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MONOTONICITY CORRECTIONS FOR NINE-POINT SCHEME OF DIFFUSION EQUATIONS*

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Abstract

In this paper, we present a nonlinear correction technique to modify the nine-point scheme proposed in [SIAM J. Sci. Comput., 30:3 (2008), 1341–1361] such that the resulted scheme preserves the positivity. We first express the flux by the cell-centered unknowns and edge unknowns based on the stencil of the nine-point scheme. Then, we use a nonlinear combination technique to get a monotone scheme. In order to obtain a cell-centered finite volume scheme, we need to use the cell-centered unknowns to locally approximate the auxiliary unknowns. We present a new method to approximate the auxiliary unknowns by using the idea of an improved multi-points flux approximation. The numerical results show that the new proposed scheme is robust, can handle some distorted grids that some existing finite volume schemes could not handle, and has higher numerical accuracy than some existing positivity-preserving finite volume schemes.

Mathematics subject classification: 52B10, 65D18, 68U05, 68U07.
Key words: Monotonicity corrections, Diffusion equation, Improved MPFA, Distorted meshes.

1. Introduction

In the numerical simulation of inertial confinement fusion, reservoir simulation and astrophysics, we often need to numerically solve the diffusion equation on the distorted meshes. To avoid non-physical oscillations in the numerical solution, we need to choose the positivity-preserving schemes. It was shown in [21] that there is no locally conservative, unconditionally positivity-preserving, linear nine-point scheme such that the discretization has a second-order accuracy and exactly reproduces the linear solution on the distorted meshes. To get a monotone scheme, some pre- and post-processing methods are proposed in [1, 4, 12, 17, 18, 20, 23, 30–32].

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On the other hand, Le Potier presents a nonlinear monotone finite volume scheme for time-dependent anisotropic diffusion problems on unstructured triangular meshes [13]. As far as we know, there have many papers so far, e.g. [3, 5, 10, 15, 19, 22, 25–27, 34], devoted to positivity-preserving nonlinear finite volume schemes to solve diffusion equations on distorted meshes. Besides, the nonlinear finite volume schemes which satisfy the stricter requirement – the discrete maximum principle, have been discussed in [2, 6, 7, 9, 14, 16, 28].

Radiation diffusion calculation occupies an important position in solving actual radiation fluid mechanics problems. In the calculation of multi-medium Lagrange radiation fluid, the flow of fluid will cause the distortion of the grid. Triangular meshes have good adaptability to the complex calculation areas, so they are often used in Lagrange radiation fluid calculations. However, the previously proposed positive-preserving finite volume schemes [25–27, 34] cannot handle highly distorted triangular meshes well, such as the triangular Kershaw meshes showed in Section 4.1. In this paper, we will propose a new nonlinear positive-preserving finite volume scheme, which can handle highly distorted triangular meshes better.

The monotone schemes in [25, 27, 34] adaptively select discrete templates, which can adapt to various large deformed meshes. However, when a certain cell has a large degree of distortion, the expression of discrete flux on some edge of the cell may not include the physical quantity on the edge. In this case, although the discrete flux design is well adapted to the geometric deformation of the cell, it fails to directly reflect the change of physical quantities on some edge of the cell, which may affect the discrete accuracy of the discrete normal flow. Besides, the expression of the discrete flux proposed in [26] contains the unknown at the midpoint of the edge. Hence, the discrete flux design can directly reflect the change of physical quantities on the edge. However, the construction process of the scheme is relatively complicated, and involves the elimination of two types of auxiliary unknowns: the vertex unknowns and the edges unknowns.

In this paper, we construct a linear flux on each cell-edge as [24, 33], which contains the unknown at the midpoint of the edge. And then, we deal with the tangential difference along the edge in the discrete flux to get a new nonlinear expression of the discrete flux that includes the cell-centered unknown and some edge unknowns. The construction process of the new proposed scheme is relatively simple, and the new expression of the discrete flux is not only suitable for the distortion of the mesh, but also directly reflects the change of the physical quantity on the edge.

The auxiliary edge unknowns should be locally approximated with the surrounding cell-centered unknowns. For a mesh with a small degree of distortion, we can use the method in [25] to approximate the auxiliary edge unknowns. However, it is found through numerical experiments that the absolute values of the interpolation coefficients obtained by the method in [25] on some distorted triangular meshes are often large, resulting in an unstable scheme. We present a new method to approximate the auxiliary edge unknowns inspired by the idea of an improved multi-points flux approximation.

However, the approximate auxiliary unknowns obtained by this new method may be negative even if the surrounding cell-centered unknowns are non-negative. We use an idea similar to [26] to assure the resulting nonlinear scheme is monotone by introducing two non-negative parameters when constructing the conservative flux. The new proposed scheme can deal with some distorted grids that the previous finite volume schemes could not handle well, such as the triangular Kershaw meshes.

The article is organized as follows. In Section 2, we introduce the nonlinear correction technique to modify the nine-point scheme. In Section 3, we give a new approach to eliminate the