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THE APPLICATION OF INTEGRAL EQUATIONS TO THE NUMERICAL SOLUTION OF NONLINEAR SINGULAR PERTURBATION PROBLEMS*

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Abstract

The nonlinear singular perturbation problem is solved numerically on non-equidistant meshes which are dense in the boundary layers. The method presented is based on the numerical solution of integral equations [1]. The fourth order uniform accuracy of the scheme is proved. A numerical experiment demonstrates the effectiveness of the method.

1. A Continuous Problem

We consider the following singularly perturbed boundary value problem:

$$\varepsilon^2 \frac{d^2u}{dx^2} = f(x,y), \quad x \in I = [0,1], \quad u(0) = u(1) = 0, \tag{1}$$

where ε is a small positive parameter. We assume that

$$\begin{aligned} &f \in C^4(I \times R), \quad g(x) \le f_u(x,u) \le G(x), \quad (x,u) \in I \times R, \\ &\min\{5g(x) - 2G(x) : x \in I\} > 0, \ 0 < r^2 < g(x), \ |g'(x)| \le L, \\ &|G'(x)| \le L, \ x \in I. \end{aligned} \tag{2}$$

According to [2], we can prove

Lemma 1. *Suppose that condition* (2) *is satisfied. There exists a unique solution* $u \in C^6(I)$ *to problem* (1), *and the following representation holds:* $u(x) = u_0(x) + V_0(x) + V_1(x)$, *where* $V_0(x) = M \exp\left(-\gamma \frac{x}{\varepsilon}\right)$, $V_1(x) = M \exp\left(-\gamma \frac{(1-x)}{\varepsilon}\right)$, *and* $|u_0^{(i)}(x)| \le M$, $i = 0, 1, \cdots, 6$, $x \in I$ *(Throughout the paper M denotes any constant independent of ε).*

The proof of the following lemma is based on the monotonicity of (1), and can be found in [3,4].

Lemma 2. *Let* (2) *be satisfied. Then, for the solution* $u \in C^6(I)$ *to problem* (1) *there holds, for* $i = 0, 1, \cdots, 6$,

$$|u^{(i)}(x)| \le \begin{cases} M\left(1 + \varepsilon^{-i} \exp\left(\gamma \frac{x}{\varepsilon}\right)\right), & 0 \le x \le \frac{1}{2}, \\ M\left(1 + \varepsilon^{-i} \exp\left(\gamma \frac{(1-x)}{\varepsilon}\right)\right), & \frac{1}{2} \le x \le 1. \end{cases}$$

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2. An Equivalent Integral Equation Problem

Introduce the non-equidistant mesh $I_h = \{x_i\}$, $0 = x_0 < x_1 < \ldots < x_n = 1$, $h_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, n$. In the subinterval $[x_{i-1}, x_{i+1}]$, we consider

$$\varepsilon^2 \frac{d^2u}{dx^2} = f(x,u), \quad u(x_{i-1}) = A, \quad \frac{du(x_{i-1})}{dx} = B.$$

Integrating once, we have

$$\varepsilon^2 u' = \varepsilon^2 B + \int_{x_{i-1}}^{x} f(t, u(t))dt;$$

hence

$$\begin{aligned}\varepsilon^2 u(x) &= \varepsilon^2 A + \varepsilon^2 B(x - x_{i-1}) + \int_{x_{i-1}}^{x} dt \int_{x_{i-1}}^{t} f(s, u(s))ds \\ &= \varepsilon^2 A + \varepsilon^2 B(x - x_{i-1}) + \int_{x_{i-1}}^{x} (x-t) f(t, u(t))dt.\end{aligned}$$

Now if we require that $u(x_{i-1}) = u_{i-1}$, $u(x_{i+1}) = u_{i+1}$, we have $u_{i-1} = u(x_{i-1}) = A$, $u_{i+1} = u(x_{i+1}) \Rightarrow A + B(x_{i+1} - x_{i-1}) + \frac{1}{\varepsilon^2}\int_{x_{i-1}}^{x_{i+1}} (x_{i+1} - t) f(t, u(t))dt$. Solving for A and B, we find that $u(x)$ satisfies the integral equation

$$\begin{aligned}\varepsilon^2 u(x) = \varepsilon^2 u_{i-1} + \varepsilon^2 (x - x_{i-1}) \frac{u_{i+1} - u_{i-1}}{x_{i+1} - x_{i-1}} + \int_{x_{i-1}}^{x} (x-t) f(t, u(t))dt \\ - \frac{x - x_{i-1}}{x_{i+1} - x_{i-1}} \int_{x_{i-1}}^{x_{i+1}} (x_{i+1} - t) f(t, u(t))dt.\end{aligned}$$

which can be rewritten in the form

$$\varepsilon^2 u(x) = \varepsilon^2 \frac{x_{i+1} - x}{x_{i+1} - x_{i-1}} u(x_{i-1}) + \varepsilon^2 \frac{x - x_{i-1}}{x_{i+1} - x_{i-1}} u(x_{i+1}) - \int_{x_{i-1}}^{x_{i+1}} K(x,t) f(t, u(t))dt, \quad (3)$$

where

$$K(x,t) = \begin{cases} (t - x_{i-1}) \dfrac{x_{i+1} - x}{x_{i+1} - x_{i-1}}, & x_{i-1} \le t \le x, \\ (x - x_{i-1}) \dfrac{x_{i+1} - t}{x_{i+1} - x_{i-1}}, & x \le t \le x_{i+1}. \end{cases}$$

The kernel is then Green's function for the problem, in the notation of classical mechanics.

3. Discretization

Letting $x = x_i$ in (3), we obtain an exact three-point difference scheme:

$$\varepsilon^2 u(x_i) + \varepsilon^2 \frac{x_{i+1} - x_i}{x_{i+1} - x_{i-1}} u(x_{i-1}) + \varepsilon^2 \frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}} u(x_{i+1}) - \int_{x_{i-1}}^{x_{i+1}} K(x_i, t) f(t, u(t))dt.$$

We denote

$$Nu(x_i) \equiv \varepsilon^2 [A_{i-1} u(x_{i-1}) + A_i u(x_i) + A_{i+1} u(x_{i+1})] + \int_{x_{i-1}}^{x_{i+1}} K(x_i, t) f(t, u(t))dt = 0, \quad (4)$$