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A CLASS OF THREE-LEVEL EXPLICIT DIFFERENCE SCHEMES*

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Abstract

A class of three-level six-point explicit schemes L_3 with two parameters s, p and accuracy $O(\tau h + h^2)$ for a dispersion equation $U_t = aU_{xxx}$ is established. The stability condition $|R| \le 1.35756176$ ($s = 3/2$, $p = 1.184153684$) for L_3 is better than $|R| < 1.1851$ in [1] and seems to be the best for schemes of the same type.

Any three-level explicit difference scheme for a dispersion equation $U_t = aU_{xxx}$ can be written in the form

$$U_{m+s}^{n+1} = \sum_{j=i}^{k} b_j U_{m+j}^n + \sum_j c_j U_{m+j}^{n-1} \qquad (*)$$

$(*)$ is referred to as an "N-point" scheme, where $N = k - i + 1$ $(k > i)$. A class of six-point schemes L_3 containing two paremeters s and p is established in this paper. Their local truncation errors are $O(\tau h + h^2)$. The optimal stability condition obtained is $|R| \le 1.35756176$ ($R = a\tau/h^3$, $\tau = \Delta t$, $h = \Delta x$), which corresponds to $s = 3/2$, $p = 1.184153684$. This stability condition is an improvement on the result $|R| \le 1.1851$ in [1] and seems to be the best condition for six-point schemes of the same type at present.

The schemes given in this note are as follows:

$$L_3: \quad U_{m+s}^{n-d+1} - U_{m+s}^{n-d} + U_{m-s}^{n+d} - U_{m-s}^{n+d-1} = 2R\sum_{j=0}^{2} C_j \left(U_{m-j+1/2}^n - U_{m-j-1/2}^n\right) \qquad (1)$$

where $a > 0$ if $d = 0$, $a < 0$ if $d = 1$, $s = 1/2, 3/2$; $C_0 = 2.5p - 3$, $C_1 = -1.25p + 1$, $C_2 = 0.25p$.

For $s = 1/2$ and $p = 1$, the schemes L_3 become H_3 in [1].

Now we analyse the stability of schemes L_3 by the Fourier method. For definiteness, put $s = 3/2$, $d = 0$ $(a > 0)$. Let

$$U_m^n = \lambda^n e^{iqx_m}, \quad i^2 = -1, \ x_m = mh, \ q\text{–real number}. \qquad (2)$$

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Substituting (2) into (1), we obtain the characteristic equation of schemes L_3 (see, [3]):

$$e^{iQ}\lambda^2 - 2F(Q)i\lambda - e^{-iQ} = 0, \quad Q = qh/2, \tag{3}$$

$$F(Q) = 2R\sum_{j=0}^{2} C_j \sin(2j+1)Q + \sin(3Q)$$

$$= Rf(y,p) + g(y), \quad y = \sin Q, \quad 0 \le Q \le \pi/2,$$

$$f(y,p) = 8y^3(py^2 - 1) = 8y^3(y-c)(y+c)/c^2, \quad p > 1, pc^2 = 1, \tag{4}$$

$$g(y) = 3y - 4y^3, \quad 0 \le y \le 1. \tag{5}$$

From equation (3) and [2,4], it follows that the stability condition of L_3 is $|Rf(y,p)+g(y)| < 1$ or

$$|R| < \sup_p \inf_{0<y\le 1} G(y,p), \tag{6}$$

$$G(y,p) = \begin{cases} -(1+g(y))/f(y,p), & 0 < y < c, \\ (1-g(y))/f(y,p), & 0 < y \le 1. \end{cases} \tag{7}$$

In order to find $\inf G(y,p)$ in the interval $0 < y \le 1$ for any fixed $p > 1$, the properities of $G(y,p)$ are discussed in the following.

1. In the case $0 < y < c$, we have

$$\partial G/\partial y = 8y^2(2y+1)W(y,p)/f(y,p)^2,$$

$$W(y,p) = py^2(-4y^2+2y+5) - 3, \tag{8}$$

$$W(0,p) = -3, \quad W(c,p) = 2(2c+1)(1-c) > 0,$$

$$\partial W/\partial y = py(16y+10)(1-y) > 0,$$

$$\partial G/\partial p = 8y^5(1+g(y))/f(y,p)^2 > 0. \tag{9}$$

From the above equalities, we see that there exists a unique zero point z of $W(y,p)$ or $\partial G/\partial y$, and z is also a unique minimum point of $G(y,p)$ for $0 < y < c$ because $G(0,p)$, $G(c,p) \to \infty$, and $G(y,p)$ is obviously a monotonically increasing function of p for any $y \in (0,c)$ (see, (9)). Thus, for arbitrary numbers p_1, p_2, c_1, c_2 satisfying $p_1 > p_2$ and $p_1c_1^2 = p_2c_2^2 = 1$, we have $c_1 < c_2$, and

$$\inf_{0<y<c_1} G(y,p_1) = G(z_1,p_1) > G(z_1,p_2) \ge \inf_{0<y<c_2} G(y,p_2) = G(z_2,p_2). \tag{10}$$

This verifies that $\inf G(y,p)$ $(0 < y < c)$ is a monotonically increasing function of $p > 1$.

