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A MIXED FINITE ELEMENT METHOD FOR MULTI-CAVITY COMPUTATION IN INCOMPRESSIBLE NONLINEAR ELASTICITY*

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Abstract

A mixed finite element method combining an iso-parametric Q_2-P_1 element and an iso-parametric P_2^+-P_1 element is developed for the computation of multiple cavities in incompressible nonlinear elasticity. The method is analytically proved to be locking-free and convergent, and it is also shown to be numerically accurate and efficient by numerical experiments. Furthermore, the newly developed accurate method enables us to find an interesting new bifurcation phenomenon in multi-cavity growth.

Mathematics subject classification: 65N12, 65N30, 74B20, 74G15, 74M99.
Key words: Multiple cavitation computation, Incompressible nonlinear elasticity, Mixed finite element method, Locking-free, Convergent.

1. Introduction

Cavitation phenomenon, which exhibits sudden dramatic growth of pre-existing small voids under loads exceeding certain criteria, is first systematically modeled and analyzed by Gent & Lindley [1] in 1958. It is considered one of the most important failure phenomenon in nonlinear elasticity, and its better understanding is crucial to explore the properties of elastic materials.

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be a simply connected domain with sufficiently smooth boundary $\partial\Omega$, and let $B_{\rho_k}(\boldsymbol{x}_k) = \{\boldsymbol{x} \in \mathbb{R}^n : |\boldsymbol{x}-\boldsymbol{x}_k| < \rho_k\}$ and $\bigcup_{k=1}^K B_{\rho_k}(\boldsymbol{x}_k) \subset \Omega$. Let $\Omega_\rho = \Omega \backslash \bigcup_{k=1}^K B_{\rho_k}(\boldsymbol{x}_k)$ be the domain occupied by an elastic body in its reference configuration, where $B_{\rho_k}(\boldsymbol{x}_k)$ denotes the pre-existing defects of radii $\rho_k \ll 1$ centered at $\boldsymbol{x}_k$, $k = 1, \cdots, K$. Then, in incompressible elastic materials, the multi-cavitation problem can be expressed as to find a deformation $\boldsymbol{u}$ to minimize the total energy

$$E(\boldsymbol{u}) = \int_{\Omega_\rho} W(\nabla \boldsymbol{u}(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}, \tag{1.1}$$

in the set of admissible deformation functions

$$\mathcal{A}_I = \Big\{\boldsymbol{u} \in W^{1,s}(\Omega_\rho; \mathbb{R}^n) \text{ is 1-to-1 a.e.} : \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{u}_0,\ \det\nabla\boldsymbol{u} = 1, \text{ a.e.}\Big\}, \tag{1.2}$$

where $W : M_+^{n\times n} \to \mathbb{R}^+$ is the stored energy density function of the material with $M_+^{n\times n}$ being the set of $n \times n$ matrices of positive determinant, and $n - 1 < s < n$ is a given Sobolev index, and where a displacement boundary condition $\boldsymbol{u} = \boldsymbol{u}_0$ is imposed on $\partial_D\Omega_\rho = \partial\Omega$, and a traction free boundary condition is imposed on $\partial_N\Omega_\rho = \cup_{k=1}^K \partial B_{\rho_k}(\boldsymbol{x}_k)$. Without loss of generality, we consider a typical energy density for nonlinear elasticity given as

$$W(F) = \mu|F|^s + d(\det F), \quad \forall F \in M_+^{n\times n}, \tag{1.3}$$

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where μ is material parameter, $|\cdot|$ denotes the Frobenius norm of a matrix, and $d(\det F) = \kappa(\det F - 1)^2 + d_1(\det F)$ with $\kappa > 0$ and $d_1 : \mathbb{R}_+ \to \mathbb{R}_+$ being a strictly convex function satisfying

$$d_1(\xi) \to +\infty \quad \text{as } \xi \to 0, \quad \text{and} \quad \frac{d_1(\xi)}{\xi} \to +\infty \quad \text{as } \xi \to +\infty. \tag{1.4}$$

Notice that, even though for incompressible nonlinear elastic materials, $d(\cdot)$ is just a constant as the determinant of any admissible deformation in $\mathcal{A}_I$ equals 1 a.e., the term plays an important role in the proof of the convergence of the numerical cavitation solutions to the mixed formulation given below.

