{n},\boldsymbol{k}\rangle^3 + 3\mathrm{i}\beta\tilde{k}\langle \boldsymbol{m}-\mathbf{2}\boldsymbol{n},\boldsymbol{k}\rangle^2 \\
& + \tilde{k}\langle \boldsymbol{m}-\mathbf{2}\boldsymbol{n},\boldsymbol{l}\rangle - \big(3\beta\tilde{k}^2 - \tilde{l} + c\beta\big)\langle \boldsymbol{m}-\mathbf{2}\boldsymbol{n},\boldsymbol{k}\rangle \\
& - \mathrm{i}\langle \boldsymbol{m}-\mathbf{2}\boldsymbol{n},\boldsymbol{k}\rangle\langle \boldsymbol{m}-\mathbf{2}\boldsymbol{n},\boldsymbol{l}\rangle + \mathrm{i}\big(\tilde{w} + \tilde{k}\tilde{l} - \beta\tilde{k}^3 - c\beta\tilde{k}\big)\Big\} \\
& \times \exp\left\{\frac{1}{2}\mathrm{i}\langle \boldsymbol{m}-\mathbf{2}\boldsymbol{n},\boldsymbol{\phi}\rangle - \frac{1}{2}\langle \boldsymbol{\tau}\boldsymbol{n},\boldsymbol{n}\rangle - \frac{1}{2}\langle \boldsymbol{\tau}(\boldsymbol{m}-\boldsymbol{n}),\boldsymbol{m}-\boldsymbol{n}\rangle\right\}.
\end{aligned}$$

Proof. Substituting (3.1)-(3.3) into (2.2)-(2.3) gives

$$\begin{aligned}
\big(3D_x^2 - c\big)f\cdot f - a^2 g g^* = & \sum_{\boldsymbol{n}\in\mathbb{Z}^{2N}}\sum_{\boldsymbol{n}'\in\mathbb{Z}^{2N}} \left\{3\langle \boldsymbol{n}-\boldsymbol{n}',\boldsymbol{k}\rangle^2 - c - a^2\exp(\mathrm{i}\langle \boldsymbol{n}-\boldsymbol{n}',\boldsymbol{\phi}\rangle)\right\} \\
& \times \exp\left\{\langle \boldsymbol{n}'+\boldsymbol{n},\boldsymbol{\eta}\rangle - \frac{1}{2}\langle \boldsymbol{\tau}\boldsymbol{n},\boldsymbol{n}\rangle - \frac{1}{2}\langle \boldsymbol{\tau}\boldsymbol{n}',\boldsymbol{n}'\rangle\right\} \\
\overset{\boldsymbol{m}=\boldsymbol{n}+\boldsymbol{n}'}{=\!=\!=} & \sum_{\boldsymbol{m}\in\mathbb{Z}^{2N}}\sum_{\boldsymbol{n}\in\mathbb{Z}^{2N}} \left\{3\langle \mathbf{2}\boldsymbol{n}-\boldsymbol{m},\boldsymbol{k}\rangle^2 - c - a^2\exp\big(\mathrm{i}\langle \mathbf{2}\boldsymbol{n}-\boldsymbol{m},\boldsymbol{\phi}\rangle\big)\right\}
\end{aligned}$$

$$\times \exp\left\{\langle \boldsymbol{m},\boldsymbol{\eta}\rangle - \frac{1}{2}\langle \tau \boldsymbol{n},\boldsymbol{n}\rangle - \frac{1}{2}\langle \tau(\boldsymbol{m}-\boldsymbol{n}),\boldsymbol{m}-\boldsymbol{n}\rangle\right\}$$
$$= \sum_{\boldsymbol{m}\in\mathbb{Z}^{2N}} F_1(\boldsymbol{m})\exp\big(\langle \boldsymbol{m},\boldsymbol{\eta}\rangle\big).$$

Analogously, we write

$$\begin{aligned}&\Big(D_t + \beta D_x^3 + 3\mathrm{i}\beta\tilde{k}D_x^2 - \big(3\beta\tilde{k}^2 - \tilde{l} + c\beta\big)D_x + \tilde{k}D_y \\ &\quad - \mathrm{i}D_xD_y + \mathrm{i}(\tilde{w} + \tilde{k}\tilde{l} - \beta\tilde{k}^3 - c\beta\tilde{k})\Big)g\cdot f \\ &= \sum_{\boldsymbol{m}\in\mathbb{Z}^{2N}} F_2(\boldsymbol{m})\exp\left(\langle \boldsymbol{m},\boldsymbol{\eta}\rangle + \frac{1}{2}\mathrm{i}\langle \boldsymbol{m},\boldsymbol{\phi}\rangle\right).\end{aligned}$$

This proves that for any $\boldsymbol{m}\in\mathbb{Z}^{2N}$, f and g will be the solutions of the bilinear HM system (2.2)-(2.3) if $F_1(\boldsymbol{m})=0$, $F_2(\boldsymbol{m})=0$.

Let $\boldsymbol{e}_j$ denote the $2N$-dimensional vector whose j-th element is 1 and all other elements are 0. Then for any $\boldsymbol{m}\in\mathbb{Z}^{2N}$, we have

$$\begin{aligned}F_1(\boldsymbol{m}-2\boldsymbol{e}_j) &= \sum_{\boldsymbol{n}\in\mathbb{Z}^{2N}} \left\{3\langle 2\boldsymbol{n}-\boldsymbol{m}+2\boldsymbol{e}_j,\boldsymbol{k}\rangle^2 - c - a^2\exp\big(\mathrm{i}\langle 2\boldsymbol{n}-\boldsymbol{m}+2\boldsymbol{e}_j,\boldsymbol{\phi}\rangle\big)\right\} \\ &\qquad \times \exp\left\{-\frac{1}{2}\langle \tau\boldsymbol{n},\boldsymbol{n}\rangle - \frac{1}{2}\langle \tau(\boldsymbol{m}-\boldsymbol{n}-2\boldsymbol{e}_j),\boldsymbol{m}-\boldsymbol{n}-2\boldsymbol{e}_j\rangle\right\} \\ &\overset{\boldsymbol{n}'=\boldsymbol{n}+\boldsymbol{e}_j}{=\!=\!=} \sum_{\boldsymbol{n}'\in\mathbb{Z}^{2N}} \left\{\langle 2\boldsymbol{n}'-\boldsymbol{m},\boldsymbol{k}\rangle^2 - c - a^2\exp\big(\mathrm{i}\langle 2\boldsymbol{n}'-\boldsymbol{m},\boldsymbol{\phi}\rangle\big)\right\} \\ &\qquad \times \exp\left\{-\frac{1}{2}\langle \tau(\boldsymbol{n}'-\boldsymbol{e}_j),\boldsymbol{n}'-\boldsymbol{e}_j\rangle - \frac{1}{2}\langle \tau(\boldsymbol{m}-\boldsymbol{n}'-\boldsymbol{e}_j),\boldsymbol{m}-\boldsymbol{n}'-\boldsymbol{e}_j\rangle\right\} \\ &= F_1(\boldsymbol{m})\cdot\exp\left\{\langle \tau\boldsymbol{e}_j,\boldsymbol{m}\rangle - \tau_{jj}\right\}. \end{aligned} \tag{3.7}$$

Similarly, we can prove

$$F_2(\boldsymbol{m}-2\boldsymbol{e}_j) = F_2(\boldsymbol{m})\cdot\exp\left\{\langle \tau\boldsymbol{e}_j,\boldsymbol{m}\rangle - \tau_{jj}\right\}. \tag{3.8}$$

Eqs. (3.7)-(3.8) imply that if $F_1(\boldsymbol{m})=0$ and $F_2(\boldsymbol{m})=0$ for all $\boldsymbol{m}$ with $m_j=0,1$, then they hold for all $\boldsymbol{m}\in\mathbb{Z}^{2N}$, and the proof is complete. □

Observing that the nonlinear algebraic equations (3.5) and (3.6) hold for all $m_j=0,1$, $j=1,2,\ldots,2N$, the total number of equations is 2×4^N. This system has parameters k_j, l_j, w_j, ϕ_j, τ_{jl}, $\tau_{j,l+N}$ for $l\geq j=1,2,\ldots,N$ and $\tilde{k}$, $\tilde{l}$, $\tilde{w}$, a, c. Among them, $\tau_{j,j+N}$, $\tilde{k}$, $\tilde{l}$, $\tilde{w}$, a and c must be real. Generally, parameters k_j, l_j, τ_{jj}, $\tilde{k}$, $\tilde{l}$, a are assumed to be known, while w_j, ϕ_j, τ_{jl}, $l>j$, $\tau_{j,l+N}$, $l\geq j$, $\tilde{w}$ and c are treated as unknowns. Then the problem of finding quasi-periodic breathers for the HM system becomes a nonlinear algebraic system with 2×4^N equations and N^2+2N+2 unknowns. It is worth noting that $\tau_{j,j+N}$, $\tilde{w}$ and c must be real while the remaining parameters are complex. To facilitate the numerical calculations, we consider real and imaginary parts separately. As a result, the total number of unknowns is $2N^2+3N+2$.

