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AN ANSWER TO THE CONJECTURE OF I. BABUŠKA AND T. JANIK*1)

Cheng Xiao-Liang
(*Hangzhou University, Hangzhou, China*)

Abstract

In this paper, we verify the conjecture in the recent paper of I. Babuška and T. Janik[1].

§1. Introduction

The p-version of the finite element method is a new method in which one fixes the mesh, possibly consisting of just one element, and increases the degree p of the piecewise polynomials in order to obtain the convergence of the approximate solution to the exact solution. There has been great progress and development on this subject since the first theoretical results were published in 1981[2]; see [3, 4] for some recent results. Recently, I. Babuška and T. Janik[1] discussed the p-version of the finite element method for the initial value problem of an ordinary differential equation containing a parameter, which is the first step to study the parabolic equation. The error estimates are uniform with respect to the parameter but they are not optimal. In this paper, the error estimates are obtained for the various parameters, which can verify the conjecture in [1].

Let $I = (-1,1), \bar{I} = [-1,1], X = L_2(I)$ be the usual space furnished with the norm

$$\|u\|_x = \left(\int_I u^2 dx\right)^{1/2}$$

Let

$$\overset{\circ}{C} = \{v \in C^\infty(\bar{I});\ v(1) = 0\}$$

where $C^\infty(\bar{I})$ is the usual space of functions with all continuous derivatives on $\bar{I}$. For any $\lambda > 0, v \in \overset{\circ}{C}$, we define

$$\|v\|_{Y_\lambda} = \| -\dot{v}/\lambda + \lambda v\|_x$$

where we denote $\dot{v} = dv/dt$. Let Y_λ be the completion of $\overset{\circ}{C}$ with respect to the norm $\|\cdot\|_{Y_\lambda}$.

On $X \times Y_\lambda, u \in X, v \in Y_\lambda$, we define the bilinear form

$$B_\lambda(u,v) = \int_I u(-\dot{v}/\lambda + \lambda v)dt. \tag{1.1}$$

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Further, let $f \in Y'_\lambda$ be a linear function on Y_λ. Consider the following problem (P_λ): Find $u_0 \in X$ such that

$$B_\lambda(u_0, v) = f(v), \quad \forall\, v \in Y_\lambda. \tag{1.2}$$

It has a unique solution.

Let $q \geq 1$ be an integer,

$$S_{q-1} = \{u \in X; u \text{ is a polynomial of degree } q-1\},$$

$$\overset{\circ}{S}_q = \{v \in Y_\lambda; v \text{ is a polynomial of degree } q\}.$$

Define the p-version of the finite element method for the problem (P_λ): Given $f \in Y'_\lambda$ and $q \geq 1$ an integer, find $u_q \in S_{q-1}$ such that

$$B_\lambda(u_q, v) = f(v), \quad \forall\, v \in \overset{\circ}{S}_q. \tag{1.3}$$

Theorem 1.1[1]. *There is a unique u_q satisfying (1.3). If $u_0 \in X$ is the exact solution of the problem (P_λ), then*

$$\|u_0 - u_q\|_x \leq (1 + 2q^{1/2}) \inf_{w \in S_{q-1}} \|u_0 - w\|_x.$$

Define

$$R_{q,\lambda}(u) = \|u - u_q\|_x / \inf_{w \in S_{q-1}} \|u - w\|_x. \tag{1.4}$$

Let $\varphi(x) < \infty, 1 < x < \infty$, be a nondecreasing function. The set $\nu(\varphi)$ will be called the set of φ-perfect solutions if the set $\nu(\varphi) \subset X$ such that

$$\sup_{u \in \nu(\varphi)} \sup_{q,\lambda} R_{q,\lambda}(u)\varphi^{-1}(q) < +\infty. \tag{1.5}$$

An especially important case is $\nu(\varphi)$ for $\varphi = 1$. In this case, the error estimates of the approximate problem (1.3) to (1.2) are optimal.

In [1], I. Babuška and T. Janik proposed a conjecture based on numerical experiments.