To relax the rather restrictive condition $\det \nabla \boldsymbol{u} = 1$, a.e. appeared in $\mathcal{A}_I$, a mixed formulation of the following form (see [2,8]) is usually used in computation:

$$(\boldsymbol{u}, p) = \arg \inf_{\boldsymbol{u} \in \mathcal{A}} \sup_{p \in L^2(\Omega_\rho)} E(\boldsymbol{u}, p), \tag{1.5}$$

where $p \in L^2(\Omega_\rho)$ is a pressure like Lagrangian multiplier introduced to relax the constraint of incompressibility, and where the Lagrangian functional $E(\boldsymbol{u}, p)$ and the set of admissible deformations $\mathcal{A}$ are defined as

$$E(\boldsymbol{u}, p) = \int_{\Omega_\rho} \left(W(\nabla \boldsymbol{u}) - p\,(\det \nabla \boldsymbol{u} - 1) \right) \mathrm{d}\boldsymbol{x}, \tag{1.6}$$

$$\mathcal{A} = \Big\{ \boldsymbol{u} \in W^{1,s}(\Omega_\rho; \mathbb{R}^n) \text{ is 1-to-1 a.e., } \boldsymbol{u}|_{\partial_D \Omega_\rho} = \boldsymbol{u}_0 \Big\}. \tag{1.7}$$

The nonlinear saddle point problem (1.6)-(1.7) with energy density (1.3) leads to the mixed displacement/traction boundary value problem of the Euler-Lagrange equation:

$$\operatorname{div}\big(D_F W(\nabla \boldsymbol{u}) - p \operatorname{cof} \nabla \boldsymbol{u}\big) = 0, \qquad \text{in } \Omega_\rho, \tag{1.8}$$

$$\det \nabla \boldsymbol{u} = 1, \qquad \text{in } \Omega_\rho, \tag{1.9}$$

$$\big(D_F W(\nabla \boldsymbol{u}) - p \operatorname{cof} \nabla \boldsymbol{u}\big)\boldsymbol{n} = \boldsymbol{0}, \qquad \text{on } \cup_{k=1}^K \partial B_{\rho_k}(\boldsymbol{x}_k), \tag{1.10}$$

$$\boldsymbol{u} = \boldsymbol{u}_0, \qquad \text{on } \partial_D \Omega_\rho. \tag{1.11}$$

where $\operatorname{cof} \nabla \boldsymbol{u}$ denotes the cofactor matrix of $\nabla \boldsymbol{u}$.

One of the main difficulties of numerical cavitation computation comes from the very large anisotropic deformation near the cavities, which, if not properly approximated, can cause mesh entanglement corresponding to nonphysical material interpenetration. In recent years, successful quadratic iso-parametric and dual-parametric finite element methods have been developed for the cavitation computation for compressible nonlinear elastic materials [3–6]. However, direct application of these methods to the case of incompressible elasticity generally encounters the barrier of the locking effect.

In the present paper, we develop a mixed finite element method combining an iso-parametric Q_2-P_1 element and an iso-parametric P_2^+-P_1 element for the computation of multiple cavities in incompressible nonlinear elasticity. A damped Newton method is applied to solve the discrete Euler-Lagrange equation. The method is proved to be stable, locking free and convergent under some reasonable assumptions. Our numerical experiments also show that the method is numerically efficient. Furthermore, the newly developed accurate method enable us to find an interesting new bifurcation phenomenon in multi-cavity growth. It is worth mentioning here, if the displacement boundary condition (1.11) is replaced by a traction boundary condition, the results of this paper still hold, and the proof is essentially the same.