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A Pre-Training Deep Learning Method for Simulating the Large Bending Deformation of Bilayer Plates

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Abstract. We propose a deep learning based method for simulating the large bending deformation of bilayer plates. Inspired by the greedy algorithm, we propose a pre-training method on a series of nested domains, which accelerate the convergence of training and find the absolute minimizer more effectively. The proposed method exhibits the capability to converge to an absolute minimizer, overcoming the limitation of gradient flow methods getting trapped in the local minimizer basins. We showcase better performance with fewer numbers of degrees of freedom for the relative energy errors and relative L^2-errors of the minimizer through numerical experiments. Furthermore, our method successfully maintains the L^2-norm of the isometric constraint, leading to an improvement of accuracy.

AMS subject classifications: 39-08, 49S05, 65Q20, 68T07, 74-10, 74G65

Key words: Deep learning, pre-training method, nonlinear elasticity, bilayer bending, isometric constraint.

1. Introduction

In the field of solid mechanics, particularly in modern nanoscale processes, the bending and deformation of plates are classical problems. Linear models are commonly used to describe small deformations, while nonlinear models are employed to capture large deformations [12]. By letting the thickness of the plate tends to zero, a series of works [16–18] have rigorously derived the dimensional reduced models from three-dimensional elasticity, under different scaling of energy with respect to the thickness, with the aid of Γ-convergence [13, 14]. Numerical methods for simulating the model formulated in [16], which coincides with the nonlinear Kirchhoff model [23], have been proposed in [2, 3, 11, 26] along the

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same line. The main challenges lie in preserving isometry during deformation that keeping relations for length and angle unchanged, similar to a piece of paper.

Bilayer plates, which consist of two compound materials with slightly different properties, present greater complexity in terms of modeling and simulation, and have garnered significant attention. These structures find wide-ranging applications in aerospace, civil engineering, and materials science [8,20,21,25,30,34,35,37,39]. When exposed to external heating or electrification, material mismatches between the two layers result in large bending deformations in bilayer plates. The study of bilayer plates involves the development of mathematical models and analytical techniques aimed at gaining a better understanding of their deformation. The classical model of bilayer [24,36,38] can bend only in one direction with uniform curvature. Building upon the work [16] that concerns curvature in two directions, certain mathematical models [28,29] formulate the large bending deformation for multi-layer plates under different scaling of energy. Meanwhile a heuristic derivation for bilayer plates may be found in [5].

We call back the model for bilayer plates in [5]: Given a domain $\Omega \subset \mathbb{R}^2$ describing the middle surface of the reference configuration of the plate, and an intrinsic spontaneous curvature tensor $Z \in \mathbb{R}^{2\times 2}$, which usually stands for the difference between the material properties of two layers, we minimize the elastic energy

$$E[u] := \frac{1}{2}\int_\Omega |H(u)+Z|^2\,\mathrm{d}x - \int_\Omega f\cdot u\,\mathrm{d}x \tag{1.1}$$

within the isometric constraint for $u:\Omega\to\mathbb{R}^3$, i.e.

$$[\nabla u(x)]^\top \nabla u(x) = \mathrm{Id}_2, \quad \text{a.e.} \quad x\in\Omega \tag{1.2}$$

with the boundary $\Gamma_D \subset \partial\Omega$ clamped — i.e.

$$u(x) = g, \quad \nabla u = \Phi \quad \text{on } \Gamma_D, \tag{1.3}$$

where $H(u)$ is the second fundamental form of the parameterized surface — i.e. $H_{ij}(u) = n\cdot\partial_i\partial_j u$, $n = \partial_1 u\times\partial_2 u$, Id_2 is a two by two identity matrix, $f,g:\Omega\to\mathbb{R}^3$and $\Phi:\Omega\to\mathbb{R}^{3\times 2}$ satisfies $\Phi^\top\Phi = \mathrm{Id}_2$.

This model, which can be viewed as a non-convex minimization problem with a nonlinear constraint, has been investigated numerically by Bartels *et al.* [4,5,7] using the framework of H^2-gradient flow iteration. In each iteration, they discretize the sub-problem in the tangent space of the admissible set to linearize the nonlinear constraint. They employed various Kirchhoff triangles for spatial discretization, and established the Γ-convergence to the stationary configuration in theory. More recently, another work [10] exploited the discontinuous Galerkin discretization to enhance the accuracy and flexibility of the algorithm. Moreover, a self-avoiding method, which ensures that the deformation does not contain self-intersections or self-contact, is proposed in [6].

Unfortunately, due to the large deformation and the non-convexity of the energy, the equilibrium configurations obtained from the above methods are highly sensitive to the discretization mesh size h and the pseudo-timestep τ in the gradient flow. The convergence may be guaranteed with sufficiently small h and τ, while naturally demands a high