Conjecture A. Let $u_0 = e^{-\lambda^2(t+1)}$. Then $u_0 \in \nu(\varphi)$ with $\varphi(x) = 1$ or $\varphi(x) \doteq \log(x)$.

In this paper, we will prove theoretically that $R_{q,\lambda}(u_0) \leq 2.0$ for any λ and any integer $q \geq 1$. That is, $u_0(t) \in \nu(\varphi)$ with $\varphi(x) = 1$.

§2. The Answer to the Conjecture

If u_0 is the exact solution of equation (1.2) and

$$u_0(t) = \sum_{k=0}^{\infty} \beta_k \hat{P}_k(t), \quad u_0^{(q-1)}(t) = \sum_{k=0}^{q-1} \beta_k \hat{P}_k(t) \tag{2.1}$$

where $\hat{P}_k(t)$ $(k = 0, 1, 2, \cdots)$ are the normalized Legendre polynomials of degree k, then from (1.2), (1.3), we get

$$B_\lambda(u_q - u_0^{(q-1)}, v) = B_\lambda(u_0 - u_0^{(q-1)}, v), \quad \forall\, v \in \overset{\circ}{S}_q. \tag{2.2}$$

Let

$$z_q(t) = u_q(t) - u_0^{(q-1)}(t) = \sum_{k=0}^{q-1} \alpha_k \hat{P}_k(t). \tag{2.3}$$

It is not difficult to see that equation (2.2) reduces to the solution of a system of linear equations[1]

$$A\alpha = F \tag{2.4}$$

where $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_{q-1})^T, F = (0, 0, \cdots, 0, \lambda\beta_q/\sqrt{(2q-1)(2q+1)})^T$,

$$A = \begin{pmatrix} a_1 & b_1 & & & 0 \\ c_2 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b_{q-1} \\ 0 & & & c_q & a_q \end{pmatrix}$$

with

$$a_1 = (\lambda^2+1)/\sqrt{2}\lambda, \quad a_i = -1/\lambda, \quad i = 2, 3, \cdots, q,$$
$$b_1 = -\lambda/\sqrt{6}, \quad b_i = -\lambda/\sqrt{(2i-1)(2i+1)}, \quad i = 2, 3, \cdots, q-1,$$
$$c_i = -\lambda/\sqrt{(2i-1)(2i-3)}, \quad i = 2, 3, \cdots, q.$$

Suppose $\alpha_0 \geq 0$; otherwise we write $z_q = u_0^{(q-1)} - u_q$ in equation (2.2) and $-F$ in equation (2.4).

Lemma 2.1. *If* $\alpha_k^* = \dfrac{1}{\sqrt{2k+1}}\alpha_k$ $(k = 0, 1, 2, \cdots, q-1)$, *where* α_k *are defined in* (2.3), *then*

$$0 \leq \alpha_0^* \leq \alpha_1^* \leq \alpha_2^* \leq \cdots \leq \alpha_{q-1}^*.$$

Proof. From the linear equation (2.4), we have

$$\alpha_1^* = (1 + 1/\lambda^2)\alpha_0^* \geq \alpha_0^*,$$
$$\alpha_2^* = \alpha_0^* + 3/\lambda^2 \alpha_1^* = (\alpha_0^* + \frac{1}{\lambda^2}\alpha_0^*) + \frac{2}{\lambda^2}\alpha_0^* + \frac{3}{\lambda^4}\alpha_0^* \geq \alpha_1^*$$

and

$$\alpha_k = -a_k/b_k \alpha_{k-1} - c_k/b_k \alpha_{k-2}, \quad k = 2, 3, \cdots, q-1,$$

that is

$$\alpha_k^* = \alpha_{k-2}^* + \frac{2k-1}{\lambda^2}\alpha_{k-1}^*, \quad k = 2, 3, \cdots, q-1.$$

By deduction $(k = 2, 3, \cdots, q-2)$,

$$\alpha_{k+1}^* - \alpha_k^* = (\alpha_{k-1}^* - \alpha_{k-2}^*) + \left(\frac{2k+1}{\lambda^2}\alpha_k^* - \frac{2k-1}{\lambda^2}\alpha_{k-1}^*\right) \geq 0.$$

