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SOLUTION OF AN OVERDETERMINED SYSTEM OF LINEAR EQUATIONS IN L_2, L_∞, L_p NORM USING L.S. TECHNIQUES *

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Abstract

A lot of curve fitting problems of experiment data lead to solution of an overdetermined system of linear equations. But it is not clear prior to that whether the data are exact or contaminated with errors of an unknown nature. Consequently we need to use not only L_2-solution of the system but also L_∞-or L_p-solution.

In this paper, we propose a universal algorithm called the Directional Perturbation Least Squares (DPLS) Algorithm, which can give optimal solutions of an overdetermined system of linear equations in L_2, L_∞, L_p $(1 \le p < 2)$ norms using only L.S. techniques (in §2). Theoretical principle of the algorithm is given in §3. Two examples are given in the end.

§1. Introduction

Let us assume that

$$X_i = (x_{i_1}, \cdots, x_{i_k})^T \in R^k, \quad Y_i = f(X_i), \quad i = 1, 2, \cdots, m$$

are the given original data. We wish to find a fitting formula linearly depending on some parameters $b = (b_1, \cdots, b_n)^T$

$$F(b, X) = \sum_{j=1}^{n} b_j \phi_j(X) = [\Phi(X)]^T b \tag{1.1}$$

such that

$$\|r(b)\|_p = \|Y - \tilde{F}(b, X)\|_p = \min_{b \in R^n} \tag{1.2}$$

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where

$$Y = (Y_1, \cdots, Y_m)^T \in R^m, \quad b = (b_1, \cdots, b_n)^T \in R^n,$$
$$\tilde{F}(b, X) = (F(b, X_1), \cdots, F(b, X_m))^T \in R^m,$$
$$r(b) = Y - \tilde{F}(b, X),$$

$\left\{\phi_i(X)\right\}_1^n$: n linearly independent given functions on the discrete set $\{X_i\}_{i=1}^m$,

$\|\cdot\|_p$: p-norm of a vector $\|r\|_p = (\sum_{i=1}^m |r_i|^p)^{1/p}$, $1 \le p \le \infty$.

Problem (1.2) can be converted to the equivalent problem of solving an overdetermined linear system

$$AX = Y \tag{1.3}$$

by minimizing the p-norm (commonly p=1,2,∞) of the residual $r(X) = Y - AX$, i.e, to find a vector $b \in R^n$ such that

$$\|Y - Ab\|_p = \min_{X \in R^n} \|Y - AX\|_p \tag{1.4}$$

where

$$A = \begin{bmatrix} \phi_1(X_1) & \cdots & \phi_n(X_1) \\ \cdots & \cdots & \cdots \\ \phi_1(X_m) & \cdots & \phi_n(X_m) \end{bmatrix} \in R^{m \times n}.$$

Different norms normally lead to different solution b . If $p = 2$, it is well-known that

$$b = A^+ Y$$

where A^+ is the pseudo-inverse of A obtained by any method, for instance the S.V.D. method. When rank $(A) \equiv r_A = n$, b is the unique solution, and when $r_A < n$, b is a uniquely definite minimal 2-norm solution; but if $p = 1$ or $p = \infty$, the problems are complicated because the function $f(X) = \|Y - AX\|_p$ is not differentiable for those values of p. Indeed , there are several good techniques available for 1-norm and ∞-norm minimization [2]. But those techniques are more complex and very different from the L.S. Technique. As J.R. Rice ([1] p.371) indicates, "the theories behind both L_1 and Chebyshev (L_∞) approximation are too complex and difficult to present here. The numerical methods that have been developed from these theories are more complex and require more computation than the methods for least squares, but one could hardly hope for them to be as simple and easy as for least squares. On the other hand, these methods are reasonably efficient, and it is practical to use them in a wide variety of applications. If the problem at hand is such that least squares might not be entirely appropriate (e.g, one may really want to minimize the