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CHARACTERIZATION AND CONSTRUCTION OF LINEAR SYMPLECTIC RK-METHODS*1)

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Abstract

A characterization of linear symplectic Runge-Kutta methods, which is based on the W-transformation of Hairer and Wanner, is presented. Using this characterization three classes of high order linear symplectic Runge-Kutta methods are constructed. They include and extend known classes of high order linear symplectic Runge-Kutta methods.

1. Introduction

The present paper is a continuation of [13] where characterizations of symmetric and symplectic Runge-Kutta methods, based on the W-transformation of Hairer and Wanner, were presented. Using the characterization of symplectic Runge-Kutta methods, two classes of high order symplectic Runge-Kutta methods were constructed there. In the present paper we shall discuss a characterization of linear symplectic Runge-Kutta methods, which is based on the W-transformation of Hairer and Wanner. Up to now only symmetric one-step methods are found to be linear symplectic in the class of high order one-step methods. We shall construct three classes of high order linear symplectic Runge-Kutta methods, which include and extend known classes of high order linear symplectic Runge-Kutta methods. In this paper we shall continue to use the notation in [13].

It is well known that the stability function of implicit Runge-Kutta methods may be written as

$$R(z)=\frac{\det(I-zA+zeb^T)}{\det(I-zA)}, \tag{1.1}$$

or

$$R(z)=1+zb^T(I-zA)^{-1}e. \tag{1.1'}$$

In [6] Feng has proved that the necessary and sufficient condition of linear symplectic schemes is

$$R(z)R(-z)=1. \tag{1.2}$$

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From [6] we can easily obtain that symmetric Runge-Kutta methods are linear symplectic. In [13] we have proved

Theorem 1.1. *An s-stage RK-method with distinct nodes c_i and $b_i \neq 0$ satisfying $B(p), C(\eta)$ and $D(\zeta)$ with $p \geq s+\zeta$ is symmetric if and only if*

a) *$\widetilde{P}c = e - c$ for the permutation matrix $\widetilde{P}$,*

b) *the transformation matrix X of the method takes the following form*

$$X = W^T BAW = \begin{pmatrix} 1/2 & -\xi_1 & & & \\ \xi_1 & \ddots & \ddots & & \\ & \ddots & 0 & -\xi_\nu & \\ & & \xi_\nu & \multicolumn{2}{|c}{R_\nu} \end{pmatrix}, \quad \text{where } \nu = \min(\eta, \zeta) \tag{1.3}$$

having the residue matrix R_ν whose (k,l)-th element $r_{kl} = 0$ if $k+l$ is even, where the (i,j)-th element of permutation matrix $\widetilde{P}$ is the Kronecker $\delta_{i,s+1-j}$.

In [9] Hairer and Wanner have found that the stability function in terms of the transformed RK-matrix $X = W^{-1}AW$ can be expressed as

$$R(z) = \frac{\det(I - zX + ze_1 e_1^T)}{\det(I - zX)}, \tag{1.4}$$

or

$$R(z) = 1 + ze_1^T(I - zX)^{-1}e_1, \tag{1.4$'$}$$

that is, $R(z)$ depends only on X and not on the underlying quadrature formula. Thus, Theorem 1.1 condition b) should be a characterization of linear symplectic Runge-Kutta methods, which is based on the W-transformation of Hairer and Wanner. Note that there exists a difference between the definition of transformation matrices

$$X^* = W^{-1}AW$$

and

$$X = W^T BAW,$$

but it is not essential. The two matrices are related by

$$X = W^T BWX^*. \tag{1.5}$$

In general, X and X^* should possess identical properties. We can obtain at least the following result :

Lemma 1.2. *For the transformation matrices specified by $X^* = W^{-1}AW$ and $X = W^T BAW$ respectively, if one of $(X - \frac{1}{2}e_1e_1^T)$ and $(X^* - \frac{1}{2}e_1e_1^T)$ satisfies condition b) in Theorem 1.1, then the other does also if and only if the (k,l)-th element of matrix $W^T BW$ vanishes if $k+l$ is odd.*

Proof. Let

$$\widetilde{I} = \begin{pmatrix} 1 & 0 & \cdots & \\ 0 & -1 & 0 & \ddots \\ & \ddots & \ddots & 0 \\ & & & (-1)^{s-1} \end{pmatrix}$$