On the other hand, the nonlinear algebraic systems generated by (3.5) and (3.6) are complex and involve certain real unknowns. Therefore, one may encounter complex solutions when solving it directly. This mixed real-complex structure also increases both computational complexity and numerical difficulty. To overcome these challenges, it is essential to reformulate the system in real forms before applying numerical algorithms.

Rewrite the integer vector $\boldsymbol{m}$ in the block form

$$\boldsymbol{m} = \begin{pmatrix} \boldsymbol{m}_A \\ \boldsymbol{m}_B \end{pmatrix}, \quad \hat{\boldsymbol{m}} = \begin{pmatrix} \boldsymbol{m}_B \\ \boldsymbol{m}_A \end{pmatrix},$$

where $\hat{\boldsymbol{m}}$ is adjoint vector of $\boldsymbol{m}$. We have the following proposition for functions $F_1(\boldsymbol{m})$ and $F_2(\boldsymbol{m})$.

Proposition 3.1. *For any* $\boldsymbol{m} \in \mathbb{Z}^{2N}$,

$$\{F_1(\boldsymbol{m})\}^* = F_1(\hat{\boldsymbol{m}}), \quad \{F_2(\boldsymbol{m})\}^* = -F_2(\hat{\boldsymbol{m}}).$$

Proof. First, we prove that the function $F_1(\boldsymbol{m})$ is even with respect to $\boldsymbol{m}$. Indeed,

$$\begin{aligned} F_1(-\boldsymbol{m}) &= \sum_{\boldsymbol{n}\in\mathbb{Z}^{2N}} \left\{3\langle 2\boldsymbol{n}+\boldsymbol{m},\boldsymbol{k}\rangle^2 - c - a^2\exp\left(\mathrm{i}\langle 2\boldsymbol{n}+\boldsymbol{m},\boldsymbol{\phi}\rangle\right)\right\} \\ &\qquad \times \exp\left\{-\frac{1}{2}\langle \tau\boldsymbol{n},\boldsymbol{n}\rangle - \frac{1}{2}\langle \tau(-\boldsymbol{n}-\boldsymbol{m}),-\boldsymbol{n}-\boldsymbol{m}\rangle\right\} \\ &\xlongequal{\boldsymbol{n}'=\boldsymbol{n}+\boldsymbol{m}} \sum_{\boldsymbol{n}'\in\mathbb{Z}^{2N}} \left\{3\langle 2\boldsymbol{n}'-\boldsymbol{m},\boldsymbol{k}\rangle^2 - c - a^2\exp\left(\mathrm{i}\langle 2\boldsymbol{n}'-\boldsymbol{m},\boldsymbol{\phi}\rangle\right)\right\} \\ &\qquad \times \exp\left\{-\frac{1}{2}\langle \tau\boldsymbol{n}',\boldsymbol{n}'\rangle - \frac{1}{2}\langle \tau(\boldsymbol{n}'-\boldsymbol{m}),\boldsymbol{n}'-\boldsymbol{m}\rangle\right\} \\ &= F_1(\boldsymbol{m}). \end{aligned} \tag{3.9}$$

Second, using the block form of vectors, we rewrite $F_1(\boldsymbol{m})$ as

$$\begin{aligned} F_1(\boldsymbol{m}) = \sum_{\boldsymbol{n}_A,\boldsymbol{n}_B\in\mathbb{Z}^N} &\Big\{3\big(\langle 2\boldsymbol{n}_A-\boldsymbol{m}_A,\boldsymbol{k}_A\rangle + \langle 2\boldsymbol{n}_B-\boldsymbol{m}_B,\boldsymbol{k}_A^*\rangle\big)^2 - c \\ &\quad - a^2\exp\big(\mathrm{i}(\langle 2\boldsymbol{n}_A-\boldsymbol{m}_A,\boldsymbol{\phi}_A\rangle + \langle 2\boldsymbol{n}_B-\boldsymbol{m}_B,\boldsymbol{\phi}_A^*\rangle)\big)\Big\} \\ &\times \exp\left\{-\frac{1}{2}\langle \tau\boldsymbol{n},\boldsymbol{n}\rangle - \frac{1}{2}\langle \tau(\boldsymbol{m}-\boldsymbol{n}),\boldsymbol{m}-\boldsymbol{n}\rangle\right\}. \end{aligned}$$

Consequently, we have

$$\begin{aligned} \big(F_1(\boldsymbol{m})\big)^* = \sum_{\boldsymbol{n}_A,\boldsymbol{n}_B\in\mathbb{Z}^N} &\Big\{3\big(\langle 2\boldsymbol{n}_A-\boldsymbol{m}_A,\boldsymbol{k}_A^*\rangle + \langle 2\boldsymbol{n}_B-\boldsymbol{m}_B,\boldsymbol{k}_A\rangle\big)^2 - c \\ &\quad - a^2\exp\big(-\mathrm{i}(\langle 2\boldsymbol{n}_A-\boldsymbol{m}_A,\boldsymbol{\phi}_A^*\rangle + \langle 2\boldsymbol{n}_B-\boldsymbol{m}_B,\boldsymbol{\phi}_A\rangle)\big)\Big\} \\ &\times \exp\left\{-\frac{1}{2}\langle \tau^*\boldsymbol{n},\boldsymbol{n}\rangle - \frac{1}{2}\langle \tau^*(\boldsymbol{m}-\boldsymbol{n}),\boldsymbol{m}-\boldsymbol{n}\rangle\right\} \end{aligned}$$

$$\overset{n'_A=-n_B}{\underset{n'_B=-n_A}{=\!=\!=}} \sum_{n'_A,n'_B\in\mathbb{Z}^N} \Big\{3\big(\langle 2n'_B+m_A,k^*_A\rangle+\langle 2n'_A+m_B,k_A\rangle\big)^2-c$$
$$-a^2\exp\big(\mathrm{i}(\langle 2n'_B+m_A,\phi^*_A\rangle+\langle 2n'_A+m_B,\phi_A\rangle)\big)\Big\}$$
$$=F_1(-\hat{m})\overset{(3.9)}{=\!=\!=}F_1(\hat{m}).$$

Analogously, one can show that the function $F_2(\boldsymbol{m})$ is even respect $\boldsymbol{m}$. Then, by using the block form of vectors, we rewrite $F_2(\boldsymbol{m})$ as

$$\begin{aligned}
F_2(m)=\sum_{n_A,n_B\in\mathbb{Z}^N} &\Big\{\big(\langle m_A-2n_A,w_A\rangle+\langle m_B-2n_B,w^*_A\rangle\big)\\
&+\beta\big(\langle m_A-2n_A,k_A\rangle+\langle m_B-2n_B,k^*_A\rangle\big)^3\\
&+3\mathrm{i}\beta\tilde{k}\big(\langle m_A-2n_A,k_A\rangle+\langle m_B-2n_B,k^*_A\rangle\big)^2\\
&+\tilde{k}\big(\langle m_A-2n_A,l_A\rangle+\langle m_B-2n_B,l^*_A\rangle\big)\\
&-\big(3\beta\tilde{k}^2-\tilde{l}+c\beta\big)\big(\langle m_A-2n_A,k_A\rangle+\langle m_B-2n_B,k^*_A\rangle\big)\\
&-\mathrm{i}\big(\langle m_A-2n_A,k_A\rangle+\langle m_B-2n_B,k^*_A\rangle\big)\\
&\times\big(\langle m_A-2n_A,l_A\rangle+\langle m_B-2n_B,l^*_A\rangle\big)\\
&+\mathrm{i}\big(\tilde{w}+\tilde{k}\tilde{l}-\beta\tilde{k}^3-c\beta\tilde{k}\big)\Big\}\\
&\times\exp\bigg\{\frac{1}{2}\mathrm{i}\big(\langle m_A-2n_A,\phi_A\rangle+\langle m_B-2n_B,\phi^*_A\rangle\big)\\
&\qquad-\frac{1}{2}\langle\tau n,n\rangle-\frac{1}{2}\langle\tau(m-n),m-n\rangle\bigg\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\big(F_2(m)\big)^*=\sum_{n_A,n_B\in\mathbb{Z}^N} &\Big\{\big(\langle m_A-2n_A,w^*_A\rangle+\langle m_B-2n_B,w_A\rangle\big)\\
&+\beta\big(\langle m_A-2n_A,k^*_A\rangle+\langle m_B-2n_B,k_A\rangle\big)^3\\
&-3\mathrm{i}\beta\tilde{k}\big(\langle m_A-2n_A,k^*_A\rangle+\langle m_B-2n_B,k_A\rangle\big)^2\\
&+\tilde{k}\big(\langle m_A-2n_A,l^*_A\rangle+\langle m_B-2n_B,l_A\rangle\big)\\
&-\big(3\beta\tilde{k}^2-\tilde{l}+c\beta\big)\big(\langle m_A-2n_A,k^*_A\rangle+\langle m_B-2n_B,k_A\rangle\big)\\
&+\mathrm{i}\big(\langle m_A-2n_A,k^*_A\rangle+\langle m_B-2n_B,k_A\rangle\big)\\
&\times\big(\langle m_A-2n_A,l^*_A\rangle+\langle m_B-2n_B,l_A\rangle\big)\\
&-\mathrm{i}\big(\tilde{w}+\tilde{k}\tilde{l}-\beta\tilde{k}^3-c\beta\tilde{k}\big)\Big\}\\
&\times\exp\bigg\{-\frac{1}{2}\mathrm{i}\big(\langle m_A-2n_A,\phi^*_A\rangle+\langle m_B-2n_B,\phi_A\rangle\big)\\
&\qquad-\frac{1}{2}\langle\tau^*n,n\rangle-\frac{1}{2}\langle\tau^*(m-n),m-n\rangle\bigg\}
\end{aligned}$$

