AN INTEGRAL METHOD FOR CONSTRUCTING BIVARIATE SPLINE FUNCTIONS * 1)

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Because of the variety of partitions which arise in higher dimensions, there so far is no unified theory about multivariate spline functions. Briefly, we have two strategies at present. One is the so-called classical approach. Since the work by one author of this paper in 1975 [1], this method has been producing many results. Some dimension formulas, expressions of B-splines and bases of some spline spaces on certain partitions were obtained (cf [2], [3]). But calculations are complicated in the case of B-splines, as both smoothness and locally supported conditions must be taken into account.

The second strategy is the polyhedral spline approach. This idea which originated with a geometric interpretation of the univariate B-splines and multivariate splines was obtained as the volume of slices of a polyhedron. Many results such as linear independence, approximation rates and properties of spline spaces have been gained. Choosing various kinds of polyhedra, we have various kinds of splines. From recurrence relations in [6], it is possible to get expressions of those splines, though the quantity of calculations is rather large and polyhedral splines sometimes have large supports.

In this paper, we will give an integral method to construct spline functions, trying to link up two strategies mentioned above. We will show the integral recursions for splines on uniform partitions. If the original spline has a minimal support, it is possible to produce minimal supported splines with more smoothness by the integral method, that is, we provide recursions for B-splines. As spline spaces with maximal smoothness which include B-splines are more useful, we also give bases of these spaces consisting of B-splines and truncated power functions. For the sake of clarity, we only discuss the case of two dimensions. The results can be applied to higher dimensions without any virtual difficulty.

We introduce some notations first. The partitions Δ_1:

$$x = i, y = j, x - y = k \text{ and } \Delta_2 : x = i, y = j, x - y = k, x + y = k,$$

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$i, j, k = 0, \pm 1, \pm 2, \cdots$, are called cross-cut triangulation and criss-cross triangulation of uniform rectangular partitions, respectively. Let

$$r'_{ij} = \Big\{(x,y) : i \le x \le i+1,\ j \le y \le j+1,\ x-i \le y-j\Big\},$$
$$r''_{ij} = \Big\{(x,y) : i \le x \le i+1,\ j \le y \le j+1,\ x-i \ge y-j\Big\},$$

which are cells of Δ_1. Let

$$\omega_{ij}^{(1)} = \Big\{(x,y) : i \le x \le i+1, j \le y \le j+1, x-i \le y-j, x-i \le j+1-y\Big\},$$
$$\omega_{ij}^{(2)} = \Big\{(x,y) : i \le x \le i+1, j \le y \le j+1, x-i \le y-j, x-i \ge j+1-y\Big\},$$
$$\omega_{ij}^{(3)} = \Big\{(x,y) : i \le x \le i+1, j \le y \le j+1, x-i \ge y-j, x-i \ge j+1-y\Big\},$$
$$\omega_{ij}^{(4)} = \Big\{(x,y) : i \le x \le i+1, j \le y \le j+1, x-i \ge y-j, x-i \le j+1-y\Big\},$$

which are cells of Δ_2. If k, μ are nonnegative integers, $S_k^\mu(\Delta_i), i = 1, 2$, is a space of bivariate pp functions in C^μ of degree k, that is

$$S_k^\mu(\Delta_i) = \Big\{s(x,y) : s(x,y) \in C^\mu(R^2); s(x,y) \in P_k,\ \text{when } (x,y) \in \text{ cells of } \Delta_i\Big\}, \quad i = 1, 2.$$

If $B(x,y) \in S_k^\mu(\Delta_i), i = 1, 2$, and T is a bounded region in R^2 so that $B(x,y) > 0$ when $(x,y) \in \text{int}\ (T)$ and $B(x,y) = 0$ when $(x,y) \bar{\in}\ \text{int}\ (T)$, we call $B(x,y)$ a locally supported spline for short. By B-spline we denote a locally supported spline with a minimal support.

From [2] we know that a necessary condition for the existence of a nontrivial locally supported spline in $S_k^\mu(\Delta_1)$ is that k, μ satisfy the inequality $k \ge (3\mu+1)/2$, and in $S_k^\mu(\Delta_2), k > (4\mu+1)/3$. We will prove that they are also sufficient conditions. Let d be the smallest integer satisfying $d > (3\mu+1)/2$ (for $\Delta_2, d > (4\mu+1)/3$); then $S_d^\mu(\Delta_i), i = 1, 2$, are spline spaces in $C^\mu(D)$ with the lowest degree which contain B-splines. The existence and recursions of B-splines in $S_d^\mu(\Delta_i)$ and in ordinary spaces will be investigated. Box-splines in $S_d^\mu(\Delta_i)$ may not have minimal supports (cf [4]), so our integral method is somehow superier to the box-spline method. Besides, we will show two examples to treat splines on non-uniform partitions with the integral method. Finally, for rectangle D and refinement $\Delta_{mn}^{(i)}$ of partition Δ_i on D, we will make bases of $S_d^\mu(\Delta_{mn}^{(i)}, D)$ consisting of mainly B-splines.

§1. Existence and Construction of B–Splines in $S_d^\mu(\Delta_1)$

Lemma 1.1 *If $B(x;y)$ is a locally supported spline in $S_d^\mu(\Delta_1)$ with support T* (shown in Fig.1) *and $A(T) = (a_1, a_2, a_3, a_4, a_5, a_6)$ where $a_i, i = 1, 2, \cdots, 6$, denotes the number of* *cells included in the ith side of hexagon T. Then*

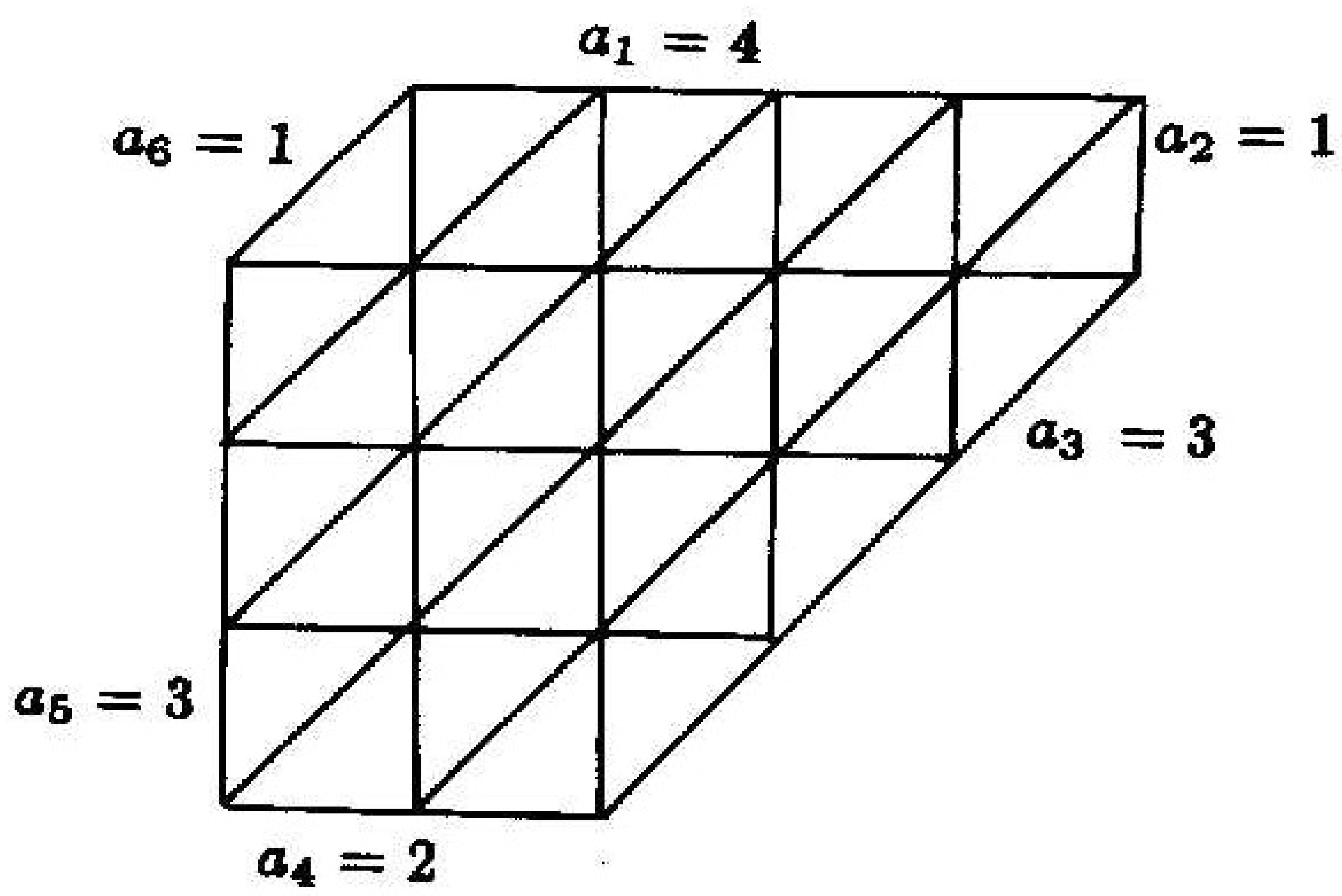

Fig. 1

$$B^{(1)}(x,y) = \int_x^{x+1} B(u,y)du,$$

$$B^{(2)}(x,y) = \int_y^{y+1} B(x,u)du,$$

$$B^{(3)}(x,y) = \int_x^{x+1} B(u,u-x+y)\ du = \int_y^{y+1} B(u-y+x,u)du$$

are all locally supported splines in $S^{\mu}_{k+1}(\Delta_1)$ *with supports* $T^{(1)}, T^{(2)}, T^{(3)}$, *respectively, where*

$$\begin{aligned} A(T^{(1)}) &= (a_1+1, a_2, a_3, a_4+1, a_5, a_6), \\ A(T^{(2)}) &= (a_1, a_2+1, a_3, a_4, a_5+1, a_6), \\ A(T^{(3)}) &= (a_1, a_2, a_3+1, a_4, a_5, a_6+1). \end{aligned}$$

The proof of the lemma is easy, so we omit it.