By the assumption and $X = W^T BWX^*$ we have

$$\tilde{I}\left(X - \frac{1}{2}e_1e_1^T\right)\tilde{I}^T = -\left(X - \frac{1}{2}e_1e_1^T\right) = -W^T BW\left(X^* - \frac{1}{2}e_1e_1^T\right).$$

On the other hand, there is

$$\begin{aligned}\tilde{I}\left(X - \tfrac{1}{2}e_1e_1^T\right)\tilde{I}^T &= \tilde{I}W^T BW\left(X^* - \tfrac{1}{2}e_1e_1^T\right)\tilde{I}^T \\ &= \tilde{I}W^T BW\tilde{I}^T\tilde{I}\left(X^* - \tfrac{1}{2}e_1e_1^T\right)\tilde{I}^T \\ &= W^T BW\tilde{I}\left(X^* - \tfrac{1}{2}e_1e_1^T\right)\tilde{I}^T.\end{aligned}$$

Thus we obtain

$$\tilde{I}\left(X^* - \frac{1}{2}e_1e_1^T\right)\tilde{I}^T + \left(X^* - \frac{1}{2}e_1e_1^T\right) = 0.$$

The reverse is also obvious.

In Section 2, we shall give a characterization of linear symplectic Runge-Kutta methods, which is based on the W-transformation of Hairer and Wanner and other results. Using these results the existing linear symplectic Runge-Kutta methods (including block implicit one-step methods and composite multistep methods) are verified. In Section 3, we construct three classes of linear symplectic Runge-Kutta methods using the characterization of linear sympletic Runge-Kutta methods. Finally, examples of new methods for two and three stages are given .

2. Characterization of linear symplectic RK-methods

Let Ω be a domain (i.e. a non-empty, open, connected set) in the oriented Euclidean space $\mathbb{R}^{2d}$ of the point $(p,q) = (p_1,\cdots,p_d;q_1,\cdots,q_d)$. If H is a sufficiently smooth real function defined in Ω, then the Hamiltonian system of differential equations with Hamiltonian H is given by

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} =: f_i(p,q), \qquad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} =: g_i(p,q), \qquad 1 \le i \le d. \tag{2.1}$$

The integer d is called the number of degrees of freedom and Ω is the phase space.

A smooth transformation $(p,q) = \psi(p^*,q^*)$ defined in Ω is said to be symplectic(with respect to symplectic matrix J) if the Jacobian $\psi' = \dfrac{\partial(p,q)}{\partial(p^*,q^*)}$ satisfies

$$\psi'^T J\psi' = \frac{\partial(p,q)^T}{\partial(p^*,q^*)}J\frac{\partial(p,q)}{\partial(p^*,q^*)} = J, \qquad \forall(p^*,q^*) \in \Omega,$$

where $J = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}$ is the so-called standard symplectic matrix.

This property is the hallmark of Hamiltonian systems.

In this paper, we restrict our interest to one-step methods. If h denotes the steplength and $(p^n;q^n)$ denotes the numerical approximations at time $t_n = nh$ to the value $(p(t_n), q(t_n))$ of a solution of (2.1), the one-step method is specified by a smooth mapping

$$(p^{n+1}, q^{n+1}) = \psi_{h,H}(p^n, q^n)$$

and $\psi_{h,H}$ is assumed to depend only smoothly on h and H. In numerically solving the Hamiltonian systems of differential equations (2.1), it is natural to require that numerical solutions should preserve the property of symplecticity. Then the numerical method

$$(p^{n+1}, q^{n+1}) = \psi_{h,H}(p^n, q^n)$$

should be a symplectic transformation. To sum up, we give the following definitions.

Definition 2.1. *A one-step method is called symplectic if, as applied to Hamiltonian systems* (2.1), *the underlying formula of generating numerical solutions* (p^{n+1}, q^{n+1}), $(p^{n+1}, q^{n+1}) = \psi_{h,H}(p^n, q^n)$, *is a symplectic transformation, that is ,*

$$\psi'^T_{h,H} J \psi'_{h,H} = \frac{\partial(p^{n+1}, q^{n+1})^T}{\partial(p^n, q^n)} J \frac{\partial(p^{n+1}, q^{n+1})}{\partial(p^n, q^n)} = J, \qquad \forall (p^n, q^n) \in \Omega \tag{2.2}$$

holds, where $\psi'_{h,H} = \dfrac{\partial(p^{n+1}, q^{n+1})^T}{\partial(p^n, q^n)}$ *is the Jacobian matrix of the transformation.*

