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HIGH ACCURACY FOR MIXES FINITE ELEMENT METHODS IN RAVIART-THOMAS ELEMENT*

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Abstract

This paper deals with Raviart-Thomas element ($Q_{2,1} \times Q_{1,2} - Q_1$ element). Apart from its global superconvergence property of fourth order, we prove that a postprocessed extrapolation can globally increased the accuracy by fifth order.

1. Introduction

We consider the mixed methods of the Neumann boundary value problem

$$\begin{aligned} \mathbf{p} + \nabla u &= 0 \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{p} &= f \quad \text{in } \Omega, \\ \mathbf{p}\cdot\mathbf{n} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset R^2$ is a bounded domain with boundaries parallel to axes, $\mathbf{n}$ is the outer unit normal to $\partial\Omega$. Denote

$$\mathbf{H}_0(\text{div}) = \{\mathbf{q} \in \mathbf{H}(\text{div}),\ \mathbf{q}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\},$$

then we can write the weak formulation of (1) as follows: Find $(u, \mathbf{p}) \in L^2(\Omega) \times \mathbf{H}_0(\text{div})$ such that

$$(\mathbf{p}, \mathbf{q}) - (u, \operatorname{div}\mathbf{q}) + (v, \operatorname{div}\mathbf{p}) = (f, v), \quad \forall (v, \mathbf{q}) \in L^2(\Omega) \times \mathbf{H}_0(\text{div}). \tag{2}$$

Let $V_h \times \mathbf{P}_h \subset L^2(\Omega) \times \mathbf{H}_0(\text{div})$ be a pair of finite element spaces with respect to T_h, a uniform rectangular mesh with the size $2h$. Then the mixed finite element approximation for (2) seeks $(u_h, \mathbf{p}_h) \in V_h \times \mathbf{P}_h$ such that

$$(\mathbf{p}_h, \mathbf{q}) - (u_h, \operatorname{div}\mathbf{q}) + (v, \operatorname{div}\mathbf{p}_h) = (f, v), \quad \forall (v, \mathbf{q}) \in V_h \times \mathbf{P}_h. \tag{3}$$

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Here we choose $V_h \times P_h$ as one of RT elements, i.e. $Q_{2,1} \times Q_{1,2} - Q_1$ element[3], which satisfies the BB-condition and is described as

$$\begin{cases} \mathbf{P}_h = \{\mathbf{q} \in \mathbf{H}_0(\mathrm{div}),\ \mathbf{q}|_e \in Q_{2,1}(e) \times Q_{1,2}(e), \quad \forall e \in T_h\}, \\ V_h = \{v \in L^2(\Omega),\ v|_e \in Q_1(e), \quad \forall e \in T_h\}, \end{cases} \tag{4}$$

where

$$Q_{m,n} = \mathrm{span}\{x^i y^j,\ 0 \le i \le m,\ 0 \le j \le n\}; \qquad Q_{m,m} = Q_m.$$

Some superconvergence results for RT element have been derived by Nakata, Weiser, Wheeler, Douglas, Milner, Wang, Ewing and Lazarov([?]-[?], [?]). The asymptotic expansion was also obtained for the lowest order RT element or $Q_{1,0} \times Q_{0,1} - Q_0$ element by Wang([?]). The aim of this paper is to obtain the global superconvergence of $O(h^4)$ and the postprocessed extrapolation result of $O(h^5)$ for $Q_{2,1} \times Q_{1,2} - Q_1$ element by using integral identity, which was created by Lin *et al*([?],[?]).

2. Global Superconvergence

For $e \in T_h$, we assume that (x_e, y_e) is the center of gravity, s_1 and s_3 of the width $2k$ are the edges along y-direction, s_2 and s_4 of the width $2h$ are the edges along x-direction. Then we can define interpolation operators j_h and i_h by

$$\begin{cases} j_h\mathbf{p}|_e \in Q_{2,1}(e) \times Q_{1,2}(e), \\ \int_{s_i} (\mathbf{p} - j_h\mathbf{p})\varphi \mathbf{n} ds = 0 \quad \forall \varphi \in P_1(s_i) \quad i = 1,2,3,4, \\ \int_e (\mathbf{p} - j_h\mathbf{p})\mathbf{q} = 0 \quad \forall \mathbf{q} \in P_1(y) \times P_1(x), \end{cases} \tag{5}$$

$$\int_e (u - i_h u) v = 0 \quad \forall v \in Q_1(e). \tag{6}$$

We immediately find from integration by parts that

$$(v, \mathrm{div}(\mathbf{p} - j_h\mathbf{p})) = 0 \quad \forall v \in V_h.$$

In fact, the projection j_h satisfying term above is Fortin's operator (see [?]) and in this paper it is locally defined. This definition can be also seen in [?] and [?]. i_h is the local L^2-projection operator. Since $\mathrm{div}\mathbf{q} \in V_h$, we can see that

$$(u - i_h u, \mathrm{div}\mathbf{q}) = 0, \quad \forall \mathbf{q} \in \mathbf{P}_h.$$

Lemma 1. *If* $\mathbf{p} \in [W^{5,r}(\Omega)]^2$, *then we have*

$$\begin{aligned} &(\mathbf{p}_h - j_h\mathbf{p}, \mathbf{q}) - (u_h - i_h u, div\mathbf{q}) + (div(\mathbf{p}_h - j_h\mathbf{p}), v) \\ = \ & \frac{2}{45} h^4 \int_\Omega (p_1)_{xxxx} q_1 + \frac{2}{45} k^4 \int_\Omega (p_2)_{yyyy} q_2 + h^5 r_h(\mathbf{p}, \mathbf{q}) \quad \forall (\mathbf{q}, v) \in \mathbf{P}_h \times V_h \end{aligned}$$

with

$$|r_h(\mathbf{p}, \mathbf{q})| \le c\|\mathbf{p}\|_{5,r} \|\mathbf{q}\|_{\mathrm{div},r'}, \quad \frac{1}{r} + \frac{1}{r'} = 1,\ 1 \le r, r' \le \infty.$$