$$
\begin{aligned}
\xlongequal[n'_B=-n_A]{n'_A=-n_B} \sum_{n'_A,n'_B\in\mathbb{Z}^N} & \Big\{\big(\langle \boldsymbol{m}_A+2\boldsymbol{n}'_B,\boldsymbol{w}^*_A\rangle+\langle \boldsymbol{m}_B+2\boldsymbol{n}'_A,\boldsymbol{w}_A\rangle\big) \\
& +\beta\big(\langle \boldsymbol{m}_A+2\boldsymbol{n}'_B,\boldsymbol{k}^*_A\rangle+\langle \boldsymbol{m}_B+2\boldsymbol{n}'_A,\boldsymbol{k}_A\rangle\big)^3 \\
& -3\mathrm{i}\beta\tilde{k}\big(\langle \boldsymbol{m}_A+2\boldsymbol{n}'_B,\boldsymbol{k}^*_A\rangle+\langle \boldsymbol{m}_B+2\boldsymbol{n}'_A,\boldsymbol{k}_A\rangle\big)^2 \\
& +\tilde{k}\big(\langle \boldsymbol{m}_A+2\boldsymbol{n}'_B,\boldsymbol{l}^*_A\rangle+\langle \boldsymbol{m}_B+2\boldsymbol{n}'_A,\boldsymbol{l}_A\rangle\big) \\
& -\big(3\beta\tilde{k}^2-\tilde{l}+c\beta\big)\big(\langle \boldsymbol{m}_A+2\boldsymbol{n}'_B,\boldsymbol{k}^*_A\rangle+\langle \boldsymbol{m}_B+2\boldsymbol{n}'_A,\boldsymbol{k}_A\rangle\big) \\
& +\mathrm{i}\big(\langle \boldsymbol{m}_A+2\boldsymbol{n}'_B,\boldsymbol{k}^*_A\rangle+\langle \boldsymbol{m}_B+2\boldsymbol{n}'_A,\boldsymbol{k}_A\rangle\big) \\
& \times\big(\langle \boldsymbol{m}_A+2\boldsymbol{n}'_B,\boldsymbol{l}^*_A\rangle+\langle \boldsymbol{m}_B+2\boldsymbol{n}'_A,\boldsymbol{l}_A\rangle\big) \\
& -\mathrm{i}\big(\tilde{w}+\tilde{k}\tilde{l}-\beta\tilde{k}^3-c\beta\tilde{k}\big)\Big\} \\
& \times\exp\bigg\{-\frac{1}{2}\mathrm{i}\big(\langle \boldsymbol{m}_A+2\boldsymbol{n}'_B,\boldsymbol{\phi}^*_A\rangle+\langle \boldsymbol{m}_B+2\boldsymbol{n}'_A,\boldsymbol{\phi}_A\rangle\big) \\
& \qquad -\frac{1}{2}\langle \tau^*\boldsymbol{n},\boldsymbol{n}\rangle-\frac{1}{2}\langle \tau^*(\boldsymbol{m}-\boldsymbol{n}),\boldsymbol{m}-\boldsymbol{n}\rangle\bigg\} \\
= & -F_2(-\hat{\boldsymbol{m}})=-F_2(\hat{\boldsymbol{m}}).
\end{aligned}
$$

The proof is complete. □

According to Proposition 3.1, if $\boldsymbol{m}=\hat{\boldsymbol{m}}$, then $F_1(\boldsymbol{m})$ is real and $F_2(\boldsymbol{m})$ is pure imaginary. While if $\boldsymbol{m}\neq\hat{\boldsymbol{m}}$, $F_1(\boldsymbol{m})$ and $F_2(\boldsymbol{m})$ are conjugate with $F_1(\hat{\boldsymbol{m}})$ and $-F_2(\hat{\boldsymbol{m}})$ respectively.

Remark 3.1. Due to the properties of $F_2(\boldsymbol{m})$, our discussion will focus on $\mathrm{i}F_2(\boldsymbol{m})$. At this point, $\mathrm{i}F_2(\boldsymbol{m})$ is real with $\boldsymbol{m}=\hat{\boldsymbol{m}}$. In contrast, for $\boldsymbol{m}\neq\hat{\boldsymbol{m}}$, it satisfies the conjugation property $\{\mathrm{i}F_2(\boldsymbol{m})\}^*=\mathrm{i}F_2(\hat{\boldsymbol{m}})$.

As a result, for each complex equation in this system, the real and imaginary parts can be separated into two independent real equations. Combining it with the original real equation ($\boldsymbol{m}=\hat{\boldsymbol{m}}$) leads to a real nonlinear algebraic system with real unknowns. This system is equivalent to the equations in Theorem 3.1. Notably, the total number of equations remains 2×4^N during this transformation process. Consequently, the problem of finding the quasi-periodic breathers can be formulated as a real nonlinear algebraic system consisting of 2×4^N equations with $2N^2+3N+2$ real unknowns. We formulate the problem as a nonlinear least square problem i.e. as the minimization of the following objective function:

$$
\boldsymbol{S}(\boldsymbol{x})=\sum \boldsymbol{P}_j^2(\boldsymbol{x}),
$$

where $\boldsymbol{x}$ is the vector of the unknowns and $\boldsymbol{P}_j$ represent all the nonlinear algebraic equations generated from (3.5)-(3.6). Using classical numerical iterative algorithms such as Gauss-Newton and Levenberg-Marquardt methods, we can solve the nonlinear least square problem and obtain unknown parameters in the theta functions. Note that related numerical examples are given in Section 4.

3.2. Asymptotic analysis

There is an asymptotic connection between quasi-periodic and regular breathers, and this relation can be effectively demonstrated by using the theta function. In what follows, we establish this relation analytically and support it by numerical examples. For simplicity, we consider the case of $N = 1$, so that (2.11) has the form

$$\theta(\boldsymbol{\eta}, \mathbf{0}, \boldsymbol{\phi} \,|\boldsymbol{\tau}) = \sum_{n_1,n_2\in\mathbb{Z}} \exp\left\{ n_1(\eta_1 + \mathrm{i}\phi_1) + n_2(\eta_1^* + \mathrm{i}\phi_1^*) - \frac{1}{2}n_1^2\tau_{11} - \frac{1}{2}n_2^2\tau_{11}^* - n_1 n_2 \tau_{12} \right\}.$$

Because of the arbitrariness of η_1^0, we can write η_1 as $\eta_1 + \tau_{11}/2$, thus obtaining

$$\begin{aligned} f &= 1 + \exp(\eta_1) + \exp(\eta_1^*) + \exp(\eta_1 + \eta_1^* - \tau_{12}) + \mathcal{O}\big(e^{-\mathscr{R}(\tau_{11})}\big), \\ g &= 1 + \exp(\eta_1 + \mathrm{i}\phi_1) + \exp\big(\eta_1^* + \mathrm{i}\phi_1^*\big) \\ &\quad + \exp\big(\eta_1 + \eta_1^* + \mathrm{i}(\phi_1 + \phi_1^*) - \tau_{12}\big) + \mathcal{O}\big(e^{-\mathscr{R}(\tau_{11})}\big). \end{aligned}$$

Setting $\exp(-\tau_{12}) = A$ and considering the limit $\mathscr{R}(\tau_{11}) \to +\infty$, we note that f and g tend towards the form of regular breathers (2.4) and (2.5).

According to Section 3.1, Theorem 3.1 consists of 8 equations formed by (3.5)-(3.6) with $\boldsymbol{m_1} = (0,0)^\mathsf{T}$, $\boldsymbol{m_2} = (1,0)^\mathsf{T}$, $\boldsymbol{m_3} = (0,1)^\mathsf{T}$, $\boldsymbol{m_4} = (1,1)^\mathsf{T}$. These equations contain an infinite series of exponential functions. Through detailed computation, we derive the following asymptotic forms of $F_1(\boldsymbol{m_1})$ and $F_2(\boldsymbol{m_1})$:

$$\begin{aligned} F_1(\boldsymbol{m_1}) &= (c + a^2) + \mathcal{O}\big(e^{-\mathscr{R}(\tau_{11})}\big), \\ F_2(\boldsymbol{m_1}) &= \mathrm{i}\big(\tilde{w} + \tilde{k}\tilde{l} - \beta\tilde{k}^3 - c\beta\tilde{k}\big) + \mathcal{O}\big(e^{-\mathscr{R}(\tau_{11})}\big). \end{aligned}$$

