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A SPECTRAL APPROXIMATION OF THE BAROTROPIC VORTICITY EQUATION*

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Abstract

A spectral scheme is considered for solving the barotropic vorticity equation. The error estimates are proved strictly. The technique used in this paper is also useful for other nonlinear problems defined on a spherical surface.

1. Introduction

The barotropic vorticity equation plays an important role in the research of weather prediction, see [1-5]. Many efforts have been made to solve this equation numerically. The early works were mainly concerned with finite-difference methods. In particular, the conservative schemes were applied successfully; see [3,4]. Since 1970s, global numerical weather prediction has developed rapidly, so it seems more natural to adopt a spectral method, see [5-8]. Because of the high accuracy of spectral approximation, this method becomes more and more attractive for long-time weather prediction. On the other hand, although strict error estimations of spectral schemes for atmospheric equations have been set up (see [7-10]), they are valid only for problems in Descartes coordinates. Indeed, as pointed out in [11], no rigorous approximation theory is available for the spectral method in spherical polar coordinates. Thus it is significant to develop the spectral method and its error analysis of the corresponding partial differential equations defined on a spherical surface for numerical weather prediction and other related problems.

In this paper, we present a spectral scheme for the barotropic vorticity equation defined on the spherical surface. In Section 2, we construct the spectral scheme by using spherical harmonic functions. In Section 3, we list a series of lemmas which play a fundamental role in the theoretical analysis. Finally we prove strictly the generalized stability and the convergence of this method in Section 4 and Section 5 respectively. The technique used in this paper is also applicable to other nonlinear problems in spherical polar coordinates.

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2. The Spectral Scheme

Let S be the unit spherical surface,

$$S = \left\{(\lambda,\theta)\big/ 0 \le \lambda < 2\pi, \quad -\frac{\pi}{2} \le \theta < \frac{\pi}{2}\right\}$$

where λ and θ are the longitude and the latitude. Let $\xi(\lambda,\theta,t), \psi(\lambda,\theta,t)$ and $\Omega > 0$ be the vorticity, the stream function and the angular velocity of the earth respectively. The gradient, the Jacobi operator and the Laplace operator are as follows:

$$\nabla\xi = \left(\frac{1}{\cos\theta}\frac{\partial\xi}{\partial\lambda}, \frac{\partial\xi}{\partial\theta}\right)^*, \quad J(\xi,\psi) = \frac{1}{\cos\theta}\left(\frac{\partial\xi}{\partial\lambda}\frac{\partial\psi}{\partial\theta} - \frac{\partial\xi}{\partial\theta}\frac{\partial\psi}{\partial\lambda}\right),$$

$$\Delta\xi = \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial\xi}{\partial\theta}\right) + \frac{1}{\cos^2\theta}\frac{\partial^2\xi}{\partial\lambda^2}.$$

The barotropic vorticity equation on S is as follows:

$$\begin{cases} \dfrac{\partial\xi}{\partial t} + J(\xi,\psi) - 2\Omega\dfrac{\partial\psi}{\partial\lambda} = 0, & (\lambda,\theta)\in S, t\in(0,T], \\ -\Delta\psi = \xi, & (\lambda,\theta)\in S, t\in[0,T], \\ \xi(\lambda,\theta,0) = \xi_0(\lambda,\theta), & (\lambda,\theta)\in S, \end{cases} \tag{2.1}$$

where the initial value $\xi_0(\lambda,\theta)$ is given. For fixed ψ, we require

$$\mu(\psi(t)) \equiv \iint_S \psi(\lambda,\theta,t)dS \equiv 0. \tag{2.2}$$

We shall consider the weak representation of (2.1). Let $D(S)$ be the set of all infinitely differentiable functions which are regular at $\theta = \pm\frac{\pi}{2}$ and have the period 2π for the variable λ. The duality of $D(S)$ is denoted by $D'(S)$. We define the generalized function $u \in D'(S)$ and its derivatives in the usual way as in [12]. Furthermore, we can define the generalized gradient, the generalized Jacobi operator and the generalized Laplace operator. For instance, if

$$\iint_S u\Delta v dS = \iint_S v\bar{u} dS, \quad \forall v \in D(S),$$

then the mapping $\overline{\Delta}$ such that $\overline{u} = \overline{\Delta}u$ is called the generalized Laplace operator. For simplicity, we denote $\overline{\Delta}$ by Δ, etc..