Recalling the definition of d, we see that $d = 3s+1$ when $\mu = 2s$, and $d = 3s+3$ when $\mu = 2s+1$. Now we discuss the existence and construction of B-splines in $S^{2s}_{3s+1}(\Delta_1)$ and $S^{2s+1}_{3s+3}(\Delta_1)$.

By $B_0(x,y)$ we denote a B-spline in $S^0_1(\Delta_1)$. Its support is shown in Fig. 2 and its representation in the ith cell is $p_i(x,y), i = 1, 2, \cdots, 6$, where

$$\begin{aligned} &p_1(x,y) = 1-y, \quad p_2(x,y) = 2-x, \\ &p_3(x,y) = 2-x+y, \quad p_4(x,y) = 1+y, \\ &p_5(x,y) = x, \quad p_6(x,y) = x-y. \end{aligned}$$

Theorem 1. *There exists a B-spline in* $S^{\mu}_k(\Delta_1)$ *if and only if* $k > (3\mu+1)/2$. *And* *we can construct locally supported splines by integral recursions as follows.*

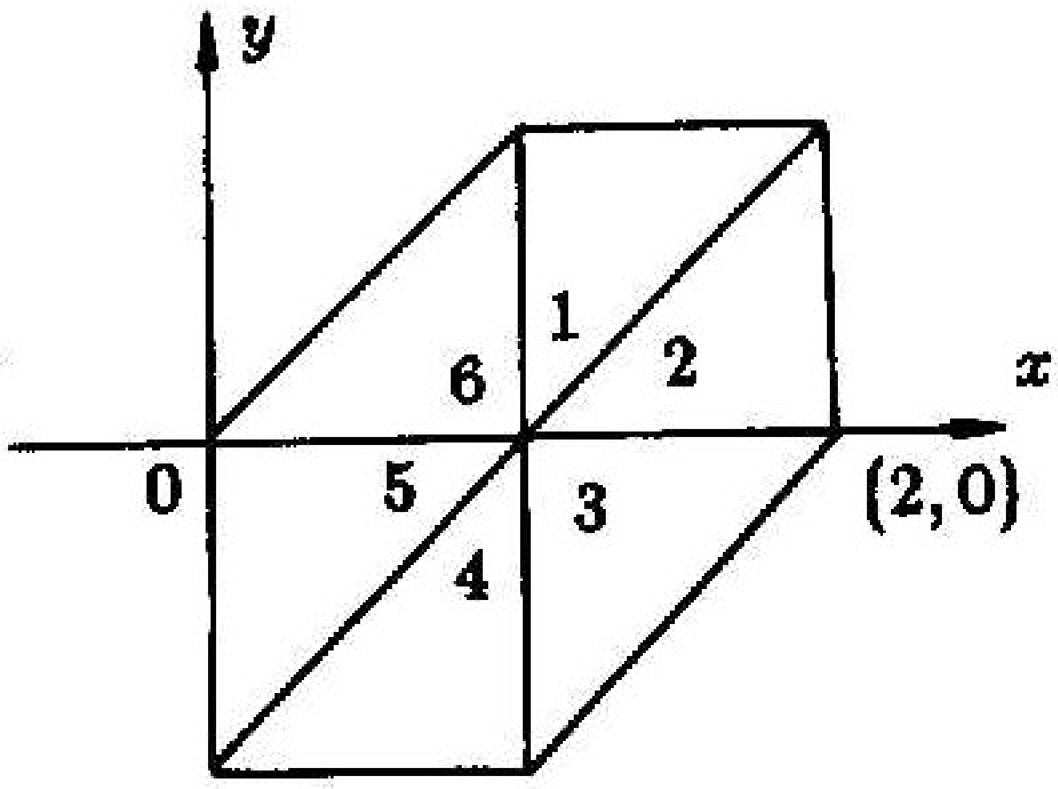

Fig. 2

1) *If a B-spline* $B_{2s}(x,y) \in S_{3s+1}^{2s}(\Delta_1)$ *with support* T_{2s}, *and* $A(T_{2s}) = (a_1, a_2, a_3, a_4, a_5, a_6)$, *then*

$$B_{2s+1}^{(1)}(x,y) = \int_{y}^{y+1} dv \int_{v-y+x}^{v-y+x+1} B_{2s}(u,v)du,$$

$$B_{2s+1}^{(2)}(x,y) = \int_{x}^{x+1} du \int_{v-x+y}^{v-x+y+1} B_{2s}(u,v)dv$$

are two linearly independent locally supported splines in $S_{3s+3}^{2s+1}(\Delta_1)$ *with supports* $T_{2s+1}^{(1)}, T_{2s+1}^{(2)}$ *respectively, where*

$$A(T_{2s+1}^{(1)}) = (a_1+1, a_2, a_3+1, a_4+1, a_5, a_6+1),$$

$$A(T_{2s+1}^{(2)}) = (a_1, a_2+1, a_3+1, a_4, a_5+1, a_6+1).$$

2) $B_{2s+2}(x,y) = \int_{x}^{x+1} B_{2s+1}^{(2)}(t,y)dt$ *is a B-spline in* $S_{3s+4}^{2s+2}(\Delta_1)$ *with support* T_{2s+2}, *and*

$$A(T_{2s+2}) = (a_1+1, a_2+1, a_3+1, a_4+1, a_5+1, a_6+1).$$

Proof. Note that $\frac{\partial}{\partial x}B_{2s+2}(x,y), \frac{\partial}{\partial y}B_{2s+2}(x,y) \in S_{3s+3}^{2s+1}(\Delta_1)$. Recalling Lemma 1.1 and the properties of the integral, we know that the results are valid. So a sequence of locally supported splines $\left\{B_{2s}(x,y), B_{2s+1}^{(1)}(x,y), B_{2s+1}^{(2)}(x,y); s = 0,1,2,\cdots\right\}$ can be constructed from $B_0(x,y)$; $B_{2s}(x,y)$ is in $S_{3s+1}^{2s}(\Delta_1)$ with support T_{2s}, and $B_{2s+1}^{(1)}(x,y), B_{2s+1}^{(2)}(x,y)$ are in $S_{3s+3}^{2s+1}(\Delta_1)$ with support $T_{2s+1}^{(1)}, T_{2s+1}^{(2)}$,

$$A(T_{2s}) = (s+1, s+1, s+1, s+1, s+1, s+1),$$

$$A(T_{2s+1}^{(1)}) = (s+2, s+1, s+2, s+2, s+1, s+2),$$

$$A(T_{2s+1}^{(2)}) = (s+1, s+2, s+2, s+1, s+2, s+2).$$

Accroding to Theorem 1, the space $S_k^\mu(\Delta_1)(k > (3\mu+1)/2)$ contains a B-spline. Take a B-spline $B_{\mu_0}(x,y) \in S_{k_0}^{\mu_0}(\Delta_1) \subset \left\{S_k^\mu(\Delta_1); k > (3\mu+1)/2\right\}$, and set $B_{0,0,0}(x,y) = B_{\mu_0}(x,y)$. Then we have the following propositions.

Proposition 1. For $k_1, k_2, k_3 = 0, 1, 2, \cdots$, if

$$B_{k_1+1,k_2,k_3}(x,y) = \int_x^{x+1} B_{k_1,k_2,k_3}(u,y)du,$$

$$B_{k_1,k_2+1,k_3}(x,y) = \int_y^{y+1} B_{k_1,k_2,k_3}(x,u)du,$$

$$B_{k_1,k_2,k_3+1}(x,y) = \int_x^{x+1} B_{k_1,k_2,k_3}(u,u-x+y)du,$$

then $B_{k_1,k_2,k_3}(x,y)$ is a locally supported spline in $S_k^\mu(\Delta_1)$, where $k = k_0 + k_1 + k_2 + k_3$, and $\mu = \mu_0 + k_1 + k_2 + k_3 - \max(k_1, k_2, k_3)$.