These expressions correspond to the relations (2.6)-(2.7) as $\mathscr{R}(\tau_{11})$ tends to $+\infty$. Notably, $F_1(\boldsymbol{m_1})$ is real but $F_2(\boldsymbol{m_1})$ is pure imaginary. Similarly, for $F_1(\boldsymbol{m_2})$ and $F_2(\boldsymbol{m_2})$, we have the following asymptotic expressions:

$$\begin{aligned} F_1(\boldsymbol{m_2}) &= 2\big(3k^2 - c - a^2\cos(\phi)\big)e^{-\tau_{11}/2} + \mathcal{O}(e^{-3\mathscr{R}(\tau_{11})/2}), \\ F_2(\boldsymbol{m_2}) &= 2\mathrm{i}\Bigg(\big(w + \beta k^3 - (3\beta\tilde{k}^2 - \tilde{l} + c\beta)k + \tilde{k}l\big)\sin\left(\frac{\phi}{2}\right) \\ &\quad + \big(3\beta\tilde{k}k^2 - kl\big)\cos\left(\frac{\phi}{2}\right)\Bigg)e^{-\tau_{11}/2} + \mathcal{O}\big(e^{-3\mathscr{R}(\tau_{11})/2}\big), \end{aligned}$$

which correspond to the relations (2.8)-(2.9). As for $F_1(\boldsymbol{m_3})$ and $F_2(\boldsymbol{m_3})$, we obtain the following results:

$$\begin{aligned} F_1(\boldsymbol{m_3}) &= 2\big(3(k^*)^2 - c - a^2\cos(\phi^*)\big)e^{-\tau_{11}^*/2} + \mathcal{O}\big(e^{-3\mathscr{R}(\tau_{11}^*)/2}\big), \\ F_2(\boldsymbol{m_3}) &= 2\mathrm{i}\Bigg(\big(w^* + \beta(k^*)^3 - (3\beta\tilde{k}^2 - \tilde{l} + c\beta)k^* + \tilde{k}l^*\big)\sin\left(\frac{\phi^*}{2}\right) \\ &\quad + \big(3\beta\tilde{k}(k^*)^2 - k^*l^*\big)\cos\left(\frac{\phi^*}{2}\right)\Bigg)e^{-\tau_{11}^*/2} + \mathcal{O}\big(e^{-3\mathscr{R}(\tau_{11}^*)/2}\big). \end{aligned}$$

It is worth noting that $F_1(\boldsymbol{m}_3)$ and $\mathrm{i}F_2(\boldsymbol{m}_3)$ are complex conjugate of $F_1(\boldsymbol{m}_2)$ and $\mathrm{i}F_2(\boldsymbol{m}_2)$, consistent with the proof of Proposition 3.1. Analogously, asymptotic expressions of $F_1(\boldsymbol{m}_4)$ and $F_2(\boldsymbol{m}_4)$ indicate that $F_1(\boldsymbol{m}_4)$ is real and $F_2(\boldsymbol{m}_4)$ is pure imaginary, furthermore we can obtain (2.10). However, these two asymptotic expressions are much more complicated, and for the sake of brevity, they are not presented here. The above analysis clearly shows that in the limit $\mathscr{R}(\tau_{11}) \to +\infty$ the quasi-periodic breathers can reduce to the regular breathers.

4. Numerical Results

Here we present numerical examples of the quasi-periodic breathers for the HM system. The detailed classification of the breather solutions is given in Section 2.1. It depends on whether certain parameters are complex, real or pure imaginary and contains three categories of breather solutions. These types of breathers correspond to three quasi-periodic breathers. A detailed classification is shown in Table 1. It is worth noting that for sufficiently large $\mathscr{R}(\tau)$, the quasi-periodic breathers tend to the regular breather solutions.

In numerical computation, we choose $\beta = 1$. The parameters k_j, l_j, τ_{jj}, $\tilde{k}$, $\tilde{l}$ and a are given, and we solve for the unknowns w_j, ϕ_j, τ_{jl}, $\tilde{w}$ and c for $j = 1, 2, \ldots, N$. The given parameters have to satisfy the conditions specified in Table 2. All the numerical experiments were conducted using Matlab R2022a (academic use) on a computer equipped with a 2.90 GHz CPU and 16 GB of main memory. The initial values for Gauss-Newton and Levenberg-Marquardt methods are randomly selected, and the errors of all the algebraic equations are within 10^{-14}. The numerical results are consistent with the classification shown in Table 1.

Table 1: Classification of the quasi-periodic breathers.

$k, l, \mathrm{i}\phi$	w	Normal τ	$\mathscr{R}(\tau) \to +\infty$
(i) complex	complex	General quasi-periodic breather	Regular breather
(ii) real	complex	General quasi-periodic breather	Regular breather
	imaginary	Quasi-periodic stationary breather	Regular stationary breather
(iii) imaginary	complex	Quasi-periodic homoclinic orbit	Regular homoclinic orbit

Table 2: Conditions for the given values of k, l and a.

k, l	Hirota-Maccari system
(i) complex	NULL
(ii) real	NULL
(iii) imaginary	$-2/3a^2 < k^2 < 0$

4.1. Quasi-periodic 1-breather

When $N = 1$, the resulting solutions are quasi-periodic 1-breathers. In this scenario, we are faced with a nonlinear algebraic system of 8 real equations and 7 real unknowns (τ_{12}, c, $\tilde{w}$, real and imaginary parts of w_1, ϕ_1).

Table 3 and Fig. 1 illustrate two representative examples of quasi-periodic 1-breathers to the HM system with complex wave numbers: $k_1 = \pi/5 + \pi i/3$, $l_1 = \pi/4 + 2\pi i/3$, and $\tilde{k} = 0.16\pi$, $\tilde{l} = 0.15\pi$, $a = 1.2$. The difference between the two examples lies in the values of $\mathscr{R}(\tau_{11})$: The first example with a normal $\mathscr{R}(\tau_{11})$ represents a general quasi-periodic breather. In the second example, τ_{11} with a much larger real part, leading to a solution that behaves like a regular breather.

As shown in Table 4 and Fig. 2, for the case where $k_1 = \pi/5$, $l_1 = \pi/4$, $\tilde{k} = 0$, $\tilde{l} = 0$ and $a = 1.2$, the value of w obtained is complex, with the solution representing a general quasi-periodic breather when the $\mathscr{R}(\tau_{11})$ is normal, and a regular breather when the real part of τ_{11} is large. However, Table 5 and Fig. 3 present a different scenario where, by appropriately selecting parameters such as $k_1 = \pi/5$, $l_1 = 20\pi/23$, $\tilde{k} = 0.6\pi$, $\tilde{l} = 0.2\pi$ and $a = 1.2$, w becomes pure imaginary. In this case, when $\mathscr{R}(\tau_{11})$ take normal values, the solution can be seen as a quasi-periodic stationary 1-breather. While quasi-periodic 1-breather behaves as a regular stationary breather with a larger $\mathscr{R}(\tau_{11})$.

Table 3: Quasi-periodic 1-breathers of the HM system with complex k, l.

τ_{11}	w_1	ϕ_1	τ_{12}	c	$\tilde{w}$
$1.9\pi + 2.7\pi i$	$0.3207 - 2.6190i$	$1.7342 - 1.7793i$	-1.6076	-1.3380	-0.9955
$7.6\pi + 4.6\pi i$	$0.3582 - 2.8764i$	$1.7281 - 1.7451i$	-1.5442	-1.4400	-0.8337

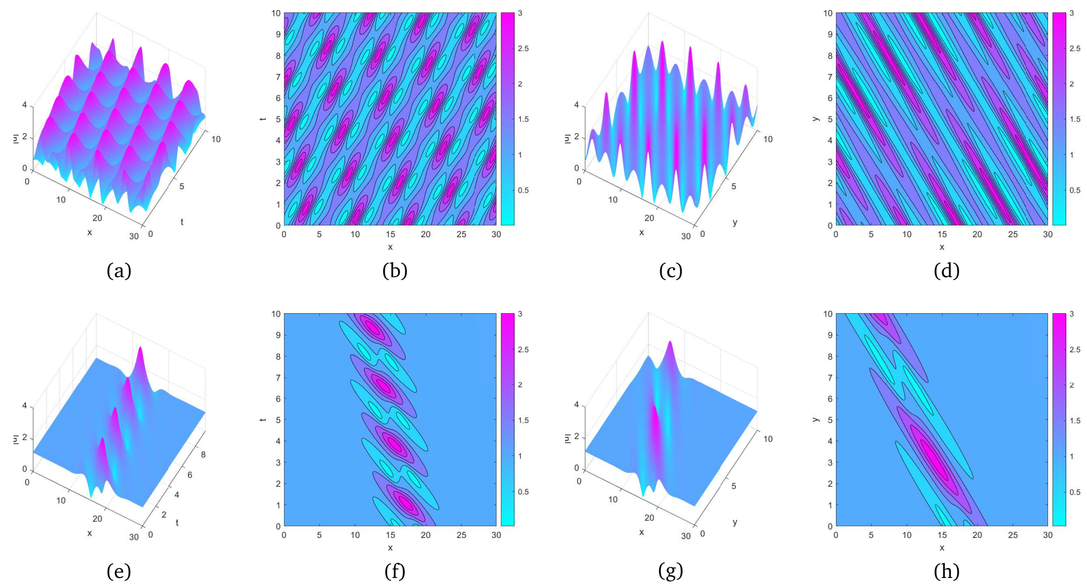

Figure 1: Quasi-periodic 1-breathers with complex wave numbers. (a)-(d): First example in Table 3. (e)-(h): Second example in Table 3. (a) 3D plot of $|u|$, $y = 0$; (b) contour plot of $|u|$, $y = 0$; (c) 3D plot of $|u|$, $t = 0$; (d) contour plot of $|u|$, $t = 0$; (e) 3D plot of $|u|$, $y = 0$; (f) contour plot of $|u|$, $y = 0$; (g) 3D plot of $|u|$, $t = 0$; (h) contour plot of $|u|$, $t = 0$.