Definition 2.2. *A one-step method is called linear symplectic if it is symplectic for linear Hamiltonian systems.*

For Hamiltonian systems there exists the relation

$$\frac{\partial(f,g)^T}{\partial(p,q)} J + J \frac{\partial(f,g)}{\partial(p,g)} = 0. \tag{2.3}$$

For notational simplicity, we assume $d = 1$ in the following. The Runge-Kutta methods with tableau

$$\begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ c_2 & a_{21} & \cdots & a_{2s} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ c_s & a_{s1} & \vdots\vdots\vdots & a_{ss} \\ \hline & b_1 & \cdots & b_s \end{array} \tag{2.4}$$

applied to the Hamiltonian systems (2.1), by virtue of the relation (2.3), we may obtain

$$\begin{aligned} &\frac{\partial(p^{n+1}, q^{n+1})^T}{\partial(p^n, q^n)} J \frac{\partial(p^{n+1}, q^{n+1})}{\partial(p^n, q^n)} \\ &= J - h^2 \textstyle\sum_{i,j=1}^s m_{i,j} \dfrac{\partial(f(p_i, q_i), g(p_i, q_i))^T}{\partial(p^n, q^n)} J \dfrac{\partial(f(p_j, q_j), g(p_j, q_j))}{\partial(p^n, q^n)}, \end{aligned} \tag{2.5}$$

where

$$m_{i,j} = b_i a_{i,j} + b_j a_{j,i} - b_i b_j, \qquad 1 \le i, j \le s,$$

that is, the so-called M matrix introduced in [2] where it is used to define algebraic stability in the study of stability criteria of implicit RK-methods. Obviously, if the coefficients of the method (2.4) satisfy

$$m_{i,j} = 0, \quad 1 \le i, j \le s, \quad \text{or} \quad M = BA + A^T B - bb^T = 0,$$

then the method is symplectic. The result was discovered independently by Lasagni[10], Sanz-Serna[12] and[14].

For linear Hamiltonian systems, in fact $\dfrac{\partial(f,g)^T}{\partial(p,q)} = C$ is a constant matrix, and then the second term on the right-hand side of (2.5) may be expanded as a series about h^2C^TJC. Finally, (2.5) becpmes

$$\frac{\partial(p^{n+1},q^{n+1})^T}{\partial(p^n,q^n)} J \frac{\partial(p^{n+1},q^{n+1})}{\partial(p^n,q^n)} = J - \sum_{k\geq 1} m_{i,j}^{(k-1)} h^{2k}(C^T)^k JC^k, \tag{2.6}$$

where $m_{i,j}^{(0)} = m_{i,j}$.

A question follows immediately; for linear symplectic methods, what is the characterization of $M^{(k)}(k=0,1,2,\cdots)$ matrix?

Using $(1.1)'$, expanding $(I-Az)^{-1}$ and $(I+zA)^{-1}$ in origin Taylor series about z and inserting them into (1.2) give the following result:

Theorem 2.1. *An s-stage implicit RK-method is linear symplectic if and only if*

$$e^T\Big(\sum_{l=0}^{2k}(-1)^l(A^T)^{2k-l}MA^l\Big)e = 0, \quad k=0,1,2,\cdots, \tag{2.7}$$

and there is

$$M^{(k)} = \Big(\sum_{l=0}^{2k}(-1)^l(A^T)^{2k-l}MA^l\Big), \quad k=0,1,2,\cdots.$$

Corollary 1. For linear symplectic RK-methods,

$$e^TMe = 0$$

is necessary but not sufficient.

For example, a 2-stage one-step method with order 2

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 0 & 1 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

satisfies $e^TMe = 0$ but is not linear symplectic.