Now, let

$$L^2(S) = \{u \in D'(S)/\|u\| < \infty\}$$

be equipped with the inner product and the norm as follows:

$$(u,v) = \iint_S uv dS, \quad \|u\| = (u,u)^{\frac{1}{2}}.$$

Furthermore,

$$H^1(S) = \left\{u\Big|u, \frac{1}{\cos\theta}\frac{\partial u}{\partial\lambda}, \frac{\partial u}{\partial\theta} \in L^2(S)\right\}$$

with the following semi-norm and norm:

$$|u|_1 = \Big(\Big\|\frac{1}{\cos\theta}\frac{\partial u}{\partial\lambda}\Big\|^2 + \Big\|\frac{\partial u}{\partial\theta}\Big\|^2\Big)^{\frac{1}{2}}, \quad \|u\|_1 = (\|u\|^2 + |u|_1^2)^{\frac{1}{2}}.$$

For positive integer r, we can define the space $H^r(S)$ with the norm $\|\cdot\|_r$ similarly. In particular, the norm of $H^2(S)$ is equivalent to (see [12])

$$(\|u\|^2 + \|\Delta u\|^2)^{\frac{1}{2}}.$$

For real $r \geq 0$, the space $H^r(S)$ is defined by the complex interpolation between spaces $H^{[r]}(S)$ and $H^{[r]+1}(S)$, $[r]$ being the integral part of r. Clearly, $H^0(S) = L^2(S)$ and $\|u\|_0 = \|u\|$. Besides, let $\|u\|_{r,\infty} = \|u\|_{C^r(S)}$ and $\|u\|_\infty = \|u\|_{0,\infty}$.

The weak representation of (2.1) is to find $(\xi,\psi) \in H^1(S)\times H^1(S)$ such that for all $v \in H^1(S)$,

$$\begin{cases} \Big(\frac{\partial}{\partial t}\xi(t), v\Big) + (J(\xi(t),\psi(t)), v) - 2\Omega(\frac{\partial\psi}{\partial\lambda}(t), v) = 0, & t\in(0,T], \\ (\nabla\psi(t),\nabla v) = (\xi(t), v), & t\in[0,T], \\ \xi(0) = \xi_0. \end{cases} \tag{2.3}$$

We now construct the spectral scheme for (2.3). Firstly, let $L_n(z)$ be the Legendre polynomial of order n, namely

$$L_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}(z^2-1)^n.$$

The normalized associated Legendre polynomial is defined as

$$L_{m,n}(z) = \sqrt{\frac{(2n+1)(n-m)!}{2(n+m)!}}(1-z^2)^{\frac{m}{2}}\frac{d^m}{dz^m}L_n(z), \quad m\geq 0, n\geq |m|,$$

$$L_{m,n}(z) = L_{-m,n}(z), \quad m<0, n\geq|m|.$$

Furthermore, the spherical harmonic function $Y_{m,n}(\lambda,\theta)$ is

$$Y_{m,n}(\lambda,\theta) = \frac{1}{\sqrt{2\pi}}e^{im\lambda}L_{m,n}(\sin\theta), \quad n\geq|m|.$$

It can be verified that (see [13])

$$-\Delta Y_{m,n}(\lambda,\theta) = n(n+1)Y_{m,n}(\lambda,\theta), \tag{2.4}$$

and

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Y_{m,n}(\lambda,\theta)Y^*_{m',n'}(\lambda,\theta)\cos\theta d\theta d\lambda = \begin{cases} 1, & \text{if } m=m', n=n', \\ 0, & \text{otherwise}. \end{cases}$$

We set

$$\hat{u}_{m,n} = \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u(\lambda,\theta)Y^*_{m,n}(\lambda,\theta)\cos\theta d\theta d\lambda.$$

Let

$$\tilde{V}_M = \text{span}\{Y_{m,n}/|m| \le M, |m| \le n \le N(m)\}.$$

where $N(m)$ determines the structure of spectral approximation. Usually we take $N(m) = M$ or $N(m) = M + |m|$. For simplicity, suppose $N(m) = M$. Let V_M be the subset of $\tilde{V}_M$ containing all real-valued functions.

Let P_M be the orthogonal projection from $L^2(S)$ onto V_M such that for any $u \in L^2(S)$,

$$(P_M u - u, v) = 0, \quad \forall v \in V_M,$$

or equivalently

$$P_M u = \sum_{|m| \le M} \sum_{n \ge |m|} \hat{u}_{m,n} Y_{m,n}(\lambda, \theta).$$

Let τ be the mesh size in the variable t, and

$$\dot{R}_\tau = \{t = k\tau/1 \le k \le [\frac{T}{\tau}]\}, \quad R_\tau = \dot{R}_\tau \bigcup \{0\}.$$

Define

$$u_t(\lambda, \theta, t) = \frac{1}{\tau}(u(\lambda, \theta, t + \tau) - u(\lambda, \theta, t)).$$

