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A SPECTRAL APPROXIMATION OF THE BAROTROPIC VORTICITY EQUATION*

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Abstract

A spectral scheme is considered for solving the barotropic vorticity equation. The error estimates are proved strictly. The technique used in this paper is also useful for other nonlinear problems defined on a spherical surface.

1. Introduction

The barotropic vorticity equation plays an important role in the research of weather prediction, see [1-5]. Many efforts have been made to solve this equation numerically. The early works were mainly concerned with finite-difference methods. In particular, the conservative schemes were applied successfully; see [3,4]. Since 1970s, global numerical weather prediction has developed rapidly, so it seems more natural to adopt a spectral method, see [5-8]. Because of the high accuracy of spectral approximation, this method becomes more and more attractive for long-time weather prediction. On the other hand, although strict error estimations of spectral schemes for atmospheric equations have been set up (see [7-10]), they are valid only for problems in Descartes coordinates. Indeed, as pointed out in [11], no rigorous approximation theory is available for the spectral method in spherical polar coordinates. Thus it is significant to develop the spectral method and its error analysis of the corresponding partial differential equations defined on a spherical surface for numerical weather prediction and other related problems.

In this paper, we present a spectral scheme for the barotropic vorticity equation defined on the spherical surface. In Section 2, we construct the spectral scheme by using spherical harmonic functions. In Section 3, we list a series of lemmas which play a fundamental role in the theoretical analysis. Finally we prove strictly the generalized stability and the convergence of this method in Section 4 and Section 5 respectively. The technique used in this paper is also applicable to other nonlinear problems in spherical polar coordinates.

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2. The Spectral Scheme

Let S be the unit spherical surface,

$$S = \left\{(\lambda, \theta)\big/ 0 \le \lambda < 2\pi, \quad -\frac{\pi}{2} \le \theta < \frac{\pi}{2}\right\}$$

where λ and θ are the longitude and the latitude. Let $\xi(\lambda,\theta,t), \psi(\lambda,\theta,t)$ and $\Omega > 0$ be the vorticity, the stream function and the angular velocity of the earth respectively. The gradient, the Jacobi operator and the Laplace operator are as follows:

$$\nabla\xi = \Big(\frac{1}{\cos\theta}\frac{\partial\xi}{\partial\lambda}, \frac{\partial\xi}{\partial\theta}\Big)^*, \quad J(\xi,\psi) = \frac{1}{\cos\theta}\Big(\frac{\partial\xi}{\partial\lambda}\frac{\partial\psi}{\partial\theta} - \frac{\partial\xi}{\partial\theta}\frac{\partial\psi}{\partial\lambda}\Big),$$

$$\Delta\xi = \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\Big(\cos\theta\frac{\partial\xi}{\partial\theta}\Big) + \frac{1}{\cos^2\theta}\frac{\partial^2\xi}{\partial\lambda^2}.$$

The barotropic vorticity equation on S is as follows:

$$\begin{cases} \dfrac{\partial\xi}{\partial t} + J(\xi,\psi) - 2\Omega\dfrac{\partial\psi}{\partial\lambda} = 0, & (\lambda,\theta) \in S, t \in (0,T], \\ -\Delta\psi = \xi, & (\lambda,\theta) \in S, t \in [0,T], \\ \xi(\lambda,\theta,0) = \xi_0(\lambda,\theta), & (\lambda,\theta) \in S, \end{cases} \tag{2.1}$$

where the initial value $\xi_0(\lambda,\theta)$ is given. For fixed ψ, we require

$$\mu(\psi(t)) \equiv \iint_S \psi(\lambda,\theta,t)dS \equiv 0. \tag{2.2}$$

We shall consider the weak representation of (2.1). Let $D(S)$ be the set of all infinitely differentiable functions which are regular at $\theta = \pm\frac{\pi}{2}$ and have the period 2π for the variable λ. The duality of $D(S)$ is denoted by $D'(S)$. We define the generalized function $u \in D'(S)$ and its derivatives in the usual way as in [12]. Furthermore, we can define the generalized gradient, the generalized Jacobi operator and the generalized Laplace operator. For instance, if

$$\iint_S u\Delta v dS = \iint_S v\bar{u} dS, \quad \forall v \in D(S),$$

then the mapping $\overline{\Delta}$ such that $\bar{u} = \overline{\Delta}u$ is called the generalized Laplace operator. For simplicity, we denote $\overline{\Delta}$ by Δ, etc..

Now, let

$$L^2(S) = \{u \in D'(S)/\|u\| < \infty\}$$

be equipped with the inner product and the norm as follows:

$$(u,v) = \iint_S uv dS, \quad \|u\| = (u,u)^{\frac{1}{2}}.$$

Furthermore,

$$H^1(S) = \Big\{u\Big|u, \frac{1}{\cos\theta}\frac{\partial u}{\partial\lambda}, \frac{\partial u}{\partial\theta} \in L^2(S)\Big\}$$