Proposition 2. Choose $B_{0,0,0}(x,y) = B_0(x,y) \in S_1^0(\Delta_1)$ in Proposition 1 . Then

$$\begin{aligned}\{B\} = \ & \Big\{B_{s+t,s,s}, B_{s+t-1,s+1,s}, \cdots, B_{s,s+t,s}, B_{s,s+t-1,s+1},\\ & B_{s,s+t-2,s+2}, \cdots, B_{s,s,s+t};\ \ t = 0,1,2,\cdots,k-3s-1\Big\}\end{aligned}$$

is a basis of $\tilde{S}_k^{2s}(\Delta_1)$, $k \ge 3s+1$;

$$\begin{aligned}\{B\} = & \Big\{B_{s+t-1,s+1,s}, B_{s+t-2,s+2,s}, \cdots, B_{s+1,s+t-1,s}, B_{s,s+t-1,s+1},\\ & B_{s,s+t-2,s+2}, \cdots, B_{s,s+1,s+t-1};\ \ t = 2,3,\cdots,k-3s-1\Big\}\end{aligned}$$

is a basis of $\tilde{S}_k^{2s+1}(\Delta_1)$, $k \ge 3s+3$. Here $\tilde{S}_k^\mu(\Delta_1)$ is a subspace of $S_k^\mu(\Delta_1)$ consisting of locally supported splines.

Proof. The linear independence of $\{B\}$ can be derived with the aid of their Fourier transforms. From dimension formulas in [2], it is easy to verify that the dimension of $\tilde{S}_k^\mu(\Delta_1)$ equals the cardinality of $\{B\}$. The conclusion then follows.

Proposition 3. Choose $B_0(x,y)$ as $B_{0,0,0}(x,y)$ in Proposition 1. Then

$$\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} B_{k_1,k_2,k_3}(x+m, y+n) = 1, \quad (x,y) \in R^2,$$

and

$$\iint_{-\infty}^{+\infty} B_{k_1,k_2,k_3}(x,y)\,dx\,dy = 1.$$

Proof. It is obvious that $B_{0,0,0}(x,y)$ has these properties. From the definition of $B_{k_1,k_2,k_3}(x,y)$ in Proposition 1, we know this proposition is true.

Proposition 4. The Fourier transform of $B_{k_1,k_2,k_3}(x,y)$ is

$$F(B_{k_1,k_2,k_3}(x,y))(u,v) = g^{k_1+1}(u)g^{k_2+1}(v)g^{k_3+1}(u+v),$$

where $g(t) = (e^{it} - 1)/it$.

Proof. Set

$$b(x) = \begin{cases} 1 & x \in (-1,0), \\ 0 & x \overline{\in} (-1,0). \end{cases}$$

Then

$$F(b(x))(t) = g(t)$$

and

$$F(B_0(x,y))(u,v) = g(u)g(v)g(u+v).$$

Noting that

$$\int_x^{x+1} B(u,y)du = B(x,y) * b(x),$$

we have established the relations.

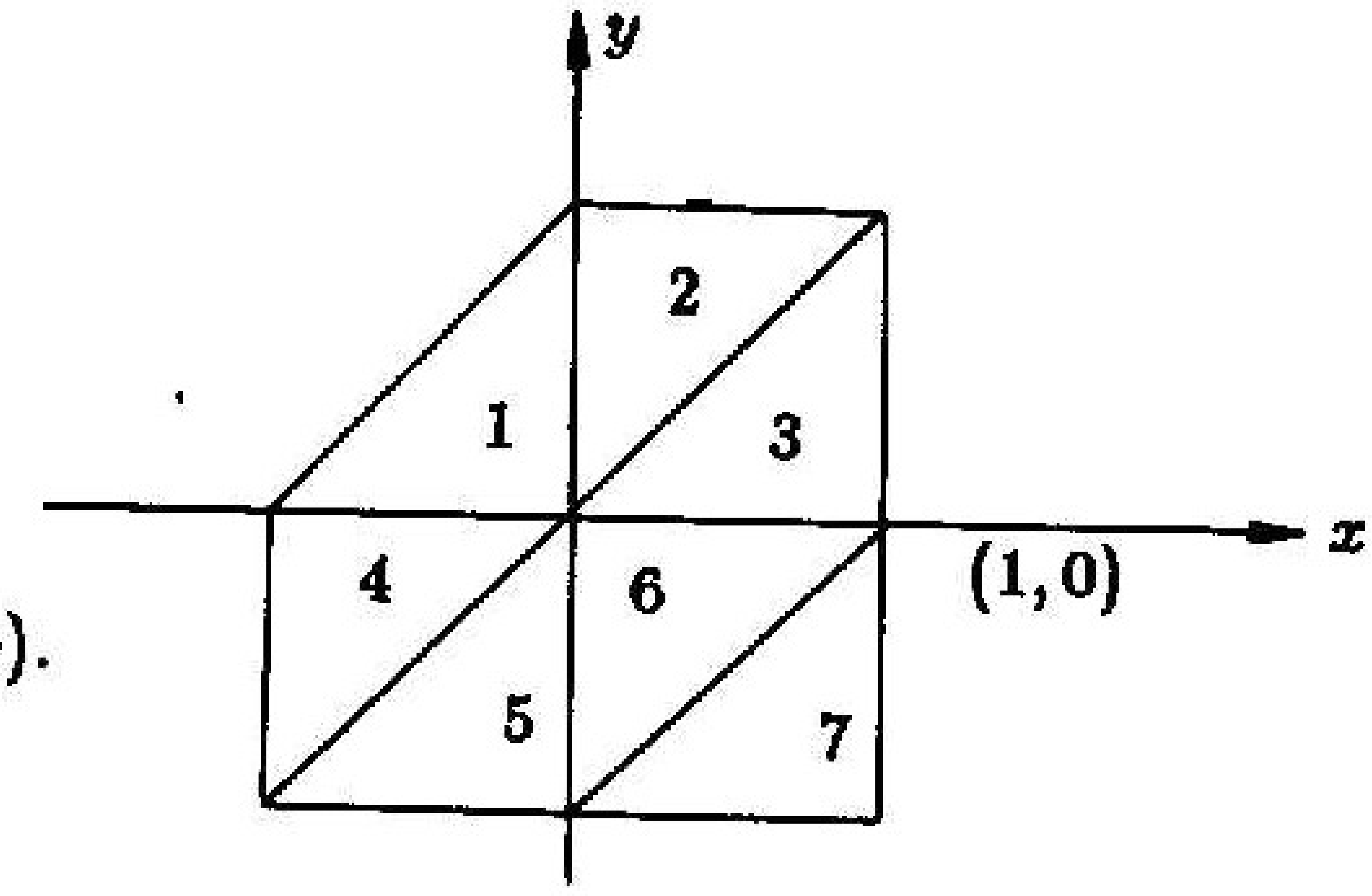

Fig. 3

Next, we will show that B-splines in $S_d^\mu(\Delta_1)$ can be derived by the integral recursions in Theorem 1.

Let $\bar{B}_0(x,y) \in S_2^0(\Delta_1)$. Its support is shown in Fig. 3 and its representations in the ith cell is $q_i(x,y), i = 1,2,\cdots,7$, where

$$q_1(x,y) = \frac{1}{2}(x-y+1)^2, \quad q_2(x,y) = (1-y)(x - \frac{y}{2} + \frac{1}{2}),$$
$$q_3(x,y) = (1-x)(\frac{3}{2}x - y + \frac{1}{2}), \quad q_4(x,y) = \frac{1}{2}(x+1)^2,$$
$$q_5(x,y) = (y+1)(x - \frac{y}{2} + \frac{1}{2}), \quad q_6(x,y) = x(1-x) + \frac{1}{2}(y-x+1)(x-y+1),$$
$$q_7(x,y) = (1-x)(y+1).$$

Proposition 5. Choose $B_0(x,y)$ as an original spline, and construct $B_{2s}(x,y)$ by integral recursions in Theorem 1. Then $B_{2s}(x,y)$ is the unique minimal supported B-spline in $S_{3s+1}^{2s}(\Delta_1), s = 0,1,2,\cdots$. Choose $\bar{B}_0(x,y)$ mentioned above as an original spline and construct $B_{2s+1}^{(1)}(x,y), B_{2s+1}^{(2)}(x,y)$ by integral recursions given in Theorem 1. Then $B_{2s+1}^{(1)}(x,y), B_{2s+1}^{(2)}(x,y)$ are B -splines in $S_{3s+3}^{2s+1}(\Delta_1), s = 0,1,2,\cdots$.

Proof. From their Fourier transforms, we know that $B_{2s}(x,y)$ is the same as $M_{s,s,s}(x,y)$ in [4] and $B_{2s+1}^{(1)}(x,y), B_{2s+1}^{(2)}(x,y)$ are the same as $N_{s+1}(x,y), N'_{s+1}(x,y)$ in [4] respectively. Therefore the conclusions are true.

Proposition 5 reveals that if we choose original splines properly, B-splines can be obtained by our integral method. Besides, we can get locally supported splines on certain nonuniform triangulations. For instance, making a partition with lines $\bar{\Delta}$: $x = 3t, x = 3t+1, y = 3t, y = 3t+1$, $t = 0,1,2,\cdots$, and then adding all upward slopings in cells of $\bar{\Delta}$, we have a nonuniform partition $\bar{\Delta}_1$. Let $B_0^*(x,y)$ be a B-spline in $S_1^0(\bar{\Delta}_1)$ (this is easy to obtain), and define $B_{2s}^*(x,y)=\int_x^{x+3}du\int_y^{y+3}B_{2(s-1)}^*(u,v)dv, s>1$. Then $B_{2s}^*(x,y)$ is a locally supported spline in $S_{2s+1}^s(\bar\Delta_1), s=0,1,2,\cdots$. Furthermore, if $\tilde\Delta$ is a partition obtained from a geometric rectangular partition consisting of lines $x=ah^i+x_0, y=bl^j+y_0$, $i,j=0,\pm1,\pm2,\cdots$, by drawing in the same diagonal in each square, we can produce B-splines in $S_{3s+1}^{2s}(\tilde\Delta)$ and $S_{3s+3}^{2s+1}(\tilde\Delta)$ when $h=l$ (cf [7]). So we say that the integral method is more flexible than the box-spline method.