Let $(\eta, \varphi) \in V_M \times V_M$ be the approximation to (ξ, ψ). δ is a parameter and $0 \le \delta \le 1$. The spectral scheme for (2.3) is as follows:

$$\begin{cases} (\eta_t(t), v) + (J(\eta(t) + \delta\tau\eta_t(t), \varphi(t)), v) - 2\Omega(\frac{\partial \varphi}{\partial \lambda}(t), v) = 0, & \forall v \in V_M, t \in \dot{R}_\tau, \\ -(\Delta\varphi(t), v) = (\eta(t), v), & \forall v \in V_M, t \in R_\tau, \\ \mu(\varphi(t)) = 0, & \forall t \in R_\tau, \\ \eta(0) = P_M \xi_0. \end{cases} \tag{2.5}$$

Clearly, if $\delta = \sigma = 0$, then (2.5) is an explicit scheme. Otherwise, it is implicit and so an iteration is needed for evaluating $\eta(t)$ at each $t \in \dot{R}_\tau$.

3. Some Lemmas

To analyze the errors, we need some fundamental estimations. In this section, we list several lemmas. Throughout this paper, we denote by c a positive constant independent of M, τ and any function, which may be different in different cases. The notation "$\subset$" means the embedding of spaces.

Lemma 1. *$H^\beta(S) \subset H^r(S)$ for $0 \le r \le \beta$ and $H^{1+\beta}(S) \subset C(S)$ for $\beta > 0$.*

Proof. The first conclusion follows from the definition directly. We now prove the second one. Let B be the unit ball in the three-dimensional Euclidean space, and w a function defined on B. We denote by $\gamma(w)$ the restriction of w on S. We can take

$H^{1+\beta}(S)$ as the trace space of $H^{\frac{3}{2}+\beta}(B)$ equipped with the norm

$$\|u\|_{H^{1+\beta}(S)} = \inf_{\substack{w\in H^{\frac{3}{2}+\beta}(B)\\ \gamma(w)=u}} \|w\|_{H^{\frac{3}{2}+\beta}(B)}.$$

By the embedding theory, $H^{\frac{3}{2}+\beta}(B) \subset C(B)$ and so for any $w \in H^{\frac{3}{2}+\beta}(B)$,

$$\|w\|_{C(B)} \leq c\|w\|_{H^{\frac{3}{2}+\beta}(B)}.$$

On the other hand, for any $u \in H^{1+\beta}(S)$, there exists $\overline{w} \in H^{\frac{3}{2}+\beta}(B)$ such that $\gamma(\overline{w}) = u$ and

$$\|u\|_{H^{1+\beta}(S)} \geq \frac{1}{2}\|\overline{w}\|_{H^{\frac{3}{2}+\beta}(B)}.$$

Therefore

$$\|u\|_{C(S)} = \sup_{x\in S}|u(x)| = \sup_{x\in S}|\overline{w}(x)| \leq \sup_{x\in B}|\overline{w}(x)| \leq c\|\overline{w}\|_{H^{\frac{3}{2}+\beta}(B)} \leq 2c\|u\|_{H^{1+\beta}(S)},$$

which implies the second conclusion.

Lemma 2. *There exists a positive constant c such that $\|u\|^2 \leq c|u|_1^2$, for all $u \in H^1(S)$ with $\mu(u) = 0$.*

Proof. By the Poincare inequality, we have

$$\|u\|^2 \leq c(\mu(u) + |u|_1^2)$$

and so the conclusion follows.

Lemma 3. *If $u \in L^2(S), v \in H^{1+\beta}(S)$ and $\beta > 0$, then*

$$\|uv\| \leq \|u\|\|v\|_{1+\beta}.$$

Proof. By Lemma 1,

$$\|uv\|^2 \leq \|u\|^2\|v\|_\infty^2 \leq c\|u\|^2\|v\|_{1+\beta}^2.$$

Lemma 4. *If $u \in V_M$ and $0 \leq r \leq \beta$, then*

$$\|u\|_\beta \leq cM^{\beta-r}\|u\|_r.$$

Proof. Let

$$u = \sum_{m=-M}^{M}\sum_{n=|m|}^{M} \hat{u}_{m,n}Y_{m,n}(\lambda,\theta).$$

By (2.4), $Y_{m,n}(\lambda,\theta)$ is the eigenfunction of the operator $-\Delta$ on S, with the eigenvalue $n(n+1)$. Thus for any $v \in H^r(S)$, the norm $\|v\|_r$ is equivalent to (see [12])

$$\Big(\sum_{m=-\infty}^{\infty}\sum_{n\geq|m|} n^r(n+1)^r|\hat{v}_{m,n}|^2\Big)^{\frac{1}{2}}. \tag{3.1}$$