Lemma 2.2. *Let* $z_q(t)$ *be the function defined in* (2.3). *Then*

$$z_q(1) - (-1)^q z_q(-1) \geq z_q(1).$$

Proof. Since

$$z_q(1) = \sum_{K=0}^{q-1} \sqrt{\frac{2k+1}{2}}\alpha_k = \sum_{k=0}^{q-1} \frac{2k+1}{\sqrt{2}}\alpha_k^*,$$

$$z_q(-1) = \sum_{k=0}^{q-1} \frac{2k+1}{\sqrt{2}}(-1)^k\alpha_k^*,$$

then

$$z_q(1) - (-1)^q z_q(-1) = \begin{cases} \sum_{k=0}^{q-1} \frac{2k+1}{\sqrt{2}}\alpha_k^*(1-(-1)^k), & q = \text{even}, \\ \sum_{k=0}^{q-1} \frac{2k+1}{\sqrt{2}}\alpha_k^*(1+(-1)^k), & q = \text{odd}, \end{cases}$$

and Lemma 2.1 implies

$$\frac{2k+1}{\sqrt{2}}\alpha_k^* \geq \frac{2k-1}{\sqrt{2}}\alpha_{k-1}^*, \quad k = 1, 2, \cdots, q-1.$$

We can get the result.

Lemma 2.3. *If z_q is the function defined in* (2.3), *then*

$$\|z_q\|_x^2 + \frac{1}{2\lambda^2}[z_q(1) - (-1)^q z_q(-1)]^2 = -z_q(1)\sqrt{2/(2q+1)}\beta_q.$$

Proof. Let $v(t) = z_q(t) - z_q(1)P_q(t) \in \overset{\circ}{S}_q$, where $P_q(t)$ is a Legendre polynomial of degree q. Then from (2.2),

$$\begin{aligned} B_\lambda(z_q, v) &= \frac{1}{2\lambda}(z_q(-1)^2 - z_q(1)^2) + \frac{1}{\lambda}z_q(1)(z_q(1) - (-1)^q z_q(-1)) + \lambda\int_{-1}^{1} z_q^2\,dt \\ &= \lambda\|z_q\|_x^2 + \frac{1}{2\lambda}(z_q(1) - (-1)^q z_q(-1))^2 \end{aligned}$$

and

$$B_\lambda(u_0 - u_0^{(q-1)}, v) = \int_{-1}^{1} \Big(\sum_{k=q}^{\infty} \beta_k \hat{P}_k(t)\Big) \cdot \Big(-\frac{\dot{v}}{\lambda} + \lambda v\Big)dt = -z_q(1)\sqrt{2/(2q+1)}\lambda\beta_q.$$

Equation (2.2) implies the result immediately.

Theorem 2.1. *Let u_0, u_q be the solution of equations* (1.2), (1.3), *respectively. Then*

$$R_{q,\lambda}(u_0) \leq 1 + \lambda/\sqrt{2q+1}|\beta_q| \Big/ \Big(\sum_{k=q}^{\infty} \beta_k^2\Big)^{1/2}. \tag{2.5}$$

Proof. By the triangle inequality,

$$|z_q(1)\sqrt{2/(2q+1)}\beta_q| \leq \frac{1}{2\lambda^2}z_q(1)^2 + \frac{\lambda^2}{2q+1}\beta_q^2.$$

By Lemmas 2.2 and 2.3,

$$\|z_q\|_x \le \lambda/\sqrt{2q+1}|\beta_q|.$$

Then we can obtain the result easily.

Especially, $u_0(t) = e^{-\lambda^2(t+1)}$ is a set of funcitons as the numerical examples in [1]. It shows that for any λ and any integer q in a practical range (numerical results),

$$R_{q,\lambda}(u_0) \le 1.5.$$

Next, we will prove theoretically that $R_{q,\lambda}(u_0) \le 2.0$ for any λ and any integer $q \ge 1$.