§2. Bases of Spline Spaces $S_d^\mu(\Delta_{mn}^{(1)},D)$

Set $D=\{(x,y): a\le x\le b, c\le y\le d\}, h=(b-a)/m, l=(d-c)/n, x_i=a+ih, y_j=c+jl$ $(i=0,1,2,\cdots,m; j=0,1,2,\cdots,n)$, and partition $\Delta_{mn}^{(1)}$ of D consists of lines

$$x=x_i, y=y_j, (x-x_i)/h=(y-y_j)/l, \quad i=0,1,2,\cdots,m; j=0,1,2,\cdots,n.$$

For every nonnegative integer μ we will give a basis of $S_d^\mu(\Delta_{mn}^{(1)},D)$.

Theorem 2. *For every integer $s\ge0$, a basis of $S_{3s+1}^{2s}(\Delta_{mn}^{(1)},D)$ is*

$$\begin{aligned}
E^{2s}=&\{B_{2s}((x-x_i)/h-2,(y-y_j)/l-1),(x-x_0)^u(y-y_j)_+^{v+2s+1},\\
&(y-y_0)^u(x_i-x)_+^{v+2s+1},((x-x_0)/h+(y-y_0)/l)^u\\
&\times((y-y_0)/l-x(x-x_0)/h+r)_+^{v+2s+1},(x-x_0)^p(y-y_0)^q;\\
&i=1,2,\cdots,m-1, j=1,2,\cdots,n-1,\\
&r=1-n,2-n,\cdots,m-1, 0\le u+v\le s,\ 0\le p+q\le 3s+1\};\\
E^{2s+1}=&\{B_{2s+1}^{(1)}((x-x_i)/h-1,(y-y_j)/l-1),B_{2s+1}^{(2)}((x-x_i)/h-1,(y-y_j)/l-1),\\
&(x-x_0)^u(y-y_j)_+^{v+2s+2},(y-y_0)^u(x_i-x)_+^{v+2s+2},((x-x_0)/h+(y-y_0)/l)^u\\
&\times(r-(x-x_0)/h+(y-y_0)/l)_+^{v+2s+2},(x-x_0)^p(y-y_0)^q;\\
&i=1,2,\cdots,m-1, j=1,2,\cdots,n-1,\ r=1-n,2-n,\cdots,m-1,\\
&0\le u+v\le s+1, 0\le p+q\le 3s+3\},
\end{aligned}$$

where $B_{2s}(x,y), B_{2s+1}^{(1)}(x,y)$ and $B_{2s+1}^{(2)}(x,y)$ are the same B-splines as in Proposition 5.

Proof. Because [2]

$$\dim S_k^\mu(\Delta_{mn}^{(1)},D)=\eta(k)+(2m+2n-3)\cdot\eta(k-\mu+1)+d(3)(m-1)(n-1),$$

where

$$\eta(t)=\frac12(t+1)(t+2),$$

$$d(n)=\frac12(k-\mu-[(\mu+1)/(n-1)])_+((n-1)k-(n+1)\mu+n-3+(n-1)[\frac{\mu+1}{n-1}]),$$

the cardinalities of E^{2s} and E^{2s+1} are equal to the dimensions of $S_{3s+1}^{2s}(\Delta_1)$ and $S_{3s+3}^{2s+1}(\Delta_1)$, respectively. So we need only to prove that E^{2s} and E^{2s+1} are linearly independent, respectively. Here we determine the linear independence of E^{2s+1}; that of E^{2s} can be got in the same way.

Assume there are $a_{ij}^{(1)}, a_{ij}^{(2)}, b_{juv}, c_{juv}, d_{ruv}, e_{pq}$ so that

$$\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}\left[a_{ij}^{(1)}B_{2s+1}^{(1)}(\frac{(x-x_i)}{h}-1,\frac{(y-y_j)}{l}-1)+a_{ij}^{(2)}B_{2s+1}^{(2)}(\frac{(x-x_i)}{h}-1,\frac{(y-y_j)}{l}-1)\right]$$

$$+\sum_{0\le u+v\le s+1}\left[\sum_{j=1}^{n-1}b_{juv}(x-x_0)^u(y-y_j)_+^{v+2s+2}+\sum_{i=1}^{m-1}c_{iuv}(y-y_0)^u(x_i-x)_+^{v+2s+1}\right.$$
$$\left.+\sum_{r=1-n}^{m-1}d_{ruv}(\frac{(x-x_0)}{h}+\frac{(y-y_0)}{l})^u(\frac{r+(y-y_0)}{l}-\frac{(x-x_0)}{h})_+^{v+2s+2}\right]$$
$$+\sum_{0\le p+q\le 3s+3}e_{pq}(x-x_0)^p(y-y_0)^q=0. \qquad (*)$$

Set

$$r_{ij}^{(1)} = \left\{(x,y): x_i\le x\le x_{i+1}, y_j\le y\le (y_{j+1}, (x-x_i)/h\le (y-y_j)/l\right\},$$
$$r_{ij}^{(2)} = \left\{(x,y): x_i\le x\le x_{i+1}, y_j\le y\le y_{j+1}, (x-x_i)/h\le (y-y_j)/l\right\},$$
$$i=0,1,\cdots,m-1; j=0,1,\cdots,n-1.$$

If $(x,y)\in r_{m-1,o}^{(2)}$, $(*)$ becomes

$$\sum_{0\le p+q\le 3s+3}e_{pq}(x-x_0)^p(y-y_0)^q=0,$$

which implies

$$e_{pq}=0,\quad 0\le p+q\le 3s+3.$$

Let $(x,y)\in r_{m-1,0}^{(1)}, r_{m-1,1}^{(2)}, r_{m-1,1}^{(1)}, r_{m-1,2}^{(2)}, r_{m-1,2}^{(1)},\cdots, r_{m-1,n-1}^{(2)}, r_{m-1,n-1}^{(1)}$, successively. From $(*)$ we infer

$$d_{ruv}=0, r=m-1,m-2,\cdots,m-n; 0\le u+v\le s+1,$$
$$b_{juv}=0, j=1,2,\cdots,n-1; 0\le u+v\le s+1.$$

Next, let $(x,y)\in r_{m-2,n-1}^{(2)}, r_{m-2,n-1}^{(1)}, r_{m-3,n-1}^{(2)}, r_{m-3,n-1}^{(1)},\cdots, r_{0,n-1}^{(2)}, r_{0,n-1}^{(1)}$, successively. From $(*)$ we infer

$$d_{ruv}=0, r=m-n-1,m-n-2,\cdots,1-n; 0\le u+v\le s+1,$$
$$c_{iuv}=0, i=1,2,\cdots,m-1, 0\le u+v\le s+1.$$

Because $B_{2s+1}^{(1)}(x,y)=\int_x^{x+1}f(u,y)du$, $B_{2s+1}^{(2)}(x,y)=\int_y^{y+1}f(x,u)du$, $B_{2s+1}^{(1)}((x-x_{m-1})/h-1,(y-y_{n-1})/l-1)$ and $B_{2s+1}^{(2)}((x-x_{m-1})/h-1, (y-y_{n-1})/l-1)$ are linearly independent of $r_{m-2,n-2}^{(1)}$, so let $(x,y)\in r_{m-2,n-2}^{(1)}, r_{m-3,n-2}^{(1)},\cdots, r_{0,n-2}^{(1)}, r_{m-2,n-3}^{(1)}, r_{m-3,n-3}^{(1)}, \cdots, r_{0,n-3}^{(1)},\cdots, r_{m-2,0}^{(1)}, r_{m-3,0}^{(1)},\cdots, r_{0,0}^{(1)}$, successively. From $(*)$ we infer

$$a_{ij}^{(1)}=a_{ij}^{(2)}=0,\quad i=1,2,\cdots,m-1; j=1,2,\cdots,n-1.$$

So the set E^{2s+1} is linearly independent.

§3. The Existence and Construction of Locally Supported Splines in $S_k^\mu(\Delta_2)$

Denote an octagon T as shown in Fig. 4 by $A(T) = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$, where a_i is the number of rectangles included in the ith dege of $T, i = 1, 2, \cdots, 8$.