Therefore

$$\|u\|_\beta^2 \le c \sum_{m=-M}^{M} \sum_{n=|m|}^{M} n^\beta (n+1)^\beta |\hat{u}_{m,n}|^2 \le cM^{2\beta-2r} \sum_{m=-M}^{M} \sum_{n=|m|}^{M} n^r (n+1)^r |\hat{u}_{m,n}|^2$$

$$\le cM^{2\beta-2r} \|u\|_r^2.$$

Lemma 5. *If $u \in H^\beta(S)$ and $r \le \beta$, then*

$$\|u - P_M u\|_r \le cM^{r-\beta} \|u\|_\beta, \quad \|P_M u\|_r \le c\|u\|_r.$$

Proof. By (3.1),

$$\|u - P_M u\|_r^2 \le c \sum_{m=-M}^{M} \sum_{n=M+1}^{\infty} n^r (n+1)^r |\hat{u}_{m,n}|^2 + c \sum_{|m|>M} \sum_{n=|m|}^{\infty} n^r (n+1)^r |\hat{u}_{m,n}|^2$$

$$\le c \sum_{m=-M}^{M} \sum_{n=M+1}^{\infty} n^r (n+1)^r |\hat{u}_{m,n}|^2 + c \sum_{|m|>M} \sum_{n=M+1}^{\infty} n^r (n+1)^r |\hat{u}_{m,n}|^2$$

$$\le cM^{2r-2\beta} \sum_{m=-\infty}^{\infty} \sum_{n=M+1}^{\infty} n^\beta (n+1)^\beta |\hat{u}_{m,n}|^2$$

$$\le cM^{2r-2\beta} \|u\|_r^2.$$

Lemma 6. *If $u, v \in H^{1+\beta}(S), \beta > 0$ and $w \in H^1(S)$, then*

$$(u, J(v,w)) + (v, J(u,w)) = 0.$$

Proof. We have

$$(u, J(v,w)) = \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u\Big(\frac{\partial v}{\partial \lambda}\frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial \theta}\frac{\partial w}{\partial \lambda}\Big) d\theta d\lambda = -\int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v\Big(\frac{\partial u}{\partial \lambda}\frac{\partial w}{\partial \theta} - \frac{\partial u}{\partial \theta}\frac{\partial w}{\partial \lambda}\Big) d\theta d\lambda$$

$$-\int_0^{2\pi} u(\lambda, \frac{\pi}{2}) v(\lambda, \frac{\pi}{2}) \frac{\partial w}{\partial \lambda}(\lambda, \frac{\pi}{2}) d\lambda + \int_0^{2\pi} u(\lambda, -\frac{\pi}{2}) v(\lambda, -\frac{\pi}{2}) \frac{\partial w}{\partial \lambda}(\lambda, -\frac{\pi}{2}) d\lambda.$$

By the regularity of w, we know that w approaches the limits independent of λ, as $\theta \to \pm\frac{\pi}{2}$ (see p.314 of [13]). It means that $\frac{\partial w}{\partial \lambda} = 0$ at $\theta = \pm\frac{\pi}{2}$, and so the conclusion follows.

Lemma 7. *For any $u \in C(0,T; L^2(S))$,*

$$2(u_t(t), u(t)) = \|u(t)\|_t^2 - \tau \|u_t(t)\|^2.$$

Lemma 8[14]**.** *Assume that*

(i) *$E(t)$ is a nonnegative function defined on R_τ,*

(ii) *ρ, b_1, b_2, d_1 and d_2 are nonnegative constants,*

(iii) *for* $t \in \dot{R}_\tau$,

$$E(t) \le \rho + \tau \sum_{t'=0}^{t-\tau} (d_1 E(t') + d_2 M^{b_1} E^{b_2+1}(t')),$$

(vi) $E(0) \le \rho$ *and* $\rho e^{(d_1+d_2)t_1} \le M^{-\frac{b_1}{b_2}}$.