It might be conjectured that the RK-method which satisfies the conditions

$$e^TMe = 0$$

and

$$e^T\Big(\sum_{l=0}^{2}(-1)^l(A^T)^{2-l}MA^l\Big)e = 0$$

is linear symplectic. The following example gives a negative answer. A 3-stage Lobatto-type method with order 4

$$
\begin{array}{c|ccc}
0 & \frac{(5+8\beta)}{60} & \frac{-(1+4\beta)}{15} & \frac{-(1+8\beta)}{60} \\
\frac{1}{2} & \frac{(11-4\beta)}{60} & \frac{(5+2\beta)}{15} & \frac{-(1+4\beta)}{60} \\
1 & \frac{(11+8\beta)}{60} & \frac{(11-4\beta)}{15} & \frac{(5+8\beta)}{60} \\
\hline
 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6}
\end{array}
$$

where $\forall\beta \in I\!R$, satisfies $e^TMe = 0$ and $e^T\left(\sum_{l=0}^{2}(-1)^l(A^T)^{2-l}MA^l\right)e = 0$, but is not linear symplectic.

Motivated by the above examples, we may conjecture that an s-stage implicit RK-method with order $p \geq s$ is linear symplectic, if

$$e^T\left(\sum_{l=0}^{2k}(-1)^l(A^T)^{2k-l}MA^l\right)e = 0, \quad k = 0, 1, 2, \cdots, s-1. \tag{2.8}$$

Obviously, even if the answer is affirmative, it is inconvenient to use the relation (2.8) to check whether a one-step method is linear symplectic.

For a special class of implicit RK-methods, namely symmetric methods, we shall give a set of conditions for checking linear symplectic methods, which is based on M matrix, later in this section. Now we discuss symmetric methods from another angle. In [13] it was shown that an s-stage RK-method is symmetric if and only if for the permutation matrix $\widetilde{P}$ whose (i,j)-th element is the Kronecker $\delta_{i,s+1-j}$,

$$A + \widetilde{P}A\widetilde{P}^T = eb^T \tag{2.9}$$

and

$$\widetilde{P}b = b \tag{2.10}$$

hold. From this, we can obtain easily the following result:

Theorem 2.2. *Symmetric Runge-Kutta methods are linear symplectic.*

Proof. By (2.9) there is

$$\det(I - zA + zeb^T) = \det(I + z\widetilde{P}A\widetilde{P}^T) = \det(I + zA).$$

Acoording to (1.1), $R(z)R(-z) = 1$ holds.

Recalling Theorem 1.1 and in combination with the result [9], that the stability function in terms of the transformed RK-matrix $X = W^{-1}AW$ depends only on X and not on the underlying quadrature formula, we can obtain

Theorem 2.3. *An s-stage high-order IRK-method with distinct nodes c_i and $b_i \neq 0$ satisfying $B(p), C(\eta)$ and $D(\zeta)$ with $p \geq s+\zeta$ is linear symplectic, if the (k,l)-th element of the residue matrix R_ν in the transformation matrix X satisfies $r_{kl} = 0$ as $k+l$ is even.*

Proof. By $W^TBe = W^Tb = e_1$ and $X = W^TBWX^*$ there is

$$W^TB(I - zA + zeb^T)W = W^TBW - zX + ze_1e_1^T$$
$$= W^TBW(I - zX^* + ze_1e_1^T).$$

Hence formula (1.1) becomes

$$R(z) = \frac{\det(I - zX^* + ze_1e_1^T)}{\det(I - zX^*)}.$$

Furthmore, by Lemma 1.2 and the property of the residue matrix R_ν we obtain

$$R(z) = \frac{\det\left(\tilde{I}(I - zX^* + ze_1e_1^T)\tilde{I}^T\right)}{\det(I - zX^*)} = \frac{\det(I + zX^*)}{\det(I - zX^*)}.$$

For symmetric RK-methods conditions (2.9) and (2.10) imply

$$\widetilde{P}c = e - c \tag{2.11}$$

and

$$M = -\widetilde{P}M\widetilde{P}^T. \tag{2.12}$$

For example, for (2.12), by (2.9) and (2.10) there is

$$BA + \widetilde{P}BA\widetilde{P}^T - bb^T = 0.$$

Then, adding

$$(BA + \widetilde{P}BA\widetilde{P}^T - bb^T)^T = 0$$

to the formula above we may obtain (2.12). But, under conditions (2.10) and (2.11), we cannot obtain (2.9) from (2.12). Then, for arbitrary s (stage of RK-method) there is the following result:

Theorem 2.4. *An s-stage RK-method is linear symplectic if* (2.10)–(2.12) *and*

$$C(\eta), \eta \geq s - 2 \quad \text{or} \quad D(\zeta), \zeta \geq s - 2 \tag{2.13}$$

hold.