Lemma 3.1. *If $B(x,y) \in S_k^\mu(\Delta_2)$, its support is T and $A(T) = (a_1, a_2, \cdots, a_8)$, then*

$$B_1(x,y) = \int_{x-1}^{x} B(u,y)du$$

and

$$B_2(x,y) = \int_{y-1}^{y} B(x,u)du$$

belong to $S_{k+1}^\mu(\Delta_2)$, and the supports of $B_1(x,y)$ and $B_2(x,y)$ are T_1 and T_2, respectively, where

$$A(T_1) = (a_1, a_2+1, a_3, a_4, a_5, a_6+1, a_7, a_8),$$
$$A(T_2) = (a_1, a_2, a_3, a_4+1, a_5, a_6, a_7, a_8+1).$$

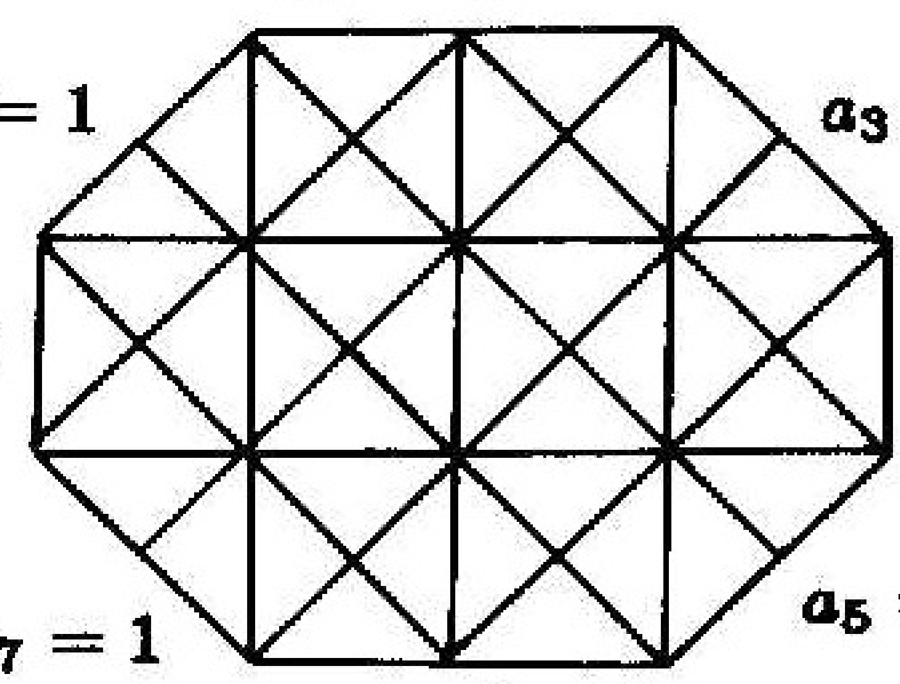

Fig. 4

Lemma 3.2. *If $B(x,y) \in S_k^\mu(\Delta_2)$ and its support is T, where $A(T) = (a_1, a_2, \cdots, a_8)$, then*

$$B_3(x,y) = \int_{x-1}^{x} B(u, y-x+u)du$$

and

$$B_4(x,y) = \int_{x}^{x+1} B(u, y+x-u)du$$

belong to $S_{k+1}^\mu(\Delta_2)$, and the supports of $B_3(x,y)$ and $B_4(x,y)$ are T_3 and T_4, respectively, where

$$A(T_3) = (a_1+1, a_2, a_3, a_4, a_5+1, a_6, a_7, a_8),$$
$$A(T_4) = (a_1, a_2, a_3+1, a_4, a_5, a_6, a_7+1, a_8).$$

The validity of Lemmas 3.1 and 3.2 are obvious.

Next, we consider the existence and construction of locally supported splines in $S_d^\mu(\Delta_2)$. When $\mu = 1$ and $d = 2$, there is a B-spline in $S_2^1(\Delta_2)$ and its support is $(1,1,1,1,1,1,1,1)$ (cf [3]). We can construct locally supported splines in $S_4^2(\Delta_2)$, $S_5^3(\Delta_2)$, $S_6^4(\Delta_2)$ in the following way (the number of linearly independent splines obtained is 3, 2, 1, respectively); then we construct those splines in $S_8^5(\Delta_2), \cdots$. Generally, if we have a locally supported spline in $S_{4k+2}^{3k+1}(\Delta_2)$, $k = 0, 1, 2, \cdots$, we will produce smoother splines with local support by three steps.

i) Construct $B_2^{(j)}(x,y) \in S_{4k+4}^{3k+2}(\Delta_2)$ from $B(x,y) \in S_{4k+2}^{3k+1}(\Delta_2)$, $j=1,2,3$, where the support of $B(x,y)$ is T,

$$A(T) = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8),$$

and

$$B_1^{(1)}(x,y) = \int_x^{x+1} B(s,y)ds,$$

$$B_1^{(2)}(x,y) = \int_y^{y+1} B(x,t)dt,$$

$$B_1^{(3)}(x,y) = \int_x^{x-1} B(s,y+x-s)ds,$$

$$B_2^{(1)}(x,y) = \int_x^{x+1} B_1^{(1)}(v, v+y-x)\,dv = \int_y^{y+1} dv \int_{v+x-y}^{v+x-y+1} B(s,v)ds,$$

$$B_2^{(2)}(x,y) = \int_x^{x+1} B_1^{(2)}(v, v+y-x)\,dv = \int_x^{x+1} dv \int_{v+y-x}^{v+y-x+1} B(v,t)dt,$$

$$B_2^{(3)}(x,y) = \int_x^{x+1} B_1^{(3)}(v, v+y-x)\,dv = \frac{1}{2}\int_{x+y}^{x+y+2} dv \int_{(v+y-x)/2}^{(v+y-x-2)/2} B(s,v-s)ds.$$

$B_2^{(j)}(x,y), j=1,2,3$, are all locally supported splines in $S_{4k+4}^{3k+2}(\Delta_2)$ with supports $T_2^{(1)}, T_2^{(2)}, T_2^{(3)}$, respectively, where

$$A(T_2^{(1)}) = (a_1, a_2+1, a_3+1, a_4, a_5, a_6+1, a_7+1, a_8),$$

$$A(T_2^{(2)}) = (a_1, a_2, a_3+1, a_4+1, a_5, a_6, a_7+1, a_8+1),$$

$$A(T_2^{(3)}) = (a_1+1, a_2, a_3+1, a_4, a_5+1, a_6, a_7+1, a_8).$$

ii) Construct locally supported splines $B_3^{(1)}(x,y)$, $B_3^{(2)}(x,y) \in S_{4k+5}^{3k+3}(\Delta_2)$ from $B_2^{(j)}(x,y) \in S_{4k+4}^{3k+2}(\Delta_2)$, $j=1,2,3$, by

$$B_3^{(1)}(x,y) = \int_x^{x-1} B_2^{(1)}(u, y+x-u)du = \int_x^{x-1} du \int_{y+x-u}^{y+x-u+1} dv \int_{v-2u-x-y}^{v-2u-x-y+1} B(s,v)ds$$

$$B_3^{(2)}(x,y) = \int_x^{x-1} B_2^{(2)}(u, y+x-u)du = \int_x^{x-1} du \int_u^{u+1} dv \int_{v-2u+x+y}^{v-2u+x+y+1} B(v,t)dt,$$

Their supports are $T_3^{(1)}$ and $T_3^{(2)}$, respectively, where

$$A(T_3^{(1)}) = (a_1+1, a_2+1, a_3+1, a_4, a_5+1, a_6+1, a_7+1, a_8),$$

$$A(T_3^{(2)}) = (a_1+1, a_2, a_3+1, a_4+1, a_5+1, a_6, a_7+1, a_8+1).$$

iii) Finally, construct

$$B_4(x,y)=\int_x^{x+1}B_3^{(2)}(s,y)ds=\int_x^{x+1}ds\int_s^{s-1}du\int_u^{u+1}dv\int_{v-2u+y+s}^{v-2u+y+s+1}B(v,t)dt.$$

It is a locally supported spline in $S_{4k+6}^{3k+4}(\Delta_2)$ with support T_4, where $A(T_4)=(a_1+1,a_2+1,a_3+1,a_4+1,a_5+1,a_6+1,a_7+1,a_8+1)$.

By these steps, we finish a cycle of constructing locally supported splines. Doing this repeatedly, we obtain all locally supported splines in $S_k^\mu(\Delta_2)$ and can prove the following theorem.

Theorem 3. *There are locally supported splines in $S_k^\mu(\Delta_2)$ iff $k>\frac{4\mu+1}{3}$. We can produce locally supported splines in $S_k^\mu(\Delta_2)$ by the above-mentioned integral recursions.*

Because of their different Fourier transforms, neither of the ten locally supported box-splines in [5] is the same as any locally supported spline constructed here. In fact, these splines have smaller supports than box-splines in the same space.

Next, we give bases of $S_k^\mu(\Delta_{mn}^{(2)},D)$, where $D=\{(x,y):a\le x\le b,c\le y\le d\},h=(b-a)/m,l=(d-c)/n,\Delta_{mn}^{(2)}$ is a partition of D consisting of lines $x=x_i=a+ih,y=y_j=c+jl,x+y=x_i+y_j,x-y=x_i-y_j,i=0,1,\cdots,m;j=0,1,\cdots,n$.

By a similar statement as used in [7], we can prove the following theorem.