Then for all $t \in R_\tau$ *and* $t \le t_1$,

$$E(t) \le \rho e^{(d_1+d_2)t}.$$

If $d_2 = 0$ *in addition, then for all* ρ *and* $t \le T$,

$$E(t) \le \rho e^{d_1 t}.$$

4. The Generalized Stability of the Scheme

As we know, nonlinear schemes are usually not stable in the sense of Lax, but might be of generalized stability (see [14,15]). We now analyze the generalized stability of scheme (2.5). Suppose that $\eta(0)$ has the error $\tilde{\eta}_0$, while the right sides of the first and the second equations of (2.5) have the errors $\tilde{f}_1$ and $\tilde{f}_2$ respectively. They induce the errors of η and φ, denoted by $\tilde{\eta}$ and $\tilde{\varphi}$. Then

$$\begin{cases} (\tilde{\eta}_t(t), v) + (J(\tilde{\eta}(t) + \delta\tau\tilde{\eta}_t(t), \varphi(t) + \tilde{\varphi}(t)), v) + (J(\eta(t) \\ \qquad + \delta\tau\eta_t(t), \tilde{\varphi}(t)), v) - 2\Omega(\frac{\partial \tilde{\varphi}}{\partial \lambda}(t), v) = (\tilde{f}_1(t), v), & \forall v \in V_M, t \in \dot{R}_\tau, \\ -(\Delta\tilde{\varphi}(t), v) = (\tilde{\eta}(t) + \tilde{f}_2(t), v), & \forall v \in V_M, t \in R_\tau, \\ \mu(\tilde{\varphi}(t)) = 0, & t \in R_\tau, \\ \tilde{\eta}(0) = \tilde{\eta}_0. \end{cases} \tag{4.1}$$

By taking $v = 2\tilde{\eta}$ in the first formula of (4.1), from Lemmas 6 and 7 we have

$$\|\tilde{\eta}(t)\|_t^2 - \tau\|\tilde{\eta}_t(t)\|^2 - 2\delta\tau(J(\tilde{\eta}(t), \varphi(t) + \tilde{\varphi}(t)), \tilde{\eta}_t(t)) + F_1(t) - 4\Omega(\frac{\partial \tilde{\varphi}}{\partial \lambda}(t), \tilde{\eta}(t)) \\ = 2(\tilde{f}_1(t), \tilde{\eta}(t)) \tag{4.2}$$

where

$$F_1(t) = 2(J(\eta(t) + \delta\tau\eta_t(t), \tilde{\varphi}(t)), \tilde{\eta}(t)).$$

Next, let d be an undetermined constant. By taking $v = d\tau\tilde{\eta}_t$ in the same formula, we get

$$d\tau\|\tilde{\eta}_t(t)\|^2 + d\tau(J(\tilde{\eta}(t), \varphi(t) + \tilde{\varphi}(t)), \tilde{\eta}_t(t)) + F_2(t) - 2\Omega d\tau(\frac{\partial \tilde{\varphi}}{\partial \lambda}(t), \tilde{\eta}_t(t)) \\ = d\tau(\tilde{f}_1(t), \tilde{\eta}_t(t)) \tag{4.3}$$

where

$$F_2(t) = d\tau(J(\eta(t) + \delta\tau\eta_t(t), \tilde{\varphi}(t)), \tilde{\eta}_t(t)).$$

We put (4.2) and (4.3) together. Then

$$\|\tilde{\eta}(t)\|_{\bar{t}}^2 + \tau(d-1)\|\tilde{\eta}_t(t)\|^2 + \sum_{j=1}^{4} F_j(t) - 2\Omega(\frac{\partial \tilde{\varphi}}{\partial \lambda}(t), \tilde{\eta}(t) + d\tau\tilde{\eta}_t(t))$$
$$= (\tilde{f}_1(t), 2\tilde{\eta}(t) + d\tau\tilde{\eta}_t(t)) \tag{4.4}$$

where

$$F_3(t) = \tau(d - 2\delta)(J(\tilde{\eta}(t), \varphi(t)), \tilde{\eta}_t(t)),$$

$$F_4(t) = \tau(d - 2\delta)(J(\tilde{\eta}(t), \tilde{\varphi}(t)), \tilde{\eta}_t(t)).$$

Furthermore, we put $v = \tilde{\varphi}$ in the second formula of (4.1), and obtain

$$|\tilde{\varphi}(t)|_1^2 \leq \frac{1}{2c}\|\tilde{\varphi}(t)\|^2 + c(\|\eta(t)\|^2 + \|\tilde{f}_2(t)\|^2).$$

Thus, Lemma 2 leads to

$$|\tilde{\varphi}(t)|_1^2 \leq c(\|\tilde{\eta}(t)\|^2 + \|\tilde{f}_2(t)\|^2). \tag{4.5}$$

Moreover, by Lemma 2 and (4.5),

$$\|\tilde{\varphi}(t)\|_2^2 \leq c(\|\tilde{\varphi}(t)\|^2 + \|\Delta\tilde{\varphi}(t)\|^2) \leq c(\|\tilde{\eta}(t)\|^2 + \|\tilde{f}_2(t)\|^2). \tag{4.6}$$

