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ITERATIVE ℓ_1 MINIMIZATION FOR NON-CONVEX COMPRESSED SENSING*

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Abstract

An algorithmic framework, based on the difference of convex functions algorithm (D-CA), is proposed for minimizing a class of concave sparse metrics for compressed sensing problems. The resulting algorithm iterates a sequence of ℓ_1 minimization problems. An exact sparse recovery theory is established to show that the proposed framework always improves on the basis pursuit (ℓ_1 minimization) and inherits robustness from it. Numerical examples on success rates of sparse solution recovery illustrate further that, unlike most existing non-convex compressed sensing solvers in the literature, our method always outperforms basis pursuit, no matter how ill-conditioned the measurement matrix is. Moreover, the iterative ℓ_1 (IL_1) algorithm lead by a wide margin the state-of-the-art algorithms on $\ell_{1/2}$ and logarithimic minimizations in the strongly coherent (highly ill-conditioned) regime, despite the same objective functions. Last but not least, in the application of magnetic resonance imaging (MRI), IL_1 algorithm easily recovers the phantom image with just 7 line projections.

Mathematics subject classification: 90C26, 65K10, 49M29.
Key words: Compressed sensing, Non-convexity, Difference of convex functions algorithm, Iterative ℓ_1 minimization.

1. Introduction

Compressed sensing (CS) techniques [5, 6, 8, 17] enable efficient reconstruction of a sparse signal under linear measurements far less than its physical dimension. Mathematically, CS aims to recover an n-dimensional vector $\bar{x} \in \mathbb{R}^n$ with few non-zero components from an under-determined linear system $Ax = A\bar{x}$ of just $m \ll n$ equations, where $A \in \mathbb{R}^{m\times n}$ is a known measurement matrix. The first CS technique is the convex ℓ_1 minimization or the so-called basis pursuit [15]:

$$\min_{x\in\mathbb{R}^n} \|x\|_1 \quad \text{s.t.} \quad Ax = A\bar{x}. \tag{1.1}$$

Breakthrough results [8] have established that when matrix A satisfies certain restricted isometry property (RIP), the solution to (1.1) is exactly $\bar{x}$. It was shown that with overwhelming probability, several random ensembles such as random Gaussian, random Bernoulli, and random partial Fourier matrices, are of RIP type [8,13,32]. Note that (1.1) is just a minimization principle rather than an algorithm for retrieving $\bar{x}$. Algorithms for solving (1.1) and its associated

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ℓ_1 regularization problem [36]:

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|Ax-b\|^2+\lambda\|x\|_1 \tag{1.2}$$

include Bregman methods [24, 43], alternating direction algorithms [3, 18, 40], iterative thresholding methods [1, 14] among others [25].

Inspired by the success of basis pursuit, researchers then began to investigate various non-convex CS models and algorithms. More and more empirical studies have shown that non-convex CS methods usually outperform basis pursuit when matrix A is RIP-like, in the sense that they require fewer linear measurements to reconstruct signals of interest. Instead of minimizing ℓ_1 norm, it is natural to consider minimization of non-convex (concave) sparse metrics, for instance, ℓ_q (quasi-)norm ($0<q<1$) [11, 12, 27], capped-ℓ_1 [30, 45], and transformed-ℓ_1 [28, 44]. Another category of CS methods in spirit rely on support detection of $\bar{x}$. To name a few, there are orthogonal matching pursuit (OMP) [37], iterative hard thresholding (IHT) [2], (re)weighted-ℓ_1 scheme [7], iterative support detection (ISD) [38], and their variations [26,31,46].

On the other hand, it has been proved that even if A is not RIP-like and contains highly correlated columns, basis pursuit still enables sparse recovery under certain conditions of $\bar{x}$ involving its support [4]. In this scenario, most of the existing non-convex CS methods, however, are not that robust to the conditioning of A, as suggested by [41]. Their success rates will drop as columns of A become more and more correlated. In [41], based on the difference of convex functions algorithm (DCA) [34,35], the authors propose DCA-ℓ_{1-2} for minimizing the difference of ℓ_1 and ℓ_2 norms [19, 42]. Extensive numerical experiments [29, 30, 41] imply that DCA-ℓ_{1-2} algorithm consistently outperforms ℓ_1 minimization, irrespective of the conditioning of A.

Stimulated by the empirical evidence found in [29, 30, 41], we propose a general DCA-based CS framework for the minimization of a class of concave sparse metrics. More precisely, we consider the reconstruction of a sparse vector $\bar{x}\in\mathbb{R}^n$ by minimizing sparsity-promoting metrics:

$$\min_{x\in\mathbb{R}^n} P(|x|) \quad \text{s.t.} \quad Ax=A\bar{x}. \tag{1.3}$$

Throughout the paper, we assume that $P(x)$ always takes the form $\sum_{i=1}^n p(x_i)$ unless otherwise stated, where p defined on $[0,+\infty)$ satisfies:

- p is concave and increasing.
- p is continuous with the right derivative $p'(0+)>0$.

The first condition encourages zeros in $|x|$ rather than small entries, since p changes rapidly around the origin; the second one is imposed for the good of the proposed algorithm, as will be seen later. A number of sparse metrics in the literature enjoy the above properties, including smoothly clipped absolute deviation (SCAD) [20], capped-ℓ_1, transformed-ℓ_1, and of course ℓ_1 itself. Although ℓ_q ($q\in(0,1)$) and logarithm functional do not meet the second condition, their smoothed versions $p(t)=(t+\varepsilon)^q$ and $p(t)=\log(t+\varepsilon)$ are differentiable at zero. These proposed properties will be essential in the algorithm design as well as in the proof of main results.

Our proposed algorithm calls for solving a sequence of minimization subproblems. The objective of each subproblem is $\|x\|_1$ plus a linear term, which is convex and tractable. We further validate robustness of this framework, by showing theoretically and numerically that it performs at least as well as basis pursuit in terms of uniform sparse recovery, independent of the conditioning of A and sparsity metric.