Table 4: Quasi-periodic 1-breathers of the HM system with real k, l and complex w.

τ_{11}	w_1	ϕ_1	τ_{12}	c	$\tilde{w}$
$1.6\pi + 2.4\pi\mathrm{i}$	$-1.1190 - 0.9603\mathrm{i}$	$0.0114 + 1.2109\mathrm{i}$	-0.3470	-1.4521	-0.0535
$7.6\pi + 4.6\pi\mathrm{i}$	$-1.1528 - 0.9142\mathrm{i}$	$0.0000 + 1.2078\mathrm{i}$	-0.3445	-1.4400	-0.0000

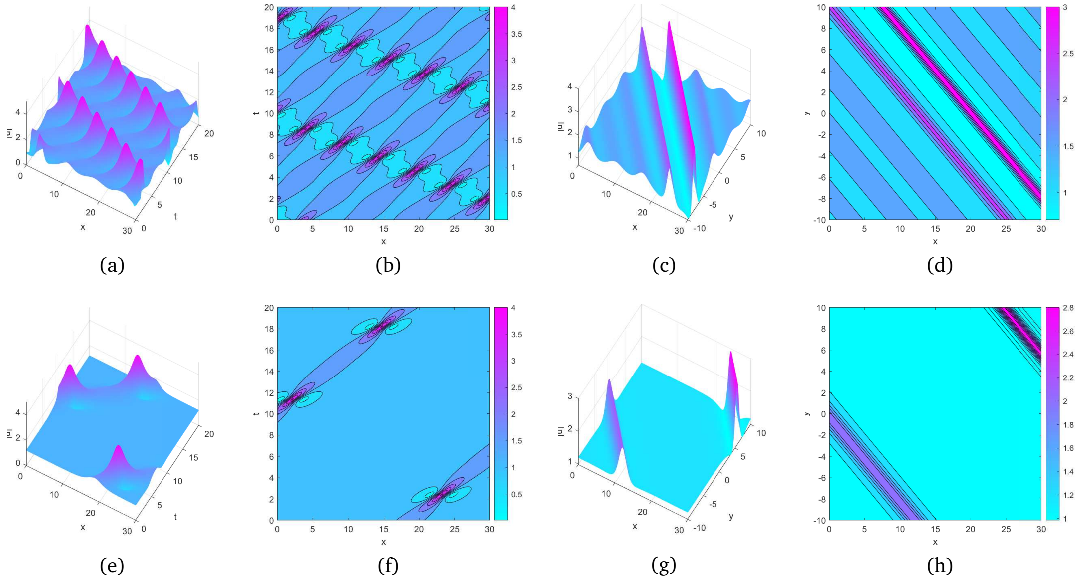

Figure 2: Quasi-periodic 1-breathers with real wave numbers. (a)-(d): First example in Table 4. (e)-(h): Second example in Table 4. (a) 3D plot of $|u|$, $y = 0$; (b) contour plot of $|u|$, $y = 0$; (c) 3D plot of $|u|$, $t = 5$; (d) contour plot of $|u|$, $t = 5$; (e) 3D plot of $|u|$, $y = 0$; (f) contour plot of $|u|$, $y = 0$; (g) 3D plot of $|u|$, $t = 10$; (h) contour plot of $|u|$, $t = 10$.

Table 6 and Fig. 4 provide two examples for the HM system with imaginary wave numbers: $k_1 = \pi\mathrm{i}/5$, $l_1 = \pi\mathrm{i}/4$, $\tilde{k} = 0.16\pi$, $\tilde{l} = 0.15\pi$ and $a = 3.2$. In the first example, $\mathscr{R}(\tau_{11})$ is set to a normal value, resulting in a quasi-periodic homoclinic solution. In the second example, τ_{11} with a much larger real part, leading to a solution that behaves like a homoclinic solution.

4.2. Quasi-periodic 2-breather

When $N = 2$, the resulting solutions are quasi-periodic 2-breathers. In this scenario, we are faced with a nonlinear algebraic system consisting of 32 real equations and 16 real unknowns (τ_{13}, τ_{24}, c, $\tilde{w}$, real and imaginary parts of w_1, w_2, ϕ_1, ϕ_2, τ_{12}, τ_{14}).

Table 7 and Fig. 5 show two examples of quasi-periodic 2-breathers to the HM system with complex wave numbers $k_1 = \pi/5 - 2\pi\mathrm{i}/5$, $l_1 = \pi/4 - 2\pi\mathrm{i}/3$, $k_2 = 2k_1$, $l_2 = 2l_1$, $\tilde{k} = 1.6\pi$, $\tilde{l} = 1.2\pi$ and $a = 5.0$. Tables 8 and 9, together with Figs. 6 and 7, present two types of examples for the HM system with real wave numbers. In the first case, with

Table 5: Quasi-periodic 1-breathers of the HM system with real k,l and imaginary w.

τ_{11}	w_1	ϕ_1	τ_{12}	c	$\tilde{w}$
$1.6\pi+2.4\pi\mathrm{i}$	$-0.0196+0.8945\mathrm{i}$	$0.0114+1.2109\mathrm{i}$	-0.3470	-1.4521	2.7116
$7.6\pi+4.6\pi\mathrm{i}$	$-0.0000+0.9556\mathrm{i}$	$0.0000+1.2078\mathrm{i}$	-0.3445	-1.4400	2.7987

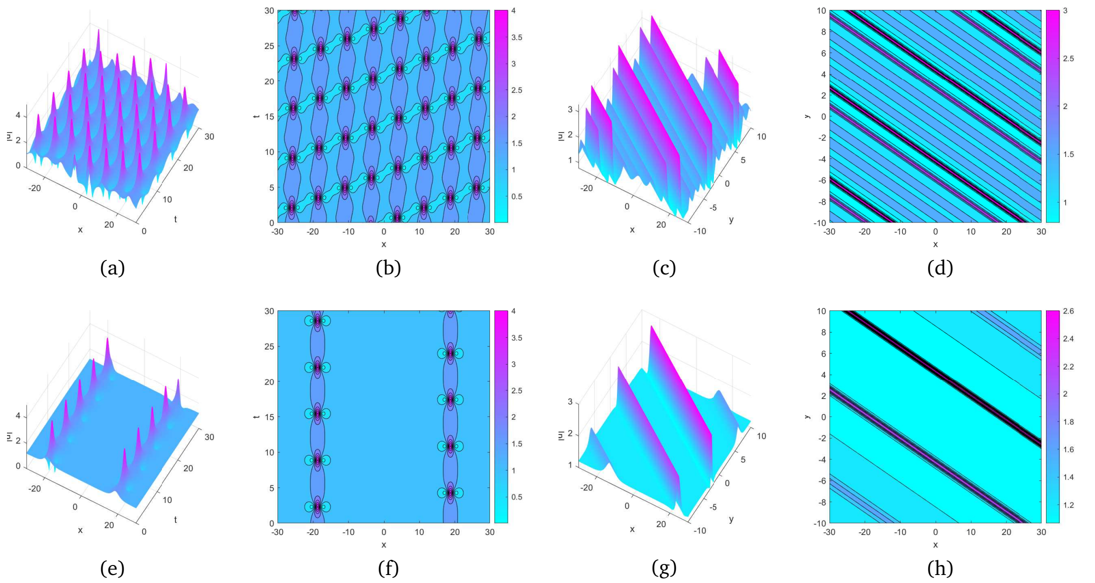

Figure 3: Quasi-periodic 1-breathers with real wave numbers. (a)-(d): First example in Table 5. (e)-(h): Second example in Table 5. (a) 3D plot of $|u|$, $y=0$; (b) contour plot of $|u|$, $y=0$; (c) 3D plot of $|u|$, $t=10$; (d) contour plot of $|u|$, $t=10$; (e) 3D plot of $|u|$, $y=0$; (f) contour plot of $|u|$, $y=0$; (g) 3D plot of $|u|$, $t=10$; (h) contour plot of $|u|$, $t=10$.

parameters $k_1=\pi/5$, $l_1=\pi/4$, $k_2=-2k_1$, $l_2=-2l_1$, $\tilde{k}=0$, $\tilde{l}=0$, and $a=5.0$, the resulting value of w is complex. In contrast, the second case uses $k_1=\pi/5$, $k_2=2k_1$, $l_1=1000\pi/539$, $l_2=2000\pi/553$, $\tilde{k}=1.6\pi$, $\tilde{l}=1.2\pi$, and the same $a=5.0$, resulting in a pure imaginary w. Table 10 and Fig. 8 provides two examples for the HM system with imaginary wave numbers: $k_1=\pi\mathrm{i}/5$, $l_1=\pi\mathrm{i}/4$, $k_2=2k_1$, $l_2=2l_1$, $\tilde{k}=1.6\pi$, $\tilde{l}=1.2\pi$ and $a=5.0$. Besides, in Tables 7-10, the first examples feature normal $\mathscr{R}(\tau_{11})$ $(\tau_{11}=1.5\pi+2.6\pi\mathrm{i})$ and $\mathscr{R}(\tau_{22})$ $(\tau_{22}=1.9\pi+2.2\pi\mathrm{i})$, whereas the second examples have much larger $\mathscr{R}(\tau_{11})$ $(\tau_{11}=6.8\pi+2.6\pi\mathrm{i})$ and $\mathscr{R}(\tau_{22})$ $(\tau_{22}=7.8\pi+2.2\pi\mathrm{i})$.

