A CLASS OF MULTIVARIATE RATIONAL INTERPOLATION FORMULAS*

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Abstract

The object of this paper is to construct a class of multivariate rational interpolation formulas that can be used to solve interpolation problems with function data given at equidistant knots of various directed lines in the higher dimensional Euclidean space. Our formulas are built up of some explicit multivariate rational functions involving three sets of free parameters so that they enjoy sufficient flexibility for interpolating functions of several variables possessing certain kinds of singularities (poles). The method adopted is an extension and modification of that described in our previous papers (cf. [3], [5]).

§ 1. Rational Interpolation $S_m(f:z)$ on a Directed Line

Denote by $\mathbb{R}^n$ and $\mathbb{C}^n$ the n-dimensional real Euclidean space and complex Euclidean space respectively. We shall adopt the following usual notations:

$$x=(x_1, \cdots, x_n)\in\mathbb{R}^n, \quad z=(z_1, \cdots, z_n)\in\mathbb{C}^n,$$

$$\langle z, \xi\rangle=\sum_{i=1}^{n} z_i\bar{\xi}_i, \quad (z, \xi\in\mathbb{C}^n),$$

$$|z|=\langle z, z\rangle^{\frac{1}{2}}=\left(\sum_{i=1}^{n} z_i\bar{z}_i\right)^{\frac{1}{2}}.$$

In particular, we write $\mathbb{C}$ instead of $\mathbb{C}'$.

Given a set of points $A_k\in\mathbb{C}^n (k=0, 1, 2, \cdots, m)$, and a function f: $\mathbb{C}^n\to\mathbb{C}$. It is easily observed that Gould-Hsu's inversion formulas[2] can be put in the following form

$$g(A_s)=\sum_{k=0}^{s}(-1)^k\binom{s}{k}\psi(k, s)f(A_k), \tag{1.1}$$

$$f(A_s)=\sum_{k=0}^{s}(-1)^k\frac{a_{k+1}+k\cdot b_{k+1}}{\psi(s, k+1)}\binom{s}{k}g(A_k), \tag{1.2}$$

where $s=0, 1, \cdots, m$, and $\{a_k\}$, $\{b_k\}\in\mathbb{C}$ are sequences of parameters and $\psi(x, k)$ is a sequence of polynomials defined by

$$\psi(x, k)=\prod_{i=1}^{k}(a_i+b_ix), \quad \psi(x, 0)\equiv1, \qquad x\in\mathbb{C},$$

in which $\{a_k\}$ and $\{b_k\}$ are chosen such that $\psi(x, k)\neq0$ for $x, k=0, 1, 2, \cdots$.

In what follows let the set of points $\{A_j\}_0^m$ take the form $A_j=A_0+jh(j=0, 1, \cdots, m)$ with $h=(h_1, \cdots, h_n)\in\mathbb{C}^n$ and $h_i(i=1, \cdots, n)$ being complex constants. Suppose that $\varphi(u, v_j)$: $\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}$ is a homogeneous function of u with $v_j\in\mathbb{C}^n$ $(j=0, 1, \cdots,$

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m), viz. $\varphi(\lambda u, v_j)=\lambda\varphi(u, v_j)$ for real λ. Then, applying the process of construction for interpolation formulas as described in [3], substituting (1.1) into the equation (1.2) with $s=m$, and replacing the discrete variable m by the continuous parameter

$$\varphi(z-A_0, v_j)/\varphi(h, v_j)\in\mathbb{C}$$

in each term of the resultant summation, we obtain

$$S_m(f;z)=\sum_{j=0}^{m} f(A_j)\sum_{k=j}^{m}\lambda_{jk}\binom{\frac{\varphi(z-A_0, v_j)}{\varphi(h, v_j)}}{k}\psi\left(\frac{\varphi(z-A_0, v_j)}{\varphi(h, v_j)}, k+1\right)^{-1}, \quad (1.3)$$

where $\lambda_{jk}=(-1)^{j+k}(a_{k+1}+kb_{k+1})\binom{k}{j}\psi(j, k), \quad j, k=0, 1, \cdots, m.$

That (1.3) is actually a multivariate interpolation formula for $f(z)$ can be proved as a theorem.

We may rewrite (1.3) in the form

$$S_m(f;z)=\sum_{j=0}^{m} f(A_j)\cdot l_j^{(m)}(z) \quad (1.4)$$

with

$$l_j^{(m)}(z)=\sum_{k=j}^{m}\lambda_{jk}\binom{\frac{\varphi(z-A_0, v_j)}{\varphi(h, v_j)}}{k}\psi\left(\frac{\varphi(z-A_0, v_j)}{\varphi(h, v_j)}, k+1\right)^{-1}. \quad (1.5)$$

Theorem 1. *The summation $S_m(f;z)$ defined by (1.4) satisfies the interpolation conditions*

$$S_m(f;A_r)=f(A_r), \qquad r=0, 1, \cdots, m. \quad (1.6)$$

Proof. In the formula (1.4) (or (1.3)) put $z=A_r$. Since $\varphi(u, \cdot)$ is a homogeneous function of u we have

$$\frac{\varphi(A_r-A_0, v_j)}{\varphi(h, v_j)}=\frac{\varphi(rh, v_j)}{\varphi(h, v_j)}=r.$$

Thus we may evaluate $l_j^{(m)}(A_r)$ as follows

$$\begin{aligned} l_j^{(m)}(A_r) &= \sum_{k=j}^{r}\lambda_{jk}\binom{r}{k}\psi(r, k+1)^{-1} \\ &= \sum_{k=j}^{r}(-1)^{k+j}c_k\binom{r}{k}\binom{k}{j}\psi(j, k)\psi(r, k+1)^{-1} \quad (c_k=a_{k+1}+kb_{k+1}) \\ &= \binom{r}{j}\sum_{k=j}^{r}(-1)^{k+j}c_k\binom{r-j}{k-j}\psi(j, k)\psi(r, k+1)^{-1} \\ &= \binom{r}{j}\sum_{k=0}^{r-j}(-1)^{k}c_{k+j}\binom{r-j}{k}\psi(j, k+j)\psi(r, k+j+1)^{-1} \\ &= \binom{r}{j}\delta_{rj}=\delta_{rj}. \end{aligned}$$

Here the final result is attained by making use of the orthogonality relation (3.2) of the paper [2]. Consequently we have

$$S_m(f;A_r)=\sum_{j=0}^{m} f(A_j)\, l_j^{(m)}(A_r)=\sum_{j=0}^{m} f(A_j)\,\delta_{rj}=f(A_r).$$