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STABLE BOUNDARY CONDITIONS AND DISCRETIZATION FOR P_{N} EQUATIONS*

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Abstract

A solution to the linear Boltzmann equation satisfies an energy bound, which reflects a natural fact: The energy of particles in a finite volume is bounded in time by the energy of particles initially occupying the volume augmented by the energy transported into the volume by particles entering the volume over time. In this paper, we present boundary conditions (BCs) for the spherical harmonic (P_{N}) approximation, which ensure that this fundamental energy bound is satisfied by the P_{N} approximation. Our BCs are compatible with the characteristic waves of P_{N} equations and determine the incoming waves uniquely. Both, energy bound and compatibility, are shown on abstract formulations of P_{N} equations and BCs to isolate the necessary structures and properties. The BCs are derived from a Marshak type formulation of BC and base on a non-classical even/odd-classification of spherical harmonic functions and a stabilization step, which is similar to the truncation of the series expansion in the P_{N} method. We show that summation by parts (SBP) finite differences on staggered grids in space and the method of simultaneous approximation terms (SAT) allows to maintain the energy bound also on the semi-discrete level.

Mathematics subject classification: 35B35, 35Q20, 35L50, 65M06, 65M12, 65M70.
Key words: Boundary conditions, Energy stability, Spherical harmonic (P_{N}) approximation, Kinetic theory, Moment method, Boltzmann, Linear transport.

1. Linear Boltzmann and Radiative Transfer Equation

The transport of particles in a background medium is governed by the *linear Boltzmann equation* [1]. In the absence of sources and absorption it takes the form

$$\frac{1}{|v(\epsilon)|}\partial_t\psi(t,\epsilon,x,\Omega)+\Omega\cdot\nabla_x\psi(t,\epsilon,x,\Omega)=\mathcal{Q}(t,\epsilon,x)\left[\psi(t,\epsilon,x,\Omega)\right], \tag{1.1}$$

where the unknown $\psi(t,\epsilon,x,\Omega)$ is the number density of particles, with respect to the measure $d\epsilon d\Omega dx$, located at x and moving in direction $\Omega\in S^2$ with energy ϵ at time t. $|v(\epsilon)|$ is the absolute velocity of a particle with energy ϵ. The scattering operator $\mathcal{Q}$ describes the change in time due to angular deflections and energy-loss as particles interact with the background medium through collisions.

Two situations that are of particular practical relevance allow to consider models of the same structure but reduced phase space.

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1. When the energy of particles is fixed we can omit the energy variable and obtain the *radiative transfer equation* [2] (RT)

$$\partial_t\psi(t,x,\Omega) + \Omega\cdot\nabla_x\psi(t,x,\Omega) = \underbrace{\int_{S^2} \sigma_s(x,\Omega'\cdot\Omega)\psi(t,x,\Omega')\,d\Omega' - \sigma_t(x)\psi(t,x,\Omega)}_{:=Q^{RT}(x)\psi(t,x,\Omega)}, \quad (1.2)$$

   where σ_s is a non-negative scattering cross-section consisting of the density of scattering centers $N_V(x)$ in the background medium such that $\sigma_s(x,\cdot)/N_V(x)$ is the probability density of the angular deflection; σ_t is the respective total cross-section. Note that σ_s, and thereby also σ_t, can depend on time.

2. Many situations allow to neglect the time dependency and it can be assumed that particles loose a significant amount of energy only by a sequence of collisions with each collision changing the energy only slightly. In these situations one usually describes the particle system in terms of the particle fluence $\hat{\psi}(x,\epsilon,\Omega) := |v(\epsilon)|\psi(x,\epsilon,\Omega)$ and employs an evolution equation in energy space called *Boltzmann equation in continuous slowing down approximation* [3] (BCSD)

$$\begin{aligned} &-\partial_\epsilon\Big(S(\epsilon,x)\hat{\psi}(\epsilon,x,\Omega)\Big) + \Omega\cdot\nabla_x\,\hat{\psi}(\epsilon,x,\Omega) \\ &= \underbrace{\int_{S^2} \tilde{\sigma}_s(\epsilon,x,\Omega'\cdot\Omega)\,\hat{\psi}(\epsilon,x,\Omega')\,d\Omega' - \tilde{\sigma}_t(\epsilon,x)\,\hat{\psi}(\epsilon,x,\Omega)}_{:=Q^{CSD}(\epsilon,x)\hat{\psi}(\epsilon,x,\Omega)}, \end{aligned} \quad (1.3)$$

   where the stopping power S describes the average energy loss per distance traveled, $\tilde{\sigma}_s$ denotes a scattering cross-section involving elastic and inelastic collisions and $\tilde{\sigma}_t$ is the joint total scattering cross-section of elastic and inelastic collisions.
   Note that through the variable transformation $\epsilon(t) = \epsilon_{\max} - t$ the BCSD can be transformed into a pseudo equation of radiation transport by setting the stopping power S to one.

From the viewpoint of this work, all three equations (Eqs. (1.1) to (1.3)) are of the same structure. We will focus on the equation for radiative transfer in the following analysis.

The boundary conditions for radiative transfer are of Dirichlet type and prescribe the incoming half of the particle distribution at the boundary:

$$\psi(t,x,\Omega) \overset{!}{=} \psi_{in}(t,x,\Omega) \qquad \forall\Omega\in S^2 : n\cdot\Omega < 0, \quad (1.4)$$

where n denotes the outward-pointing normal vector at $x\in\partial G$ and, with $(\partial G\times S^2)_- := \{(x,\Omega)\in\partial G\times S^2 : n\cdot\Omega<0\}$, $\psi_{in} : \mathbb{R}_+\times(\partial G\times S^2)_- \to \mathbb{R}_+$ is a given distribution of incoming particles. It is common sense, that the energy of particles in a bounded domain $G\subset\mathbb{R}^3$ at time T is bounded from above by the energy of the particles that initially ($t=0$) occupied the domain augmented by the energy of the particles that entered through the boundary. This natural fact is mimicked by the following *energy bound* for a solution ψ of radiative transfer equation (1.2), see also [4, 5]

$$\|\psi(T,\cdot)\|^2_{L^2(G\times S^2)} \le \|\psi_0\|^2_{L^2(G\times S^2)} + \|\psi_{\text{in}}\|^2_{L^2((0,T)\times(\partial G\times S^2)_-)}. \quad (1.5)$$