Theorem 4. *The basis of $S_{4k+4}^{3k+2}(\Delta_{mn}^{(2)},D)$ is*

$$\begin{aligned}p^{3k+2}= &\Big\{B_{3k+2}^{(1)}((x-x_i)/h,(y-y_j)/l),B_{3k+2}^{(2)}((x-x_i)/h,(y-y_j)/l),\\ &B_{3k+2}^{(3)}((x-x_i)/h,(y-y_j)/l),(x-x_m)^s(y_j-y)_+^{t+3k+3},\\ &(i=1,2,\cdots,m,j=1,2,\cdots,n-1,0\le s+t\le k+1,(s,t)\ne(s_{3k+2}^r,t_{3k+2}^r),\\ &r=1,2,3);(y-y_0)^s(x_i-x)_+^{3k+3+t}(i=1,2,\cdots,m-1,0\le s+t\le k+1);\\ &(x-x_0+y-y_0)^s((x-x_0)/h+(y-y_0)/l-u)_+^{t+3k+3}(u=1,2,\cdots,\\ &m+n-1,0\le s+t\le k+1);(x-x_0+y-y_0)^s((y-y_0)/l\\ &-(x-x_0)/h-v)_+^{t+3k+3}(v=1-n,2-n,\cdots,m-1-n,0\le s+t\le k+1);\\ &(x-x_0)^p(y-y_0)^q(0\le p+q\le 4k+4)\Big\};\end{aligned}$$

the basis of $S^{3k+3}_{4k+5}(\Delta^{(2)}_{mn}, D)$ *is*

$$p^{3k+3} = \Big\{B^{(1)}_{3k+3}((x-x_i)/h,(y-y_j)/l), B^{(2)}_{3k+3}((x-x_i)/h,(y-y_j)/l)$$
$$(i=1,2,\cdots,m, j=1,2,\cdots,n-1); (x-x_m)^s(y_j-y)_+^{t+3k+4}(j=1,2,$$
$$\cdots,n-1, 0\le s+t\le k+1, (s,t)\ne(s^r_{3k+3},t^r_{3k+3})r=1,2); (y-y_0)^s$$
$$(x_i-x)_+^{3k+4+t}(i=1,2,\cdots,m-1, 0\le s+t\le k+1); (x-x_0+y-y_0)^s$$
$$((x-x_0)/h+(y-y_0)/l-u)_+^{3k+4+t}(u=1,2,\cdots,m+n-1, 0\le s+t$$
$$\le k+1); (x-x_0+y-y_0)^s(-(x-x_0)/h+(y-y_0)/l-v)_+^{3k+4+t}(v=1-n,2-n,$$
$$\cdots m-1-n, 0\le s+t\le k+1); (x-x_0+y-y_0)^s((x-x_0)/h$$
$$-(y-y_0)/l+v)_+^{t+3k+4}(v=m-n,m-n+1,\cdots,m-1, 0\le s+t\le k+1);$$
$$(x-x_0)^p(y-y_0)^q(0\le p+q\le 4k+5)\Big\};$$

the basis of $S^{3k+4}_{4k+6}(\Delta^{(2)}_{mn}, D)$ *is*

$$p^{3k+4} = \Big\{B_{3k+4}((x-x_i)/h,(y-y_j)/l)(i=1,2,\cdots,m, j=1,2,\cdots,n-1);$$
$$(x-x_m)^s(y_j-y)_+^{3k+5+t}(j=1,2,\cdots,n-1, 0\le s+t\le k+1);$$
$$(y-y_0)^s(x_i-x)_+^{3k+5+t}(i=1,2,\cdots,m-1, 0\le s+t\le k+1,$$
$$(s,t)\ne(s_{3k+4},t_{3k+4})); (x-x_0-y+y_0)^s((x-x_0)/h+(y-y_0)/l$$
$$-u)_+^{t+3k+5}(u=1,2,\cdots,m+n-1, 0\le s+t\le k+1); (x-x_0+y-y_0)^s$$
$$((y-y_0)/l-(x-x_0)/h-v)_+^{t+3k+5}(v=1-n,2-n,\cdots,m-1-n,$$
$$0\le s+t\le k+1); (x-x_0+y-y_0)^s((x-x_0)/h-(y-y_0)/l+v)_+^{t+3k+5}$$
$$(v=m-n,m-n+1,\cdots,m-1, 0\le s+t\le k+1); (x-x_0)^p(y-y_0)^q$$
$$(0\le p+q\le 4k+6)\Big\},$$

where $(s^r_{3k+2}, t^r_{3k+2}) = (b_r, c_r), r=1,2,3,$ *which satisfy*

$$\det\begin{pmatrix} z^{(1)}_{b_1,c_1} & z^{(1)}_{b_2,c_2} & z^{(1)}_{b_3,c_3} \\ z^{(2)}_{b_1,c_1} & z^{(2)}_{b_2,c_2} & z^{(2)}_{b_3,c_3} \\ z^{(3)}_{b_1,c_1} & z^{(3)}_{b_2,c_2} & z^{(3)}_{b_3,c_3} \end{pmatrix} \ne 0,$$

and $z^{(r)}_{bc}, r=1,2,3,$ *come from*

$$B^{(r)}_{3k+2} = \sum_{0\le b+c\le k+1} z^{(r)}_{bc} x^b y^{c+3k+3}, (x,y)\in\omega^{(1)}_{m-1,n-2}, r=1,2,3,$$

and $(s^r_{3k+3}, t^r_{3k+3}) = (d_r, e_r), r = 1, 2,$ *which satisfy*

$$\det \begin{pmatrix} w^{(1)}_{d_1,e_1} & w^{(1)}_{d_2,e_2} \\ w^{(2)}_{d_1,e_1} & w^{(2)}_{d_2,e_2} \end{pmatrix} \neq 0,$$

$w^{(r)}_{de}, r = 1, 2,$ *come from*

$$B^{(r)}_{3k+3} = \sum_{0 \le d+e \le k+1} w^{(r)}_{de} x^d y^{e+3k+4}, (x, y) \in \omega^{(1)}_{m-1,n-1}, r = 1, 2,$$

and $(s_{3k+4}, t_{3k+4}) = (g, h)$ *comes from*

$$B_{3k+4} = \sum_{0 \le g+h \le k+1} v_{gh} x^g y^{h+3k+5}, (x, y) \in \omega^{(2)}_{m-1,n-2},$$

and $v_{gh} \neq 0$.

§4 Symmetric B-spline functions

If the support of a spline is symmetric with respect to its centre and grid lines passing through the centre, we call the spline a symmetric spline and its support a symmetric support. A locally supported spline with minimal symmetric support is called a symmetric B-spline. In this section, we will consider construction of symmetric B-splines using an integral operator which is an accumulation of four integral operators shown in Lemmas 3.1 and 3.2 that is,

$$(Uf)(x, y) = \int_{x-1}^{x} dv \int_{v-1}^{v} du \int_{x+y-v-1}^{x+y-v} dt \int_{u-1}^{u} f(s, t) ds,$$

if $f(x, y) \in S^{\mu}_k(\Delta_2)$.

Choosing original splines $B^{0,1}_1(x, y), B^{0,2}_1(x, y) \in S^0_1(\Delta_2)$, $B^1_2(x, y) \in S^1_2(\Delta_2)$ (cf [3]) and $B^{2,1}_4(x, y), B^{2,2}_4(x, y), B^{2,3}_4(x, y) \in S^2_4(\Delta_2)$ (cf [8]) (they are all symmetric splines), we will construct symmetric B-splines in $S^{3k}_{4k+1}(\Delta_2), S^{3k+1}_{4k+2}(\Delta_2), S^{3k+2}_{4k+4}(\Delta_2)$, respectively.

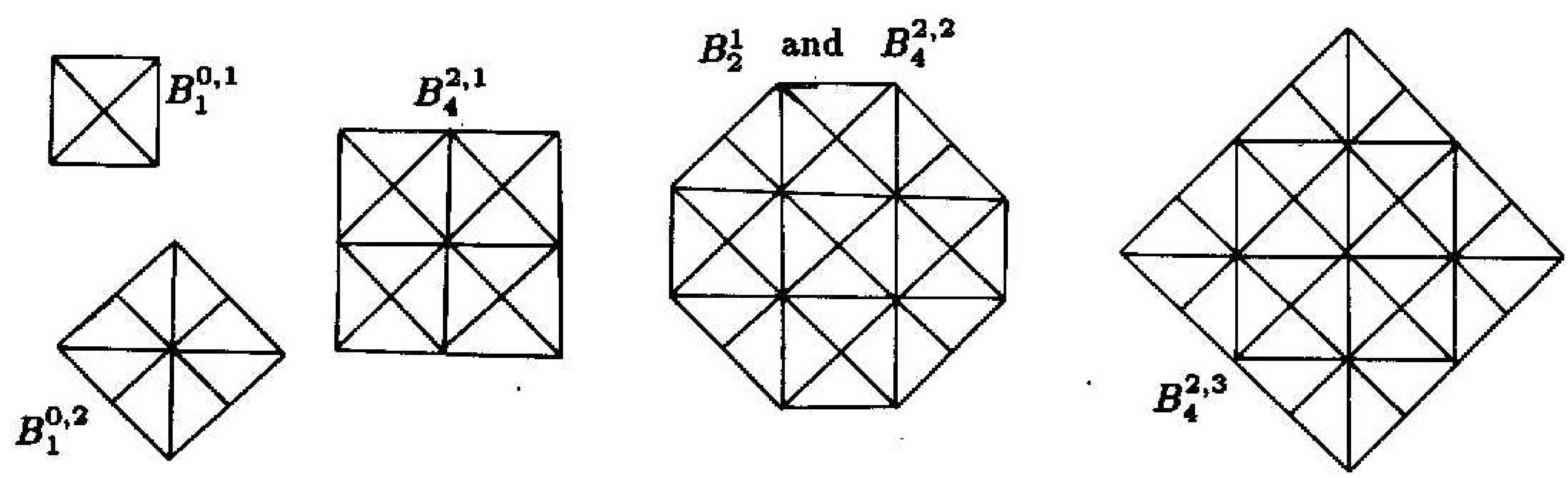

Fig. 5

Definition. If $c(x,y) \in S_k^\mu(\Delta_2)$ and has support as shown in Fig. 6, we call it a simple spline and denote it by

$$c(x,y) = (p_1(x,y), p_2(x,y), p_3(x,y)).$$

For a rectangle

$$D = \Big\{(x,y) : 0 \le x \le m, 0 \le y \le m\Big\},$$

It is easy to prove the following lemmas.

