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CONVERGENCE OF AN ALTERNATING A-ϕ SCHEME FOR QUASI-MAGNETOSTATIC EDDY CURRENT PROBLEM *1)

Chang-feng Ma
(*Department of Mathematics, Huazhong University of Science and Technology, Wuhan 430074, China*)
(*Department of Computational Science and Mathematics, Guilin University of Electronic Technology, Guilin 541004, China*)

Abstract

We propose in this paper an alternating $\boldsymbol{A}$-ϕ method for the quasi-magnetostatic eddy current problem by means of finite element approximations. Bounds for continuous and discrete error in finite time are given. And it is verified that provided the time step τ is sufficiently small, the proposed algorithm yields for finite time T an error of $\mathcal{O}(h+\tau^{1/2})$ in the L^2-norm for the magnetic field $\boldsymbol{H}(=\mu^{-1}\nabla\times\boldsymbol{A})$, where h is the mesh size, μ the magnetic permeability.

Mathematics subject classification: 65N30.
Key words: Eddy current problem, Alternating $\boldsymbol{A}$-ϕ method, Finite element approximation, Error estimates.

1. Introduction

The quasi-magnetostatic eddy current model arises from Maxwell's equations as an approximation by neglecting the displacement current (see [1]-[4], [6]-[7], [10]-[11]):

$$\nabla\times\boldsymbol{E}+\mu\frac{\partial\boldsymbol{H}}{\partial t}=0, \tag{1.1}$$

$$\nabla\times\boldsymbol{H}=\boldsymbol{J}=\sigma\boldsymbol{E}+\boldsymbol{J}_s, \tag{1.2}$$

where μ is the magnetic permeability of the solution domain and σ the spatially varying electrical conductivity, and $\boldsymbol{J}_s$ is source electric currents density. A constitutive equation

$$\boldsymbol{B}=\mu\boldsymbol{H}$$

relates the magnetic induction and magnetic field vectors. The divergence-free conditions

$$\nabla\cdot\boldsymbol{B}=0 \ \text{ and } \ \nabla\cdot\boldsymbol{J}=0$$

are also imposed, indicating no point sources or sinks of electric current or magnetic induction exists inside the solution domain Ω. This is reasonable for low-frequency, high-conductivity applications like electrical machines. A number of different formulations have proposed [2, 3, 4, 6, 7, 8, 14, 15]. We consider in this paper the above eddy-current model (1.1)-(1.2) by introducing the magnetic vector potential $\boldsymbol{A}=\boldsymbol{A}(\boldsymbol{x},t)$ and the electrical scalar potential $\phi=\phi(\boldsymbol{x},t)$ as primary unknown.

As a matter of fact, the magnetic $\boldsymbol{H}$ can be expressed, in light of $\nabla\cdot\boldsymbol{B}=0$ and $\boldsymbol{B}=\mu\boldsymbol{H}$, as follows

$$\boldsymbol{H}=\frac{1}{\mu}\nabla\times\boldsymbol{A}. \tag{1.3}$$

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Combining equations (1.1) and (1.3) we have

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla \phi. \tag{1.4}$$

Thus, in term of the $\boldsymbol{A}$-ϕ potentials, equation (1.2) becomes the following **curl-curl** equation:

$$\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \nabla \times (\upsilon \nabla \times \boldsymbol{A}) + \sigma \nabla \phi = \boldsymbol{J}_s. \tag{1.5}$$

Here υ is the inverse of the magnetic permeability μ (magnetic susceptibility). To maintain a divergence-free current density $\boldsymbol{J}$, the following auxiliary equation

$$\nabla \cdot (\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla \phi - \boldsymbol{J}_s) = 0 \tag{1.6}$$

must be solved simultaneously with equation (1.5).

In the following context we shall concentrate our attention on the finite element error analysis of the following initial-boundary value problem of equations (1.5) and (1.6):

$$\begin{cases} \sigma \frac{\partial \boldsymbol{A}}{\partial t} + \nabla \times (\upsilon \nabla \times \boldsymbol{A}) + \sigma \nabla \phi = \boldsymbol{J}_s, \ \Omega \times [0, T], \\ \nabla \cdot (\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \sigma \nabla \phi - \boldsymbol{J}_s) = 0, \ \Omega \times [0, T], \\ \boldsymbol{A} \times \boldsymbol{n} = \boldsymbol{0}, \quad \partial \Omega \times [0, T], \\ \boldsymbol{A}(\cdot, 0) = \boldsymbol{A}_0, \quad \Omega. \end{cases} \tag{1.7}$$

Here $\Omega \subset \mathbb{R}^3$ is a sufficiently smooth simply-connected and bounded polyhedral computational domain with boundary $\Gamma = \partial\Omega$ and $\boldsymbol{n}$ the unit normal vector to Γ. Though the equations are initially posed on the entire space $\mathbb{R}^3$, we can switch to a bounded domain by introducing an artificial boundary sufficiently removed from the region of interest. This is commonplace in engineering simulations (see [9]).

For the sake of simplicity, we confine ourselves to in this paper linear isotropic, that is, $\upsilon \in L^\infty(\Omega)$ is a bounded uniformly positive scalar function of the spatial variable $\boldsymbol{x} \in \Omega$ only. Hence, for some $\underline{\upsilon}$, $\bar{\upsilon} > 0$ holds $0 < \underline{\upsilon} \le \upsilon \le \bar{\upsilon}$ a.e. in Ω and conductivity $\sigma \in L^\infty(\Omega)$ holds $\sigma \ge 0$ a.e. in Ω.

It is important to note that the magnetic vector potential $\boldsymbol{A}$ lacks physical meaning. The really interesting quantity is the magnetic field $\boldsymbol{H} = \upsilon \nabla \times \boldsymbol{A}$. This is reason that we can use an un-gauged formulation as in (1.7), which does not impose a constraint on $\nabla \cdot \boldsymbol{A}$ on the solution domain Ω. Obviously, this forfeits uniqueness of the solution in parts of the domain where $\sigma = 0$, but the solution for $\boldsymbol{H}$ remains unique everywhere.

The $\boldsymbol{A}$-ϕ method by means of finite element approximation has been applied in the magnetostatic eddy current computation far and wide in the last two decades. The numerical results indicate that the method is a fairly valid one simulating the quasi-magnetostatic eddy current model. It is worth our while mentioning that, however, the literature on the error estimates of this method can so far not be found yet. We will in our paper devote ourselves to finite element error analysis of the proposed so-called decoupled $\boldsymbol{A}$-ϕ scheme. It is shown that provided the time step τ is small enough, the proposed algorithm yields for finite time T an error of $\mathcal{O}(h + \tau^{1/2})$ in the L^2-norm for the magnetic field $\boldsymbol{H}(= \upsilon \nabla \times \boldsymbol{A})$, where h is the mesh size.

The contents of this paper is organized as follows. In section 2, we describe the decoupled $\boldsymbol{A}$-ϕ scheme in detail. We give two forms of the method: semi-discrete finite element implicit scheme in time first and fully discrete finite element approximation. Section 3 devotes to the error estimates of the magnetic field $\boldsymbol{H}(= \upsilon \nabla \times \boldsymbol{A})$ and electrical field $\boldsymbol{E}(= -\boldsymbol{A}_t - \nabla \phi)$ with appropriate regularity assumptions.