We are going to estimate $|F_j(t)|$. Let $\varepsilon > 0$ and

$$|||u|||_r = \max_{0\leq t\leq T}\|u(t)\|_r, \quad |||u|||_{r,\infty} = \max_{0\leq t\leq T}\|u(t)\|_{r,\infty}, \quad \text{etc.}$$

By (4.5), we know that for any $\beta > 0$,

$$|F_1(t)| \leq c|||\eta|||_{1,\infty}^2(\|\tilde{\eta}(t)\|^2 + |\tilde{\varphi}(t)|_1^2) \leq c|||\eta|||_{1,\infty}^2(\|\tilde{\eta}(t)\|^2 + \|\tilde{f}_2(t)\|^2).$$

Similarly,

$$|F_2(t)| \leq \varepsilon\tau\|\tilde{\eta}_t(t)\|^2 + \frac{c\tau d^2}{\varepsilon}|||\eta|||_{1,\infty}^2(\|\tilde{\eta}(t)\|^2 + \|\tilde{f}_2(t)\|^2).$$

By Lemma 4,

$$|F_3(t)| \leq \varepsilon\tau\|\tilde{\eta}_t(t)\|^2 + \frac{c\tau(d-2\delta)^2}{\varepsilon}|||\varphi|||_{1,\infty}^2|\tilde{\eta}(t)|_1^2$$
$$\leq \varepsilon\tau\|\tilde{\eta}_t(t)\|^2 + \frac{c\tau M^2(d-2\delta)^2}{\varepsilon}|||\varphi|||_{1,\infty}^2\|\tilde{\eta}(t)\|^2.$$

Furthermore, we have from Lemma 3 and (4.6) that for $\beta > 0$,

$$|F_4(t)| \leq \varepsilon\tau\|\tilde{\eta}_t(t)\|^2 + \frac{c\tau(d-2\delta)^2}{\varepsilon}\|\tilde{\varphi}(t)\|_{2+\beta}^2|\tilde{\eta}(t)|_1^2$$

$$\leq \varepsilon\tau\|\tilde{\eta}_t(t)\|^2 + \frac{c\tau M^{2+\beta}(d-2\delta)^2}{\varepsilon}\|\tilde{\eta}(t)\|^2\|\tilde{\varphi}(t)\|_2^2$$

$$\leq \varepsilon\tau\|\tilde{\eta}_t(t)\|^2 + \frac{c\tau M^{2+\beta}(d-2\delta)^2}{\varepsilon}(\|\tilde{\eta}(t)\|^4 + \|\tilde{f}_2(t)\|^2\|\tilde{\eta}(t)\|^2).$$

Finally, we have

$$\left|2\Omega\Big(\frac{\partial\tilde{\varphi}}{\partial\lambda}(t), \tilde{\eta}(t) + d\tau\tilde{\eta}_t(t)\Big)\right| \leq \varepsilon\tau\|\tilde{\eta}_t(t)\|^2 + c\Big(1+\frac{\tau d^2}{\varepsilon}\Big)(\|\tilde{\eta}(t)\|^2 + \|\tilde{f}_2(t)\|^2),$$

$$|(\tilde{f}_1(t), \tilde{\eta}(t) + d\tau\tilde{\eta}_t(t))| \leq \varepsilon\tau\|\tilde{\eta}_t(t)\|^2 + c\|\tilde{\eta}(t)\|^2 + c\Big(1+\frac{\tau d^2}{\varepsilon}\Big)\|\tilde{f}_1(t)\|^2.$$

By substituting the above estimates into (4.4), we obtain

$$\|\tilde{\eta}(t)\|_t^2 + \tau(d-1-5\varepsilon)\|\tilde{\eta}_t(t)\|^2 \leq Q_1(t)\|\tilde{\eta}(t)\|^2 + Q_2\|\tilde{\eta}(t)\|^4 + F(t) \tag{4.7}$$

with

$$Q_1(t) = c + c(1+\frac{\tau d^2}{\varepsilon})|||\eta|||_{1,\infty}^2 + \frac{c\tau M^2(d-2\delta)^2}{\varepsilon}|||\varphi|||_{1,\infty}^2$$

$$+ \frac{c\tau M^{2+\beta}(d-2\delta)^2}{\varepsilon}\|\tilde{f}_2(t)\|^2,$$

$$Q_2 = \frac{c\tau M^{2+\beta}(d-2\delta)^2}{\varepsilon},$$

$$F(t) = c(1+\frac{\tau d^2}{\varepsilon})\|\tilde{f}_1(t)\|^2 + c(1+\frac{\tau d^2}{\varepsilon})(|||\eta|||_{1,\infty}^2 + 1)\|\tilde{f}_2(t)\|^2.$$