Proof. Let $\widetilde{X} = (\widetilde{P}W)^TBA(\widetilde{P}W)$. Since $\widetilde{P}c = e - c$, by the symmetry of Legendre polynomials we have $\widetilde{P}P_k(c) = (-1)^kP_k(c)$ for $k = 0(1)s - 1$. It then follows that

$$\widetilde{X}_{kl} = (-1)^{k+l}X_{kl}. \tag{2.14}$$

Condition (2.12) may be rewritten as

$$BA + \widetilde{P}BA\widetilde{P}^T - bb^T = -(BA + \widetilde{P}BA\widetilde{P}^T - bb^T)^T.$$

We have

$$W^T(BA + \widetilde{P}BA\widetilde{P}^T - bb^T)W = -(W^T(BA + \widetilde{P}BA\widetilde{P}^T - bb^T)W)^T$$
$$\Longrightarrow X + \widetilde{X} - e_1e_1^T = -(X + \widetilde{X} - e_1e_1^T)^T.$$

(for more details see [13]), that is, the matrix $X + \widetilde{X} - e_1e_1^T$ is skew symmetric. Furthermore, according to conditions (2.13) and (2.14), there is

$$X + \widetilde{X} - e_1e_1^T = 0$$

and the transformation matrix X must satisfy the condition b) in Theorem 1.1. Therefore, the method is linear symplectic.

It may be seen from the proof above that the matrix $(BA + \widetilde{P}BA\widetilde{P}^T - bb^T)$ is skew symmetric and $(BA + \widetilde{P}BA\widetilde{P}^T - bb^T)e = 0$ if the condition $s \le 3$ is used in place of (2.13) in Theorem 2.4 such conclusion holds still.

Up to now Lobatto III A and III B methods are found to be linear symplectic in the class of high order RK-methods(besides symplectic RK-methods). Their linear symplecticness can be verified easily by one of Theorems 2.2–2.4 because the methods are symmetric. In addition, for example, A-stable block implicit one-step methods[15] and A-stable composite multistep methods[1] with the form

$$\bar{y}_m = ey_n + h\bar{d}f_n + h\bar{B}F(\bar{y}_m), \quad n = mr, \quad m = 0, 1, 2, \cdots,$$

where $\bar{d} = (d_1, d_2, \cdots, d_r)^T, \bar{y}_m = (y_{n+1}, y_{n+2}, \cdots, y_{n+r})^T, F(\bar{y}_m) = (f_{n+1}, f_{n+2}, \cdots, f_{n+r})$ and $\bar{B} = (b_{ij})_{r*r}$, are also linear symplectic. In fact, A-stable r-block one-step methods or A-stable r-step composite multistep methods above may be converted into an $(r+1)$-stage symmetric RK-method which possesses a special Runge-Kutta tableau

$$\begin{array}{c|cccc}
0 & 0 & 0 & \cdots & 0 \\
\frac{1}{r} & \frac{d_1}{r} & \frac{b_{11}}{r} & \cdots & \frac{b_{1r}}{r} \\
\vdots & \vdots & \vdots & & \vdots \\
1 & \frac{d_r}{r} & \frac{b_{r1}}{r} & \cdots & \frac{b_{rr}}{r} \\
\hline
 & \frac{d_r}{r} & \frac{b_{r1}}{r} & \cdots & \frac{b_{rr}}{r}
\end{array} \quad . \qquad (2.15)$$

By (2.15) and the property of the transformation matrix X of linear symplectic methods, such linear symplectic r-block one-step methods or r-step composite multistep methods can reach at most $2r$-th order.

3. The Construction of linear symplectic RK-methods

We construct a family of s-stage IRK-methods satisfying at least $B(2s-2), C(s-2)$ and $D(s-2)$, based on the combination

$$M(x) = P_s(x) + \frac{\sqrt{2s+1}}{\sqrt{2s-1}}\alpha P_{s-1}(x) + \frac{\sqrt{2s+1}}{\sqrt{2s-3}}\beta P_{s-2}(x), \qquad (3.1)$$

which is linear symplectic, where $P_{s-i}(x), \quad i = 0, 1, 2$, are Legendre polynomials of degree $s - i$, $i = 0, 1, 2$, respectively. Here we assume that the roots of $M(x)$, c, are real and distinct, the weights are determined by $B(2s-2)$. Furthermore, by the definition of W, we compute a matrix W and then choose the transformation matrix as

$$X = \begin{pmatrix} 1/2 & -\xi_1 & & & \\ \xi_1 & 0 & \ddots & & \\ & \ddots & \ddots & -\xi_{s-2} & \\ & & \xi_{s-2} & 0 & -\xi_{s-1}\sigma_1 \\ & & & \xi_{s-1}\sigma_2 & 0 \end{pmatrix} \tag{3.2}$$

where $\xi_k = \dfrac{1}{2\sqrt{4k^2-1}}$ and $\sigma_1, \sigma_2 \in I\!R$.