Lemma 4.1. *For simple splines*

$$c_1(x,y) = (-y, x, 0), c_2(x,y) = (0, y+x, y-x),$$

a basis of $S_1^0(\Delta_2, D)$ *is*

$$\Big\{c_1(x-i, y-j)(i = 1, 2, \cdots, m, j = -1, -2, \cdots, n-1);$$
$$c_2(x-i, y-j)(i = 0, 1, \cdots, m, j = 0, 1, \cdots, n-1); c_2(x-m, y+1)\Big\}.$$

Lemma 4.2 *For a simple spline*

$$c_3(x,y) = (2y^2, 2y^2 - (x+y)^2, (x-y)^2),$$

a basis of $S_2^1(\Delta_2, D)$ *is*

$$\Big\{c_3(x-i, y-j)(i = 0, 1, \cdots, m+1,$$
$$j = -2, -1, \cdots, n-1, (i,j) \ne (m+1, -2))\Big\}.$$

Lemma 4.3. *For simple splines*

$$c_4(x,y) = (0, (x+y)^3(3x-y), (x-y)^3(3x+y)),$$
$$c_5(x,y) = (4y^4, (x-y)^4 + x^3(4x+16y), (x-y)^4),$$
$$c_6(x,y) = (y^3(2x+y), x^3(x+2y), 0),$$

a basis of $S_4^2(\Delta_2, D)$ *is*

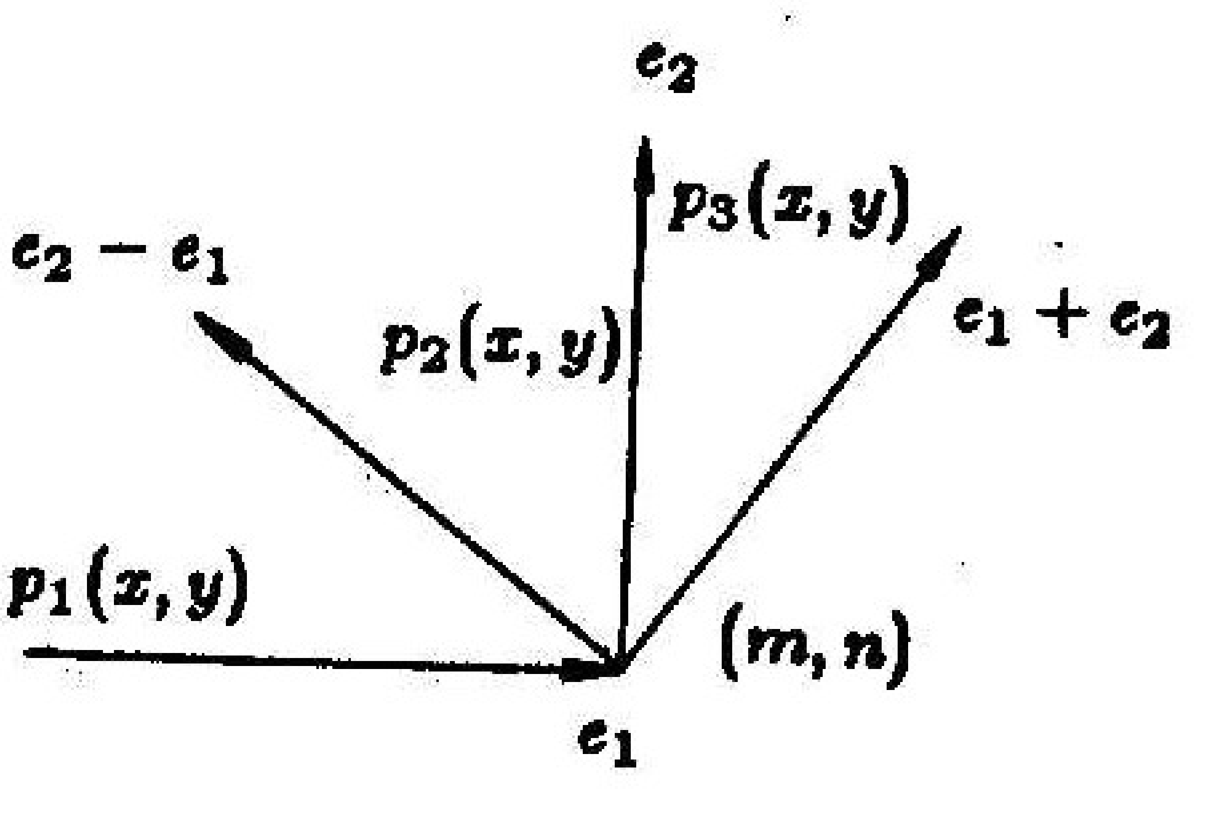

Fig. 6

$$\Big\{c_4(x-i, y-j)(i = 0, 1, \cdots, m+1, j = -2, -1, \cdots, n-1, (i,j)$$
$$\ne (m+1; n-1), (m+1, -2));$$
$$c_5(x-i, y-j)(i = 0, 1, \cdots, m+1, j = -2, -1, \cdots, n-1, (i,j)$$
$$\ne (m+1, -1), (m+1, -2));$$
$$c_6(x-i, y-j)(i = 1, 2, \cdots, m+1, j = -2, -1, \cdots, n-1, (i,j)$$
$$\ne (m+1, -1), (m+1, -2))\Big\}.$$

Define integral operator

$$I_1(\cdot)=\int_x^{+\infty}\cdot dx,\quad I_2(\cdot)=\int_{-\infty}^{y}\cdot dy,\quad I_3(\cdot)=\int_{-\infty}^{x}\cdot dx,\quad I_4(\cdot)=\int_y^{+\infty}\cdot dy,$$

where I_1, I_2, I_3, I_4 are integrations along directions $e_1, e_2, e_1+e_2, e_1-e_2$, respectively. One can verify that if $p(x,y)$ is a simple spline in $S^{\mu}_{k}(\Delta_2)$, then $(I_1I_2I_3I_4)^n p(x,y)$ is a simple spline in $S^{3n+\mu}_{4n+k}(\Delta_2), n=0,1,\cdots$, where the order of integrations is exchangeable. Furthermore, we have the following results.

Lemma 4.4. *Translates of simple splines $(I_1I_2I_3I_4)^r c_1(x,y)$ and $(I_1I_2I_3I_4)^r c_2(x,y)$ can form a basis of $S^{3r}_{4r+1}(\Delta_2,D)$; those of $(I_1I_2I_3I_4)^r c_4(x,y)$, $(I_1I_2I_3I_4)^r c_5(x,y)$ and $(I_1I_2I_3I_4)^r c_6(x,y)$, a basis of $S^{3r+2}_{4r+4}(\Delta_2,D)$; those of $(I_1I_2I_3I_4)^r c_3(x,y)$, a basis of $S^{3r+1}_{4r+2}(\Delta_2,D)$.*

Proof. We need only to note that $\left(\frac{\partial^4}{\partial x^3\partial y}-\frac{\partial^4}{\partial x\partial y^3}\right)f\in S^{\mu-3}_{k-4}(\Delta_2)$ by definition of the smoothing cofactor in [1] where $f\in S^{\mu}_{k}(\Delta_2)$, $k>4, \mu\geq 3$. Then we infer the conclusion by induction.

Writing out their expressions, we see that the first and third parts of $(I_1I_2I_3I_4)^r c_1(x,y)$, $(I_1I_2I_3I_4)^r c_2(x,y)$, $(I_1I_2I_3I_4)^r c_3(x,y)$, $(I_1I_2I_3I_4)^r c_4(x,y)$, $(I_1I_2I_3I_4)^r c_5(x,y)$, and $(I_1I_2I_3I_4)^r c_6(x,y)$ have factors y^{3r+1} and $(x-y)^{3r+2}$, y^{3r+2} and $(x-y)^{3r+1}$, y^{3r+2} and $(x-y)^{3r+2}$, y^{3r+5} and $(x-y)^{3r+3}$, y^{3r+4} and $(x-y)^{3r+4}$, y^{3r+3} and $(x-y)^{3r+5}$, respectively. We will use this factor in the proof later.