Now, let $d = p_0 + 1 + 5\varepsilon, p_0 > 0$ and

$$E(t) = \|\tilde{\eta}(t)\|^2 + p_0\tau^2\sum_{t'=0}^{t-\tau}\|\tilde{\eta}_t(t')\|^2, \quad \rho(t) = \|\tilde{\eta}_0\|^2 + \tau\sum_{t'=0}^{t-\tau}F(t').$$

By summing up (4.7) for $t \in R_\tau$, we get

$$E(t) \leq \rho(t) + \tau\sum_{t'=0}^{t-\tau}(Q_1(t')\|\tilde{\eta}(t')\|^2 + Q_2\|\tilde{\eta}(t')\|^4). \tag{4.8}$$

In particular, if $\delta > \frac{1}{2}$, then we can take $d = 2\delta$ and $Q_2 = 0$, etc.

Finally, we apply Lemma 8 to (4.8), and obtain the following result.

Theorem 1. *Assume that $\delta > \frac{1}{2}$ or the following conditions are fulfilled:*

(1) $\delta \leq \frac{1}{2}, p \geq 2$ and $\tau \leq cM^{-p}$,

(2) $\|\tilde{f}_2(t)\|^2 \leq cM^{p-\beta-2}$, where β is an arbitrarily small positive constant,

(3) $\rho(t_1)e^{d_1 t} \leq d_2 M^{p-\beta-2}$, where d_j are positive constants depending only on $|||\eta|||_{1,\infty}$ and $|||\varphi|||_{1,\infty}$.

Then for all $t \leq t_1$,

$$E(t) \leq \rho(t)e^{d_2 t}. \tag{4.9}$$

Remark 1. In the first case, $\delta > 1/2$ and thus we can take $d = 2\delta$. In the second case, $\delta \le \frac{1}{2}$ and so $d - 2\delta \neq 0$. Thus we require that $p > 2$ or $p = 2, \|\tilde{f}_2(t)\|^2 \le CM^{-\beta}$ and $\rho(t) \le cM^{-\beta}$. Then for all $\tau, \tilde{f}_2(t), \rho(t)$ and $t \le T$, (4.9) holds. The above statements mean that scheme (2.5) is of generalized stability with different indices for different values of δ.

5. The Convergence

This section is for the convergence of scheme (2.5). Clearly, $P_M\left(\frac{\partial u}{\partial \lambda}\right) = \frac{\partial}{\partial \lambda}(P_M u)$. Moreover by (2.4),

$$P_M(\Delta u(\lambda,\theta)) = -\sum_{|m|\le M}\sum_{n\ge|m|} n(n+1)\hat{u}_{m,n}Y_{m,n}(\lambda,\theta)$$

$$= \sum_{|m|\le M}\sum_{n\ge|m|} \hat{u}_{m,n}\Delta Y_{m,n}(\lambda,\theta) = \Delta(P_M u(\lambda,\theta)).$$

Let $\xi^{(M)} = P_M\xi$ and $\psi^{(M)} = P_M\psi$. Then (2.3) leads to

$$\begin{cases} (\xi_t^{(M)}(t), v) + (J(\xi^{(M)}(t) + \delta\tau\xi_t^{(M)}(t), \psi^{(M)}(t)), v) & \\ \qquad -2\Omega(\frac{\partial \psi^{(M)}}{\partial \lambda}(t), v) = \sum_{j=1}^{4}(\tilde{g}_j(t), v), & \forall v \in V_M, t \in \dot{R}_\tau, \\ -(\Delta\psi^{(M)}(t), v) = (\xi^{(M)}(t), v), & \forall v \in V_M, t \in R_\tau, \\ \mu(\psi^{(M)}(t)) = 0, & t \in R_\tau, \\ \xi^{(M)}(0) = P_M\xi_0 & \end{cases} \tag{5.1}$$

with

$$\tilde{g}_1(t) = \xi_t^{(M)}(t) - \frac{\partial \xi}{\partial t}(t), \qquad \tilde{g}_2(t) = \delta\tau J(\xi_t^{(M)}(t), \psi^{(M)}(t)),$$

$$\tilde{g}_3(t) = J(\xi^{(M)}(t) - \xi(t), \psi^{(M)}(t)), \quad \tilde{g}_4(t) = J(\xi(t), \psi^{(M)}(t) - \psi(t)).$$