Lemma 2.4. *For* $u_0(t) = e^{-\lambda^2(t+1)}$, *if* $u_0(t) = \sum_{k=0}^{\infty} \beta_k \hat{P}_k(t)$, *then*

$$\sum_{k=n}^{\infty} \beta_k^2 = -\lambda^2 \beta_{n-1}\beta_n/\sqrt{(2n-1)(2n+1)}, \quad n = 1, 2, \cdots.$$

Proof. Since

$$\begin{aligned}\beta_k &= \int_{-1}^{1} e^{-\lambda^2(t+1)} \hat{P}_k(t)dt = \sqrt{\frac{2k+1}{2}} \int_{-1}^{1} e^{-\lambda^2(t+1)} \frac{1}{2k+1}(P'_{k+1} - P'_{k-1})dt \\ &= \lambda^2(\beta_{k+1}/\sqrt{(2k+1)(2k+3)} - \beta_{k-1}/\sqrt{(2k-1)(2k+1)},\end{aligned}$$

then

$$\beta_k^2 = \lambda^2(\beta_k\beta_{k+1}/\sqrt{(2k+1)(2k+3)} - \beta_{k-1}\beta_k/\sqrt{(2k-1)(2k+1)}).$$

For $k \to \infty, \beta_k \to 0$,

$$\sum_{k=n}^{\infty} \beta_k^2 = -\lambda^2 \beta_{n-1}\beta_n/\sqrt{(2n-1)(2n+1)}.$$

Lemma 2.5. *For* $u_0(t) = e^{-\lambda^2(t+1)} = \sum_{k=0}^{\infty} \beta_k \hat{P}_k(t)$, *let* $\beta_k^* = (-1)^k \frac{\beta_k}{\sqrt{2k+1}}$. *Then*

$$\beta_0^* > \beta_1^* > \beta_2^* > \cdots > \beta_n^* > \beta_{n+1}^* > \cdots > 0.$$

Proof. The result $\beta_k^* > 0$ $(k = 0, 1, 2, \cdots)$ can be proved easily from Lemma 2.4 ($\beta_0^* > 0$), and

$$\begin{aligned}\beta_{n+1}^* - \beta_n^* &= \frac{1}{\sqrt{2}}(-1)^{n+1} \int_{-1}^{1} e^{-\lambda^2(t+1)}(P_{n+1} + P_n)dt \\ &= \frac{1}{\sqrt{2}}(-1)^{n+1} \frac{1}{n!2^n} \int_{-1}^{1} e^{-\lambda^2(t+1)} \frac{d^n}{dt^n}[(1+t)^{n+1}(t-1)^n]dt \\ &= -\frac{\lambda^{2n}}{\sqrt{2}\, n!2^n} \int_{-1}^{1} e^{-\lambda^2(t+1)}(1+t)^{n+1}(1-t)^n dt < 0\end{aligned}$$

This completes the proof.

From Lemma 2.4 and Lemma 2.5, we can get the inequality

$$|\beta_q| \le \frac{\sqrt{2q+1}}{\lambda}\Big(\sum_{k=q}^{\infty} \beta_k^2\Big)^{1/2} \tag{2.6}$$

and (2.5)–(2.6) implies

$$R_{q,\lambda}(u_0) \leq 2.0$$

for any $\lambda, q \geq 1$.

Now we verify the conjecture which plays an important role in [1].

Theorem 2.2. *For* $u_0(t) = e^{-\lambda^2(t+1)}$, $u_0 \in \nu(\varphi), \varphi = 1$.

References

[1] I. Babuška and T. Janik, The *h-p* version of the finite element method for parabolic equations: Part I. The *p*-version in time, *Numer. Methods for Partial Differ. Equat.*, **5** (1989), 363–399.

[2] I. Babuška, B.A. Szaba and I.N. Katz, The *p*-version of the finite element method. *SIAM J. Numer. Anal.*, **18** (1981), 515–545.

[3] W. Gui and I. Babuška, The *h, p* and *h-p* versions of the finite element method for one dimensional problem, *Numer. Math.*, **49** (1986), 577–612.

[4] I. Babuška and B.Q. Guo, The *h-p* version of the finite element method for problems with nonhomogeneous essential boundary condition, *Comput. Meth. in Mech. and Enging.*, **74** (1989), 1–28.