Define the area shown in Fig. 7 by Ω_{MN} (where $\Omega_{MN}=\Omega_M\cup\Omega_N$ and shadowy triangles ω_1 and ω_2 do not belong to Ω_{MN}).

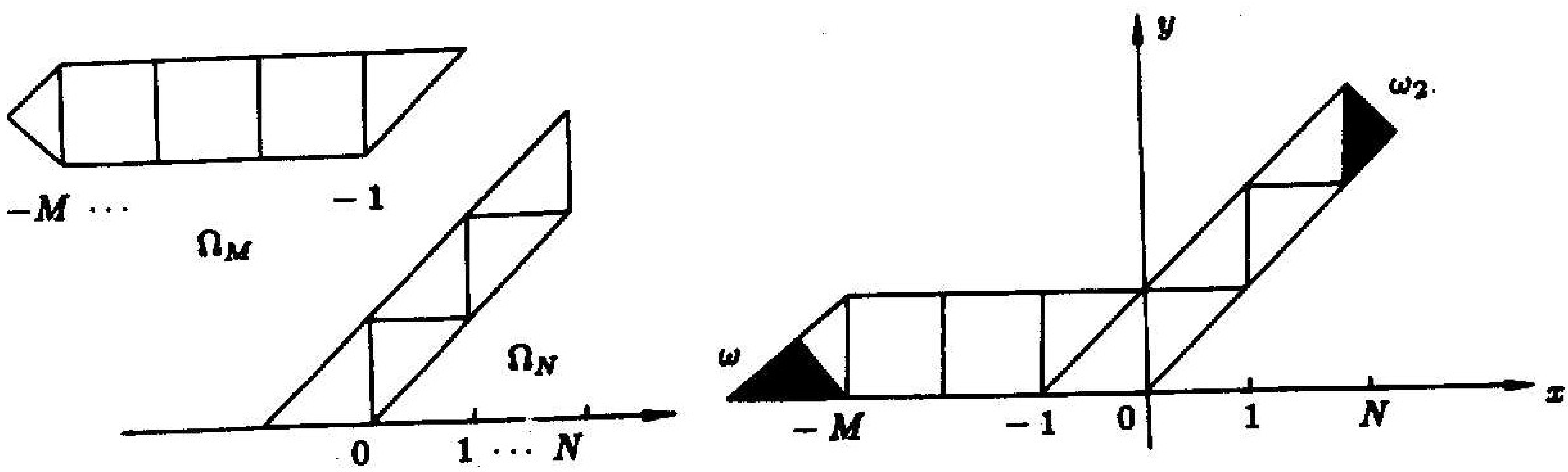

Fig. 7

Definition. *Spline spaces $S^{\mu}_{k}(\Omega_{MN})$ consist of such splines that when $f(x,y)\in S^{\mu}_{k}(\Omega_{MN})$, there is an $F(x,y)\in S^{\mu}_{k}(\Delta_2)$ so that*

$$F(x,y)=\begin{cases}0, & (x,y)\in\omega_1\cup\omega_2\cup\Big\{(x,y):y<0\Big\}\cup\Big\{(x,y):y-x<0\Big\},\\ f(x,y), & (x,y)\in\Omega_{MN}.\end{cases}$$

Lemma 4.5. *If $p_k(x,y) = x^k + s_{k-1}x^{k-1} + \cdots + s_1 x + s_0$, and $t_1, t_2, \cdots, t_N$ are all real numbers, then*

$$\sum_{i=0}^{N} t_i p_k(x-i) = 0$$

if and only if $t_i, i = 0, 1, \cdots, N$ satisfy

$$\begin{cases} t_0 + t_1 + t_2 + \cdots + t_N = 0, \\ t_1 + 2t_2 + \cdots + Nt_N = 0, \\ t_1 + 2^2 t_2 + \cdots N^2 t_N = 0, \\ \vdots \\ t_1 + 2^k t_2 + \cdots + N^k t_N = 0. \end{cases}$$

Proof. Because

$$\sum_{i=0}^{N} t_1 \sum_{j=0}^{K} s_j (x-i)^j = \sum_{g=0}^{K} x^g \sum_{j=g}^{K} s_j C_j^g \sum_{i=0}^{N} t_i (-i)^{j-g},$$

let $g = k$, and compare the coefficients of x^k on both sides; let $g = k-1$, and compare the coefficients of x^{k-1} on both sides; and so on. Then, we get the result.

Theorem 5. *There exist symmetric B-splines $B_{1,k}^{0,r}(x,y), r = 1,2$, in $S_{4k+1}^{3k}(\Delta_2)$, $k = 0, 1, \cdots$, where*

$$B_{1,0}^{0,r}(x,y) = B_1^{0,r}(x,y), B_{1,k}^{0,r}(x,y) = U^k(B_1^{0,r}(x,y)), r = 1,2; k = 0,1,\cdots.$$

Proof. We know from the above discussion that $B_{1,k}^{0,1}(x,y)$ and $B_{1,k}^{0,2}(x,y)$ are linearly independent. For any $f(x,y) \in S_{4k+1}^{3k}(\Omega_{MN})$, we have from lemma 4.4

$$\begin{aligned} f(x,y) = \ & q_M c_1^k(x+M,y) + \cdots + q_1 c_1^k(x+1,y) + a_0 c_1^k(x,y) + \cdots + a_N c_1^k(x-N, \\ & y-N) + C_{-M} c_2^k(x+M,y) + \cdots + C_{-1} c_2^k(x+1,y) + C_0 c_2^k(x,y) + \cdots \\ & + C_N c_2^k(x-N, y-N) \end{aligned}$$

when $(x,y) \in \Omega_{MN}$, where $c_i^k(x,y) = (I_1 I_2 I_3 I_4)^k c_i(x,y), i = 1, 2,$. So with aforesaid facts, we get

$$\begin{aligned} c_1^k(x,y) &= (x^k + \cdots) y^{3k+1} + d_0 (x^{k-1} + \cdots) y^{3k+2} + o(y^{3k+2}), \\ c_2^k(x,y) &= (x^{k-1} + \cdots) y^{3k+2} + o(y^{3k+2}), \end{aligned}$$

when $(x,y) \in \omega_2$; and

$$c_1^k(x,y) = (x-y)^{3k+2} r_{k-1}(x,y), c_2^k(x,y) = (x-y)^{3k+1} r_k(x,y),$$

when $(x,y) \in \omega_1$. There is no factor $(x-y)$ in $r_{k-1}(x,y)$ and $r_k(x,y)$.

Let $f(x,y) = 0, (x,y) \in \omega_r, r = 1,2$, which, by Lemma 4.5, is equivalent to the following systems of equations having non-zero solutions

$$\begin{cases} a_1 + 2^{k-1}a_2 + \cdots + N^{k-1}a_N = 0, \\ a_1 + 2^{k-2}a_2 + \cdots + N^{k-2}a_N = 0, \\ \vdots \\ a_1 + 2a_2 + \cdots + Na_N = 0, \\ a_0 + a_1 + a_2 + \cdots + a_N = 0, \\ q_M + \cdots + q_1 + a_0 = 0, \\ Mq_M + \cdots + q_1 = 0, \\ \vdots \\ M^k q_M + \cdots + q_1 = 0, \end{cases} \tag{I}$$

$$\begin{cases} C_1 + 2^k C_2 + \cdots + N^k C_N = 0, \\ C_1 + 2^{k-1}C_2 + \cdots + N^{k-1}C_N = 0, \\ \vdots \\ C_1 + 2C_2 + \cdots + NC_N = 0, \\ C_0 + C_1 + C_2 + \cdots + C_N = 0, \\ C_{-M} + \cdots + C_{-1} + C_0 = 0, \\ MC_{-M} + \cdots + C_{-1} = 0, \\ \vdots \\ M^k C_{-M} + \cdots + C_{-1} = 0. \end{cases} \tag{II}$$

If a non-zero solution exists, for the system of equations (I) the smallest permissible value of (M, N) is $(k+1, k)$; for (II) it is $(k, k+1)$. So the splines corresponding to $f(x,y)$ are $B_{1,k}^{0,1}(x,y)$ and $B_{1,k}^{0,2}(x,y) \in S_{4k+1}^{3k}(\Delta_2)$ with supports $(k+1, k, k+1, k, k+1, k, k+1, k)$ and $(k, k+1, k, k+1, k, k+1, k, k+1)$ respectively, $k = 0, 1, \cdots$. The proof is finished.

Similarly, we can prove the following conclusions.

Theorem 6. *There exist symmetric B-splines* $B_{4,k}^{2,r}(x,y), r = 1,2,3$, *in* $S_{4k+4}^{3k+2}(\Delta_2)$, $k = 0, 1, \cdots$, *where* $B_{4,0}^{2,r}(x,y) = B_4^{2,r}(x,y)$, $r = 1,2,3$, *and* $B_{4,k}^{2,r}(x,y) = U^k(B_4^{2,r}(x,y)), r = 1,2,3$.

Theorem 7. *There exists a symmetric B-spline* $B_{2,k}^1(x,y)$ *in* $S_{4k+2}^{3k+1}(\Delta_2), k = 0, 1, \cdots$, *where*

$$B_{2,0}^1(x,y) = B_2^1(x,y), B_{2,k}^1(x,y) = U^k(B_2^1(x,y)), k = 0, 1, \cdots.$$

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