Table 6: Quasi-periodic 1-breathers of the HM system with imaginary k,l.

τ_{11}	w_1	ϕ_1	τ_{12}	c	$\tilde{w}$
$1.6\pi+2.4\pi\mathrm{i}$	$-0.4470-6.4108\mathrm{i}$	$-0.4855-0.0007\mathrm{i}$	-0.0595	-10.2420	-5.2760
$7.6\pi+4.6\pi\mathrm{i}$	$-0.4111-6.4005\mathrm{i}$	$-0.4857-0.0000\mathrm{i}$	-0.0596	-10.2400	-5.2571

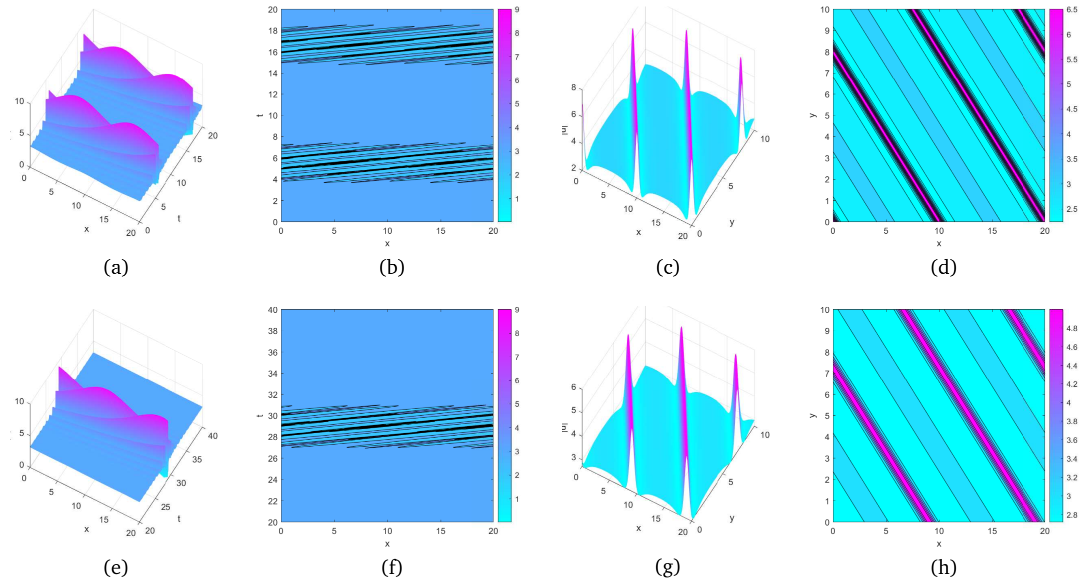

Figure 4: Quasi-periodic 1-breathers with imaginary wave numbers. (a)-(d): First example in Table 6. (e)-(h): Second example in Table 6. (a) 3D plot of $|u|$, $y=0$; (b) contour plot of $|u|$, $y=0$; (c) 3D plot of $|u|$, $t=5$; (d) contour plot of $|u|$, $t=5$; (e) 3D plot of $|u|$, $y=0$; (f) contour plot of $|u|$, $y=0$; (g) 3D plot of $|u|$, $t=30$; (h) contour plot of $|u|$, $t=30$.

Table 7: Quasi-periodic 2-breathers of the HM system with complex k,l. The first example with normal $\mathscr{R}(\tau_{11})$ and $\mathscr{R}(\tau_{22})$. The second example with large $\mathscr{R}(\tau_{11})$ and $\mathscr{R}(\tau_{22})$.

w_1	ϕ_1	τ_{12}	τ_{13}	c
$97.4473-17.8700i$	$0.6148+0.3229i$	$0.1213+0.2215i$	-0.1217	-25.2525
$96.4784-18.2093i$	$0.6175+0.3217i$	$0.1229+44.2062i$	-0.1226	-25.0000
w_2	ϕ_2	τ_{14}	τ_{24}	$\tilde{w}$
$-64.7895-132.8832i$	$-1.2239-0.7303i$	$-0.8033+4.6696i$	-0.5282	-21.4827
$-62.5622-133.2172i$	$-1.2280-0.7364i$	$-0.7911-1.6584i$	-0.5360	-17.6116

Table 8: Quasi-periodic 2-breathers of the HM system with real k,l and complex w. The first example with normal $\mathscr{R}(\tau_{11})$ and $\mathscr{R}(\tau_{22})$. The second example with large $\mathscr{R}(\tau_{11})$ and $\mathscr{R}(\tau_{22})$.

w_1	ϕ_1	τ_{12}	τ_{13}	c
$-15.9719-3.3020i$	$0.0002+0.3065i$	$-2.1879-0.0928i$	-0.0234	-25.0143
$-15.9560+3.2441i$	$-0.0000-0.3066i$	$-2.2433-0.0000i$	-0.0234	-25.0000
w_2	ϕ_2	τ_{14}	τ_{24}	$\tilde{w}$
$33.3000+6.7734i$	$-0.0003-0.6068i$	$0.0461-0.0000i$	-0.0907	-0.0537
$33.4003-6.7097i$	$0.0000+0.6063i$	$0.0460-6.2832i$	-0.0905	0.0000

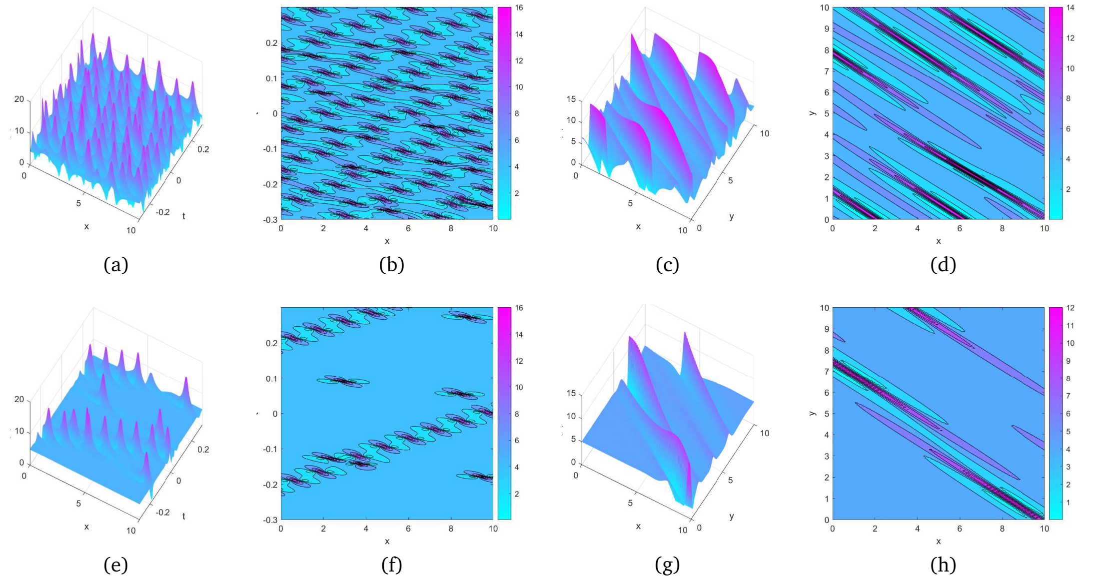

Figure 5: Quasi-periodic 2-breathers with complex wave numbers. (a)-(d): First example in Table 7. (e)-(h): Second example in Table 7. (a) 3D plot of $|u|$, $y=0$; (b) contour plot of $|u|$, $y=0$; (c) 3D plot of $|u|$, $t=0$; (d) contour plot of $|u|$, $t=0$; (e) 3D plot of $|u|$, $y=0$; (f) contour plot of $|u|$, $y=0$; (g) 3D plot of $|u|$, $t=0$; (h) contour plot of $|u|$, $t=0$.