According to Theorem 2.3 and Butcher's fundamental theorem [3], the four-parameter family of IRK-methods with coefficients $A = WXW^TB$, which is linear symplectic and of order at least $2s-2$, is constructed (see [9] IV.5. for details). Besides such results with the special choice of parameters (α, β, σ_1 and σ_2) we can obtain:

a) $\alpha = \beta = 0$ corresponding to s-stage Gauss-type method :

1) order $2s$ if $\sigma_1 = \sigma_2 = 1$[3] ;

2) order $2s-2$ with $B(2s), C(s-2)$ and $D(s-2)$, if $\sigma_1, \sigma_2 \neq 1$;

3) order $2s-2$ with $B(2s), C(s-1)$ and $D(s-2)$, if $\sigma_1 \neq 1$ and $\sigma_2 = 1$;

4) order $2s-2$ with $B(2s), C(s-2)$ and $D(s-1)$, if $\sigma_1 = 1$ and $\sigma_2 \neq 1$;

b) $\alpha = 0$ and $\beta = -1$ corresponding to s-stage Lobatto-type methods with order $2s-2$:

1) Lobatto III A method if $\sigma_1 = 0$ and $\sigma_2 = \dfrac{1}{b^T P_{s-1}^2(c)}$ [5] ;

2) Lobatto III B method if $\sigma_1 = \dfrac{1}{b^T P_{s-1}^2(c)}$ and $\sigma_2 = 0$[5] ;

3) Lobatto III E method if $\sigma_1 = \sigma_2 = \dfrac{1}{b^T P_{s-1}^2(c)}$ [11],[4] ;

4) Lobatto III S method if $\sigma_1 = \sigma_2 \neq \dfrac{1}{b^T P_{s-1}^2(c)}$ and $s \geq 3$[4] ;

5) Lobatto III X method if $\sigma_1 \neq \sigma_2$ and, besides 1) and 2) ;

c) $\beta = 0$ and $\alpha = 1$ corresponding to s-stage Radau I type method :

1) Radau I B method with order $2s-1$ satisfying $B(2s-1), C(s-1)$ and $D(s-1)$ if $\sigma_1 = \sigma_2 = 1$[13] ;

2) Called as Radau I C method with order $2s-2$ satisfying $B(2s-1)$, $C(s-1)$ and $D(s-2)$ if $\sigma_1 = 0$ and $\sigma_2 = 1$;

3) Called as Radau I D method with order $2s-2$ satisfying $B(2s-1)$, $C(s-2)$ and $D(s-1)$ if $\sigma_1 = 1$ and $\sigma_2 = 0$;

d) $\beta = 0$ and $\alpha = -1$ corresponding to s-stage Radau II type method :

1) Radau II B method with order $2s-1$ satisfying $B(2s-1)$, $C(s-1)$ and $D(s-1)$ if $\sigma_1 = \sigma_2 = 1$ [13] ;

2) Called Radau II C method with order $2s-2$ satisfying $B(2s-1)$, $C(s-1)$ and $D(s-2)$ if $\sigma_1 = 0$ and $\sigma_2 = 1$;

3) Called Radau II D method with order $2s-2$ satisfying $B(2s-1)$, $C(s-2)$ and $D(s-1)$ if $\sigma_1 = 1$ and $\sigma_2 = 0$.

The examples of new methods of Gauss type and Radau type for two and three stages are given in the following.

The 2-and 3-stage Gauss type methods with order 2 and 4 are

$$1 \quad \sqrt{3} \quad \Big| \quad 1\Big(1+(\sigma_1-\sigma_2)\sqrt{3}\Big) \qquad 1\Big(1-(\sigma_1+\sigma_2)\sqrt{3}\Big)$$