2. In the case of $c < y \le 1$, we have

$$\partial G/\partial y = 8y^2H(y,p)/f(y,p)^2,$$

$$H(y,p) = pL(y) - 6y + 3, \quad 1 < p < p_0, \ c_0 < y \le 1, \tag{11}$$

$$L(y) = -8y^5 + 12y^3 - 5y^2 = 4y^2(2y-1)(y_1 - y)(y - y_2),$$

$$y_1 = (\sqrt{21}-1)/4, \quad y_2 = -(\sqrt{21}+1)/4,$$

$$p_0 = p(z) = \min_{0.5<y<y_1} (6y-3)/L(y) = 3.510857143, \quad z = 5/8,$$

$$c_0 = c(z) = 0.533695352 > 1/2, \quad pc^2 = 1,$$

$$\partial G/\partial p = -8y^5(1-g(y))/f(y,p)^2 < 0, \tag{12}$$

$$\lim G(y,p) = 0, \quad y \to 1/2 \text{ for any } p > 1. \tag{13}$$

From (11)–(13), with no loss of generality, we may assume that $c \geq c_0$ or $p \leq p_0$. Under this assumption, there are $H(y,p) < 0$, $\partial G/\partial y < 0$ and so $G(y,p)$ is a monotonically decreasing function of y for $c < y \leq 1$, Consequently,

$$\inf_{0<y\leq 1} G(y,p) = G(1,p) = 1/(4p-4), \quad 1 < p \leq p_0. \tag{14}$$

From the above discussion, we see that if we choose a parameter p such that

$$G(1,p) = \inf_{0<y<c} G(y,p) = G(z,p), \tag{15}$$

then stability condition (6) will be the best.

In order to calculate the minimum point z, solve for p from the equation $W(y,p) = 0$ (see (8)):

$$p = 3/(5y^2 + 2y^3 - 4y^4), \quad y = z. \tag{16}$$

Substituting (16) into (15), we obtain

$$(y+1)^2(y-1)(2y+1)A(y) = 0,$$

$$A(y) = 4y^3 - 8y^2 + 3 = 0, \quad 0 < y < c. \tag{17}$$

From $A(-1) = -9$, $A(0) = 3$, $A(1) = -1$ and $A(2) = 3$, it follows that there exists a unique zero point z of $A(y)$ in the interval $(0,1)$. The following numerical results are calculated by the formulas (17), (16), (14), (4), (5) and (6).

$$z = 0.7859966342, \qquad A(z) = 2E-10,$$

$$p^* = p(z) = 1.184153684, \qquad c^* = c(z) = 0.918958638, \quad pc^2 = 1, \tag{18}$$

$$G(1,p) = 1.357561766, \qquad G(z,p) = 1.357561765,$$

$$|R| \leq 1.35756176. \tag{19}$$

Up to now, we obtained the stability condition (18)–(19) of L_3 with $s = 3/2$, $d = 0$. Using the same method, it is easy to prove that (18)–(19) is also the stability condition of L_3 with $s = 3/2$, $d = 1$.

By repeating the above procedure, we can show that the best stability condition of L_3 in the case of $s = 1/2$ is $p = 1$, $|R| \leq 1.1851$, that is the result of [1].

Example. Use a difference method to solve the initial-boundary value problem for the equation $U_t = aU_{xxx}$ $(a = 1, -1)$; its solution is

$$U(x,t) = \cos(x - at), \quad 0 \leq x \leq 1, \quad t > 0. \tag{20}$$

The numerical computation was carried out on a pocket computer PC–1500. At the mesh point, U_m^n denotes the approximate solution computed by difference scheme L_3

($s = 3/2$, $h = 0.01$, $p = p^*$, $M = 1/h$), in the order of $m = 0, 1, 2, \ldots, M$ for $a > 0$ and $m = M, M-1, \ldots, 1, 0$ for $a < 0$. The necessary boundary values and initial values in computation were calculated by (20). Some data of errors $U_m^n - U(X_m, T_n)$ are listed in Tables 1–4. From the tables, we see that the data in Tables 1–2 are of numerical stability and in Tables 3–4 are of numerical unstability. These results all coincide with the stability condition (18)–(19).

Table 1. $a = 1$, $R = 1.357561$

m n	2	202	402	602
31	$4.3E-11$	$4.80E-09$	$1.09E-08$	$1.73E-08$
51	$9.5E-11$	$7.83E-09$	$1.71E-08$	$2.71E-08$
71	$1.94E-10$	$1.18E-08$	$2.35E-08$	$3.78E-08$

Table 2. $a = -1$, $R = -1.357561$

m n	2	202	402	602
25	$1.9E-11$	$-4.66E-09$	$-6.92E-09$	$-1.82E-08$
50	$-9.4E-11$	$-1.03E-08$	$-1.72E-08$	$-2.07E-08$
75	$-1.42E-10$	$-1.63E-08$	$-3.40E-08$	$-4.97E-08$

Table 3. $a = 1$, $R = 1.36$

m n	2	202	402	602
31	$5.5E-11$	$-4.53E-07$	$-5.60E-03$	-67.79
51	$1.11E-10$	$-1.08E-06$	$-9.98E-03$	-125.38
71	$2.25E-10$	$-9.47E-07$	$-1.18E-02$	-150.12

Table 4. $a = -1$, $R = -1.36$

m n	2	202	402	602
25	$2E-11$	$-5.35E-07$	$-5.81E-03$	-75.49
50	$-6E-11$	$6.52E-07$	$5.17E-03$	63.89
75	$-2.23E-10$	$-8.89E-08$	$-2.11E-03$	-24.96

References

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