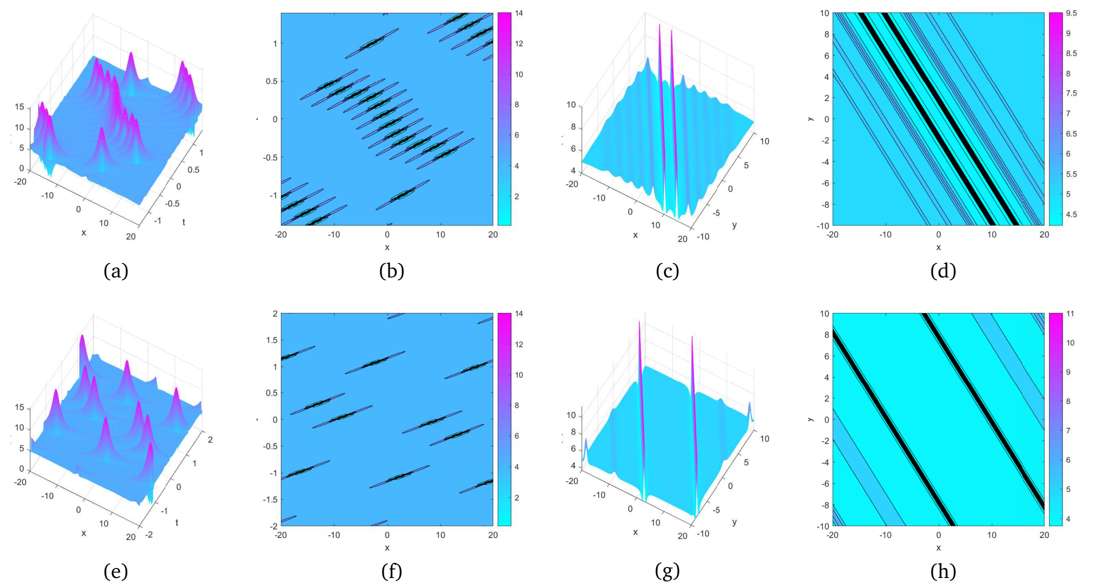

Figure 6: Quasi-periodic 2-breathers with real wave numbers. (a)-(d): The first example in Table 8. (e)-(h): Second example in Table 8. (a) 3D plot of $|u|$, $y=0$; (b) contour plot of $|u|$, $y=0$; (c) 3D plot of $|u|$, $t=0$; (d) contour plot of $|u|$, $t=0$; (e) 3D plot of $|u|$, $y=0$; (f) contour plot of $|u|$, $y=0$; (g) 3D plot of $|u|$, $t=0$; (h) contour plot of $|u|$, $t=0$.

Table 9: Quasi-periodic 2-breathers of the HM system with real k,l and imaginary w. The first example with normal $\mathscr{R}(\tau_{11})$ and $\mathscr{R}(\tau_{22})$. The second example with large $\mathscr{R}(\tau_{11})$ and $\mathscr{R}(\tau_{22})$.

w_1	ϕ_1	τ_{12}	τ_{13}	c
$0.0084-15.1186i$	$-0.0002-0.3065i$	$-0.0461+0.0000i$	-0.0234	-25.0133
$-0.0000+15.0583i$	$0.0000+0.3066i$	$2.2432+6.2832i$	-0.0234	-25.0000
w_2	ϕ_2	τ_{14}	τ_{24}	$\tilde{w}$
$0.0344+32.2543i$	$0.0003+0.6068i$	$2.2044+0.0046i$	-0.0907	-17.7419
$0.0000+32.4091i$	$0.0000+0.6063i$	$-0.0460+0.0000i$	-0.0905	-17.6116

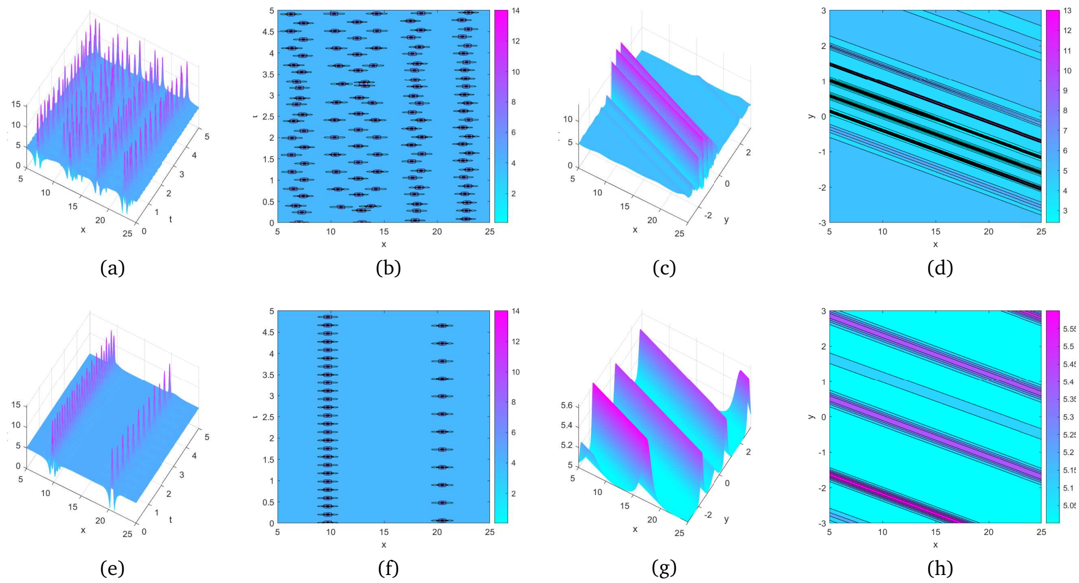

Figure 7: Quasi-periodic stationary 2-breathers with real wave numbers. (a)-(d): First example in Table 9. (e)-(h): Second example in Table 9. (a) 3D plot of $|u|$, $y=0$; (b) contour plot of $|u|$, $y=0$; (c) 3D plot of $|u|$, $t=2$; (d) contour plot of $|u|$, $t=2$; (e) 3D plot of $|u|$, $y=0$; (f) contour plot of $|u|$, $y=0$; (g) 3D plot of $|u|$, $t=4$; (h) contour plot of $|u|$, $t=4$.

Table 10: Quasi-periodic 2-breathers of the HM system with imaginary k,l. The first example with normal $\mathscr{R}(\tau_{11})$ and $\mathscr{R}(\tau_{22})$. The second example with large $\mathscr{R}(\tau_{11})$ and $\mathscr{R}(\tau_{22})$.

w_1	ϕ_1	τ_{12}	τ_{13}	c
$-35.0681+25.9352i$	$-0.3091-0.0002i$	$0.0488+0.0001i$	-0.0240	-25.0133
$-35.0518+25.8492i$	$-0.3090-0.0000i$	$0.0488+0.0000i$	-0.0240	-25.0000
w_2	ϕ_2	τ_{14}	τ_{24}	$\tilde{w}$
$67.6009+52.9592i$	$0.6253+0.0004i$	$-2.1186-0.0116i$	-0.0993	-17.7079
$67.5041+53.1867i$	$0.6258+0.0000i$	$-2.1484-12.5664i$	-0.0995	-17.6116

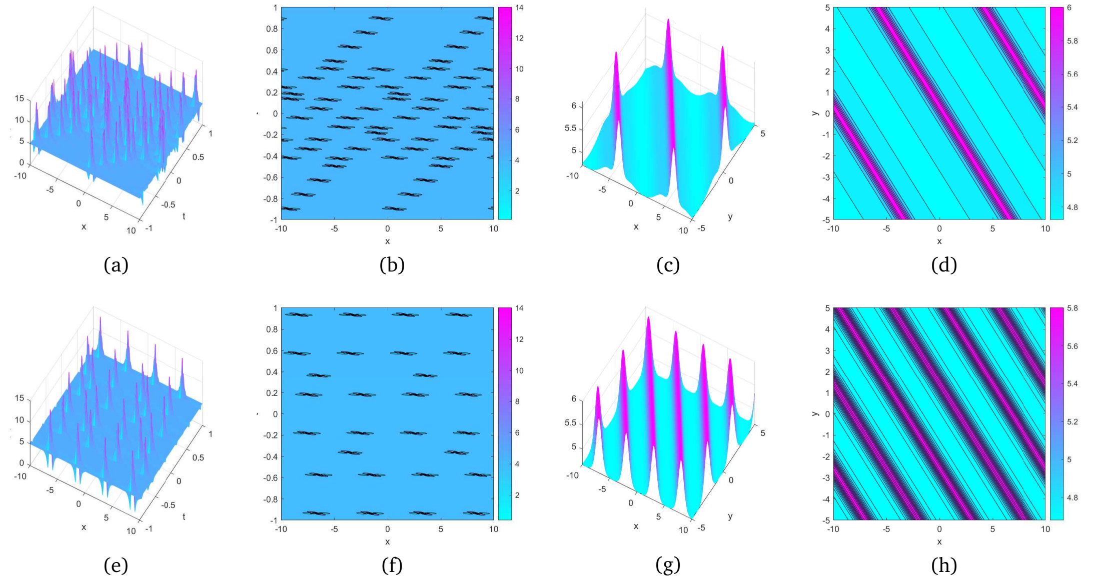

Figure 8: Quasi-periodic 2-breathers with imaginary wave numbers. (a)-(d): First example in Table 10. (e)-(h): Second example in Table 10, behaving like a homoclinic orbit solution. (a) 3D plot of $|u|$, $y=0$; (b) contour plot of $|u|$, $y=0$; (c) 3D plot of $|u|$, $t=0.2$; (d) contour plot of $|u|$, $t=0.2$; (e) 3D plot of $|u|$, $y=0$; (f) contour plot of $|u|$, $y=0$; (g) 3D plot of $|u|$, $t=0.2$; (h) contour plot of $|u|$, $t=0.2$.

