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A POSTERIORI ERROR ANALYSIS FOR MIXED FINITE ELEMENT SOLUTION OF THE TWO–DIMENSIONAL STATIONARY STOKES PROBLEM*

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Abstract

In this paper we present a posteriori error estimator in a suitable norm of mixed finite element solution for the two-dimensional stationary Stokes problem. The estimator is optimal in the sense that, up to multiplicative constants, the upper and lower bounds of the error are the same. The constants are independent of the mesh and the true solution of the problem.

§1. Introduction

The stationary Stokes problem arises from stationary flow of an incompressible viscous fluid with a small Renord's number. However, it is the basis of handling the complete Navier-Stokes equations. In this paper we consider solving the Stokes problem by the mixed finite element method. It is difficult to solve the Stokes problems with singularities, including, for example, corner singularities. The adaptive finite element method is, however, a class of effective method for solving the boundary value problems with singularities. A posteriori error estimator of finite element methods presents an appreciation for the computed results. It is the basis of the adaptive refinement mesh algorithm.

So far, the theorecal problems of a posteriori error analysis for finite element methods of the boundary value problems of one-dimensional elliptic equations have been solved by I. Babuska and others (see [9]–[12]). For two-dimensional problems there are also a large number of works by I. Babuska and his co-workers. They have presented a posteriori error indicator of the finite element method for the Dirichlet problem of Poisson equation ([1]–[5]). The indicator in [3], for example, is based on solving local Dirichlet problems in the patch of elements surrounding each vertex in the finite element mesh. And in [1], using conforming bilinear square elements, I. Babuska and A. Miller show that error indicators can be based on jumps in the normal derivative of the computed solution at interelement boundaries. Such schemes as a rule require less computation than the ones involving the solution of local Dirichlet problems. E Wei-nan and Huang Hong-ci extended some results of I. Babuska to the case of more general conforming elements (see [7] and [12]). In [6] R. E. Bank and A. Weiser present error indicators for the Neumann problem of an elliptic equation by solving a local Neumann problem in each finite element.

In this paper, a posteriori error estimator of mixed finite element solution for the two-dimensional stationary Stokes problem is presented in a suitable norm. The estimator is optimal in the sense that, up to multiplicative constants, the upper and lower bounds of the error are the same. The constants are independent of the mesh and the true solution of the Stokes problem. Moreover, they are not large in practice. The estimator is based on solving local Stokes problems in the patch of elements surrounding each vertex in the finite

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element mesh. In this scheme, the Dirichlet boundary conditions ensure well-posedness of the local Stokes problems. The estimator we obtain finally consists of the indicator at each element and those indicators which are based on the computation of the norm of the local residual of the Stokes equation and the jumps in the computed pressure solution and in the normal derivative of the computed velocity solution at interelement boundaries. Two numerical examples in this paper support the above theorecal results.

In Section 2, we shall introduce the results of the mathematical theory and the mixed finite element methods of the Stokes problem, explain the basic notation used in this paper and present a partition of the solution domain, and give an approximate computing method for the LBB constant of each element. In Section 3, the main theorems in the paper will be proved and a posteriori error estimator will be presented. Finally, two numerical examples in Section 4 support the results in Section 3.

§2. Preliminaries

Let Ω be a bounded connected domain in R^2 with a smooth boundary Γ. We consider the Dirichlet problem of Stokes equation

$$\begin{cases} -\nu\cdot\nabla^2\vec{u}+\vec{\text{grad}}\cdot p=\vec{f} & \text{in } \Omega, \\ \text{div}\cdot\vec{u}=g & \text{in } \Omega, \\ \vec{u}=0 & \text{on } \Gamma \end{cases} \tag{2.1}$$

where the velocity function $\vec{u}(x)$ and the pressure function $p(x)$ are unknown, ν is a viscous coefficient, and $\vec{f}(x)$ and $g(x)$ are the given functions. For simplicity, we assume that $\nu=1$.

In this paper, $H^m(\Omega)$ with m being an integer denotes the usual Sobolev space. $H_0^1(\Omega)$ denotes the space in which the functions are in $H^1(\Omega)$ and their traces are zero. The norm of $H_0^1(\Omega)$ is defined as

$$|u|_{1,\Omega}=\left\{\iint_\Omega[(\partial u/\partial x_1)^2+(\partial u/\partial x_2)^2]dx\right\}^{1/2}, \qquad u\in H_0^1(\Omega).$$

Correspondingly, the norm of $(H_0^1(\Omega))^2$ is defined as

$$|\vec{u}|_{1,\Omega}=[|u_1|_{1,\Omega}^2+|u|_{1,\Omega}^2]^{1/2}, \qquad \vec{u}=(u_1,u_2)\in(H_0^1(\Omega))^2.$$

Denote $V(H_0^1(\Omega))^2$ and $W=L_0^2(\Omega)=\{q\in L^2(\Omega);\ \iint_\Omega q(x)dx=0\}$.

The variational form of (2.1) is

$$\begin{cases} \text{Find } [\vec{u},p]\in V\times W, \text{ such that} & \\ \quad a(\vec{u},\vec{v})+b(\vec{v},p)=(\vec{f},\vec{v}) & \text{for all } \vec{v}\in V, \\ \quad b(\vec{u},q)=(-g,q) & \text{for all } q\in W \end{cases} \tag{2.2}$$

where

$$a(\vec{u},\vec{v})=\iint_\Omega(\vec{\text{grad}}u_1\cdot\vec{\text{grad}}\ v_1+\vec{\text{grad}}\ u_2\cdot\vec{\text{grad}}\ v_2)dx,$$

$$b(\vec{u},q)=-\iint_\Omega \text{div}\cdot\vec{u}(x)q(x)dx,$$

$$(\vec{f},\vec{v})=\iint_\Omega\vec{f}\cdot\vec{v}dx \qquad \text{and} \qquad (g,q)=\iint_\Omega g\cdot q dx.$$