ERROR EXPANSION FOR FEM AND SUPERCONVERGENCE UNDER NATURAL ASSUMPTION*

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In this paper, we derive the error expansion for finite element method under natural assumption and discuss the superconvergence as a special case of error expansion.

§1. Introduction

In a survey by Krizek and Neittaanmaki various types of superconvergence for FEM were discussed at some cases. But we can not expect the superconvergence for the displacement of linear finite element solution. At that case we can raise the convergence accuracy considerably using Richardson extrapolation. On extrapolation for FEM Chinese - German group has obtained a lot of results under some assumptions. See a survey by Rannacher. In this paper, we try to unit the discussion of superconvergence with the one of extrapolation. We deduce the error expansion under natural assumptions and discuss the superconvergence as a special case of error expansion. Consider the model problem

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \quad \text{on} \partial\Omega \tag{1.1}$$

where $\Omega \subset R^2$ is a convex domain and has smooth or piecewise smooth boundary. We use the finite element space over piecewise uniform or piecewise almost uniform triangulation to construct the finite element solution of problem (1.1). Let u^h and u^I be the linear finite element solution and interpolation of the true solution respectively. We derive the following expansion

$$u^h - u^I = \sum_{K=1}^{n-1} h^{2k} e_k^h + r^h \tag{1.2}$$

where the coefficients e_k^h are the finite element projections of the weak solution of problem (1.1) with defferent right hand side, and the remainder r^h satisfies

$$\|r^h\|_{1,\infty,\Omega_0} \le c \, |\ln h| h^{\alpha_n}$$

where α_n depends on the situation and Ω_0 is the subdomain of Ω which we shall discribe in details.

*Received November 30, 1987.

As a special case with $n = 1$ we have the superconvergence estimate

$$\|u^h - u^I\|_{1,\infty,\Omega_0} \le ch^{\alpha_1}|\ln h|.$$

Replacing ∇u^h by some kind of average gradient $\bar{\nabla} u^h$ we obtain

$$(\bar{\nabla} u^h - \nabla u)(p) = O(h^{\alpha_1}|\ln h|)$$

for any nodal point p in Ω_0.

With $n = 2$ in (1.2) we get

$$u^h - u^I = h^2 w^I + r^h \tag{1.3}$$

from which we have

$$\begin{aligned} &\frac{1}{3}(4u^{h/2} - u^h)(p) - u(p) = O(h^{\alpha_2}|\ln h|), \\ &\frac{1}{3}\bar{\nabla}(4u^{h/2} - u^h)(p) - \nabla u(p) = O(h^{\alpha_2}|\ln h|). \end{aligned} \tag{1.4}$$

For the proof of (1.3) we emphasize the situation when Ω is a polygonal domain and u is of usual smoothness. At that case $\alpha_2 = 2\bar{\beta} - \varepsilon$ where ε is any positive number and $\bar{\beta}$ depends on the interior angle of Ω. In the last section we prove (1.4) with $\alpha_2 = 4$ when Ω is a smooth domain.

§2. Error Expansion on a Convex Polygonal Domain

Let $\Omega \subset R^2$ be a convex polygonal domain with the corner points $\{0_j\}$. We consider the problem (1.1) and its finite element solution. Choosing an arbitrary point o in $\bar{\Omega}$ and linking o with each corner point o_j, we subdivide Ω into several macro-triangles $\{\Omega_j\}$ with edges $\Gamma_j = \overline{oo}_j$. Let $T_h = \{K\}$ be a regular triangulation of Ω. Suppose that the restriction T_j to each Ω_j of T_h is uniform, i.e. each side of each triangle in T_j is parallel to one of three fixed direction vector. Assume that $S_h \subset H_0^1$ is the standard piecewise linear finite element space and u^h is the finite element solution. In order to evaluate the error expansion we consider the integral

$$\begin{aligned} I(u,v) &= \int_\Omega \nabla(u^h - u^I)\nabla v dx = \int_\Omega \nabla(u - u^I)\nabla v dx \\ &= \sum_j \int_{\Omega_j} \nabla(u - u^I)\nabla v dx, \quad \forall v \in S^h. \end{aligned} \tag{2.1}$$

It suffices to expand the integral

$$\int_{\Omega_j} \nabla(u - u^I)\nabla v dx.$$