Furthermore, let $\tilde{\xi} = \eta - \xi^{(M)}$ and $\tilde{\psi} = \varphi - \psi^{(M)}$. Then by (2.5) and (5.1),

$$\begin{cases} (\tilde{\xi}_t(t), v) + (J(\tilde{\xi}(t) + \delta\tau\tilde{\xi}_t(t), \psi^{(M)}(t) + \tilde{\psi}(t)), v) + (J(\xi^{(M)}(t) & \\ \qquad +\delta\tau\xi_t^{(M)}(t), \tilde{\psi}(t)), v) - 2\Omega(\frac{\partial \tilde{\psi}}{\partial \lambda}(t), v) = -\sum_{j=1}^{4}(\tilde{g}_j(t), v), & \forall v \in V_M, t \in \dot{R}_\tau, \\ -(\Delta\tilde{\psi}(t), v) = (\tilde{\xi}(t), v), & \forall v \in V_M, t \in R_\tau, \\ \mu(\tilde{\psi}(t)) = 0, & t \in R_\tau, \\ \tilde{\xi}(0) = 0. & \end{cases} \tag{5.2}$$

We know from (4.1) and (5.2) that for the convergence, we have to estimate only

$\|\tilde{g}_j(t)\|^2$. According to

$$\xi_t(t)-\frac{\partial\xi}{\partial t}(t)=\frac{1}{\tau}\int_t^{t+\tau}(t+\tau-t')\frac{\partial^2\xi}{\partial t^2}(t')dt',$$

we have

$$\tau\sum_{t'=0}^{t-\tau}\|\tilde{g}_1(t')\|^2\le c\tau^2\|\xi\|^2_{H^2(0,t;L^2(S))}.$$

Next, we have from Lemmas 3 and 5 that, for any $\beta>0$,

$$\|\tilde{g}_2(t)\|^2\le c\tau^2|||\psi^{(M)}|||^2_{2+\beta}|||\frac{\partial\xi^{(M)}}{\partial t}|||^2_1\le c\tau^2|||\psi|||^2_{2+\beta}|||\frac{\partial\xi}{\partial t}|||^2_1.$$

Since $\xi=-\Delta\psi$, we get

$$\|\tilde{g}_2(t)\|^2\le c\tau^2|||\xi|||^2_\beta|||\frac{\partial\xi}{\partial t}|||^2_1.$$

Furthermore, Lemmas 3 and 5 lead to

$$\|\tilde{g}_3(t)\|^2\le c\|\psi^{(M)}(t)\|^2_{2+\beta}\|\xi^{(M)}(t)-\xi(t)\|^2_1\le cM^{-2r}|||\psi|||^2_{2+\beta}|||\xi|||^2_{1+r}$$
$$\le cM^{-2r}|||\xi|||^2_\beta|||\xi|||^2_{1+r},$$

$$\|\tilde{g}_4(t)\|^2\le c\|\xi^{(M)}(t)\|^2_1\|\psi^{(M)}(t)-\psi(t)\|^2_{2+\beta}\le cM^{-2r}|||\xi|||^2_1|||\xi|||^2_{\beta+r}.$$

Therefore,

$$\tilde{\rho}(t)=\tau\sum_{j=1}^{4}\sum_{t'=0}^{t-\tau}\|\tilde{g}_j(t')\|^2\le c\tau^2\|\xi\|^2_{H^2(0,T;L^2(S))}+c\tau^2|||\xi|||^2_\beta\|\xi\|^2_{C(0,T;H^1(S))}$$
$$+cM^{-2r}|||\xi|||^2_{1+r}|||\xi|||^2_{\beta+r}.$$

Clearly, if $\tau=O(M^{-p})$, then $\tilde{\rho}(t)=O(M^{-2p}+M^{-2r})$. In particular, if $p>\frac{2+\beta}{3}$ and $r>1+\frac{\beta}{2}-\frac{p}{2}$, then $\tilde{\rho}(t)=o(M^{p-\beta-2})$.

By an argument similar to the proof of Theorem 1, we have the following conclusion.

Theorem 2. *Assume that*

(i) $\delta>\frac{1}{2}$ *and* $\tau=o(1)$, *or* $\delta\le\frac{1}{2}$, $\tau\le cM^{-p}$ *and* $p\ge 2$,

(ii) $\xi\in C(0,T;H^{1+r}(S))\bigcap C^1(0,T;H^1(S))\bigcap H^2(0,T;L^2(S))$ *and* $r>0$.

Then for all $t\le T$,

$$\|\tilde{\xi}(t)\|^2\le d_4(\tau^2+M^{-2r})$$

where d_4 *is a positive constant depending only on the norms of* ξ *in the spaces mentioned in the above.*

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