5. Conclusions and Discussions

In this paper, we use a direct approach based on Hirota's bilinear method and theta function identities to analyze quasi-periodic breather solutions and the dynamic behavior of the HM system. Through a rigorous asymptotic analysis, we have shown that the quasi-periodic breathers could be reduced to the regular breathers as diagonal elements of the Riemann matrix tend to infinity. In addition, we classify the quasi-periodic breathers into three types: general quasi-periodic breathers, quasi-periodic stationary breathers and quasi-periodic homoclinic orbit. Several numerical examples are provided by applying the Levenberg-Marquardt method to the cases $N=1$ and $N=2$. These examples also serve to confirm the results of the theoretical analysis.

It is known that the investigation of the nonlocal integrable systems has become one of the most popular topics in soliton theory and nonlinear mathematical physics — cf. [2,15,16,23], and the general N-breather solutions for the nonlocal HM system have been obtained via the KP hierarchy reduction approach [36]. Observing that the quasi-periodic wave solutions to the nonlocal nonlinear Schrödinger equation have been studied [32,40], but as far as we know, there are very few results about the quasi-periodic breathers in the nonlocal case. Thus, it is also interesting and meaningful to explore the direct approach to computing the quasi-periodic breathers for the nonlocal integrable equations that possess breathers, such as the nonlocal HM system. We would like to study this problem in the future.

Acknowledgments

The authors would like to thank referees for valuable comments.

This work is supported by the National Natural Science Foundation of China (Grant No. 12071447).

References

[1] M.J. Ablowitz and P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press (1991).
[2] M.J. Ablowitz and Z.H. Musslimani, *Integrable nonlocal nonlinear Schrödinger equation*, Phys. Rev. Lett. **110**, 064105 (2013).
[3] N.N. Akhmediev and V.I. Korneev, *Modulation instability andperiodic solutions of the nonlinear Schrödinger equation*, Teoret. Mat. Fiz. **69**, 189–194 (1986).
[4] C.L. Bai and H. Zhao, *Complex hyperbolic-function method and its applications to nonlinear equations*, Phys. Lett. A **355**, 32–38 (2006).
[5] Y. Chen and Z. Yan, *The Weierstrass elliptic function expansion method and its applications in nonlinear wave equations*, Chaos Solitons Fractals **29**, 948–964 (2003).
[6] F. Coppini, P.G. Grinevich and P.M. Santini, *The periodic N breather anomalous wave solution of the Davey-Stewartson equations; first appearance, recurrence, and blow up properties*, J. Phys. A **57**, 015208 (2024).
[7] S.T. Demiray, Y. Pandir and H. Bulut, *All exact travelling wave solutions of Hirota equation and Hirota-Maccari system*, Optik **127**, 1848–1859 (2016).
[8] B.A. Dubrovin, *Theta functions and non-linear equations*, Russian Math. Surveys **36**, 11–92 (1981).
[9] E.G. Fan, *Uniformly constructing a series of explicit exact solutions to nonlinear equations in mathematical physics*, Chaos Solitons Fractals **16**, 819–839 (2003).
[10] E.G. Fan, K.W. Chow and J.H. Li, *On doubly periodic standing wave solutions of the coupled Higgs field equation*, Stud. Appl. Math. **128**, 86–105 (2012).
[11] X. Geng and H.H. Dai, *Algebro-geometric solutions of (2+1)-dimensional coupled modified Kadomtsev-Petviashvili equations*, J. Math. Phys. **41**, 337–348 (2000).
[12] J. Hietarinta, *Recent results from the search for bilinear equations having three-soliton solutions*, in: *Nonlinear Evolution Equations: Integrability and spectral methods*, A. Degasperis, A.P. Fordy and M. Lakshmanan (Eds), Manchester U.P., pp. 307–317 (1990).
[13] R. Hirota, *Exact envelope-soliton solutions of a nonlinear wave equation*, J. Math. Phys. **14**, 805–809 (1973).
[14] R. Hirota, *The Direct Method in Soliton Theory*, Cambridge University Press (2004).
[15] J.L. Ji and Z.N. Zhu, *On a nonlocal modified Korteweg-de Vries equation: Integrability, Darboux transformation and soliton solutions*, Commun. Nonlinear Sci. Numer. Simul. **42**, 699–708 (2017).
[16] M. Jia and S.Y. Lou, *Integrable nonlinear Klein-Gordon systems with PT nonlocality and/or space-time exchange nonlocality*, Appl. Math. Lett. **130**, 108018 (2022).
[17] C. Kalla, *Breathers and solitons of generalized nonlinear Schrödinger equations as degenerations of algebro-geometric solutions*, J. Phys. A **44**, 335210 (2011).
[18] C. Kalla and C. Klein, *On the numerical evaluation of algebro-geometric solutions to integrable equations*, Nonlinearity **25**, 569–596 (2012).

[19] Y. Kodama, *KP Solitons and the Grassmannians: Combinatorics and Geometry of Two-Dimensional Wave Patterns*, Springer (2017).
[20] B.G. Konopelchenko, *Solitons in Multidimensions: Inverse Spectral Transform Method*, World Scientific (1993).
[21] E.A. Kuznetsov, *Solitons in a parametrically unstable plasma*, Sov. Phys. Dokl. **22**, 507–508 (1977).
[22] L. Luo and E. Fan, *Bilinear approach to the quasi-periodic wave solutions of modified Nizhnik-Novikov-Vesselov equation in (2+1)-dimensions*, Phys. Lett. A **374**, 3001–3006 (2010).
[23] W.X. Ma, *Inverse scattering and soliton solutions of nonlocal reverse-spacetime nonlinear Schrödinger equations*, Proc. Amer. Math. Soc. **149**, 251–263 (2021).
[24] W.X. Ma, R. Zhou and L. Gao, *Exact one-periodic and two-periodic wave solutions to Hirota bilinear equations in (2+1) dimensions*, Mod. Phys. Lett. A **24**, 1677–1688 (2009).
[25] Y.C. Ma, *The perturbed plane-wave solutions of the cubic Schrödinger equation*, Stud. Appl. Math. **60**, 43–58 (1979).
[26] A. Maccari, *A generalized Hirota equation in 2+1 dimensions*, J. Math. Phys. **39**, 6547–6551 (1998).
[27] W.Q. Peng, S.F. Tian, X.B. Wang and T.T. Zhang, *Characteristics of rogue waves on a periodic background for the Hirota equation*, Wave Motion **93**, 102454 (2020).
[28] C. Shi, B. Liu and B.F. Feng, *General soliton solutions to the coupled Hirota equation via the Kadomtsev-Petviashvili reduction*, Chaos Solitons Fractals **197**, 116400 (2025).
[29] J.Q. Sun, X.B. Hu and Y.N. Zhang, *Quasi-periodic breathers and rogue waves to the focusing Davey-Stewartson equation*, Phys. D **460**, 134084 (2024).
[30] Y. Tao and J. He, *Multisolitons, breathers, and rogue waves for the Hirota equation generated by the Darboux transformation*, Phys. Rev. E **85**, 026601 (2012).
[31] R. Wang, Y. Zhang, X. Chen and R. Ye, *The rational and semi-rational solutions to the Hirota Maccari system*, Nonlinear Dynam. **100**, 2767–2778 (2020).
[32] X.B. Wang, *Quasi-periodic wave solutions of the nonlocal coupled nonlinear Schrödinger equation*, Appl. Math. Lett. **132**, 108086 (2022).
[33] A.M. Wazwaz, *Abundant soliton and periodic wave solutions for the coupled Higgs field equation, the Maccari system and the Hirota-Maccari system*, Physica Scripta **85**, 065011 (2012).
[34] P. Xia, Y. Zhang, H. Zhang and Y. Zhuang, *Some novel dynamical behaviours of localized solitary waves for the Hirota-Maccari system*, Nonlinear Dynam. **108**, 533–541 (2022).
[35] P. Xin, Z. Zhao and Y. Wang, *Quasi-periodic breathers and their dynamics to the Fokas system in nonlinear optics*, Wave Motion **133**, 103449 (2025).
[36] X. Yang, Y. Zhang and W. Li, *General high-order solitons and breathers with a periodic wave background in the nonlocal Hirota-Maccari equation*, Nonlinear Dynam. **112**, 4803–4813 (2024).
[37] X. Yu, Y.T. Gao and Z.Y. Sun, *N-soliton solutions for the (2+1)-dimensional Hirota-Maccari equation in fluids, plasmas and optical fibers*, J. Math. Anal. Appl. **378**, 519–527 (2011).
[38] G. Zhang, S. Chen and Z. Yan, *Focusing and defocusing Hirota equations with non-zero boundary conditions: Inverse scattering transforms and soliton solutions*, Commun. Nonlinear Sci. Numer. Simul. **80**, 104927 (2020).
[39] H. Zhang, *A direct algebraic method applied to obtain complex solutions of some nonlinear partial differential equations*, Chaos Solitons Fractals **39**, 1020–1026 (2009).
[40] Z.L. Zhao and Y. Wang, *N-periodic wave solutions of the (2+1)-dimensional integrable nonlocal nonlinear Schrödinger equations*, Wave Motion **136**, 103526 (2025).
[41] Z.L. Zhao, Y. Wang and P. Xin, *Numerical calculation and characteristics of quasi-periodic breathers to the Kadomtsev-Petviashvili-based system*, Phys. D **472**, 134497 (2025).