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A FINITE DIFFERENCE METHOD FOR TWO DIMENSIONAL ELLIPTIC INTERFACE PROBLEMS WITH IMPERFECT CONTACT*

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Abstract

In this paper two dimensional elliptic interface problem with imperfect contact is considered, which is featured by the implicit jump condition imposed on the imperfect contact interface, and the jumping quantity of the unknown is related to the flux across the interface. A finite difference method is constructed for the 2D elliptic interface problems with straight and curve interface shapes. Then, the stability and convergence analysis are given for the constructed scheme. Further, in particular case, it is proved to be monotone. Numerical examples for elliptic interface problems with straight and curve interface shapes are tested to verify the performance of the scheme. The numerical results demonstrate that it obtains approximately second-order accuracy for elliptic interface equations with implicit jump condition.

Mathematics subject classification: 65N06, 65B99.
Key words: Finite difference method, Elliptic interface problem, Imperfect contact.

1. Introduction

We consider the elliptic interface problem with imperfect contact

$$\begin{cases} -\nabla\cdot\beta(x,y)\nabla u(x,y)+c(x,y)u(x,y)=f(x,y), & x\in\Omega\backslash\Gamma, \\ u(x,y)=g(x,y), & x\in\partial\Omega \end{cases} \tag{1.1}$$

together with the following implicit jump conditions across the interface Γ:

$$\begin{cases} [u]=u^+-u^-=\lambda\beta^+\nabla u^+\cdot\vec{n}, \\ \left[\beta\dfrac{\partial u}{\partial\vec{n}}\right]=\beta^+\nabla u^+\cdot\vec{n}-\beta^-\nabla u^-\cdot\vec{n}=0, \end{cases} \tag{1.2}$$

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where $u^{\pm}(x,y) = u(x,y)|_{\Omega^{\pm}}$ and $\vec{n}$ is a unit normal to the interface pointing from Ω^- to Ω^+. Without loss of generality, we assume that $\Omega \subset \mathbb{R}^2$ is a rectangular domain, and the interface Γ is a smooth straight or curve line which separates Ω into two sub-domains Ω^- and Ω^+.

The interface problem (1.1)-(1.2) arises in many important scientific and engineering applications. Examples include incompressible two-phase flow with surface tension featuring jumps in pressure and pressure gradient across the interface [29, 31], temperature discontinuity between gas and cooling solid surface [13], heat conduction between materials of different heat capacity and conductivity and interface diffusion process [24, 32]. It is also applied to model the conjugate heat transfer problem in thermodynamic processes between materials that are thermally coupled through non-adiabatic contacts [9], heat transfer in composite media, heat transfer in building [36], transient behavior for the thermoelastic contact of two rods of dissimilar materials [1]. Moreover, the dielectric heat conduction problem of solid spherical particles dispersed in the continuous phase [25], the calculation of temperature distribution in multi-layer thin film structures [30], e.g. X-ray lithography, laser annealing, laser processing, etc, can also be described by the elliptic interface problem.

The solution of the elliptic interface problems is often discontinuous due to discontinuous coefficients or singular sources across the interface. Various numerical methods are provided for solving these kind of problems. According to the geometric relationship between the computational grid and the material interface, the numerical methods can be generally divided into two categories:

(1) The interface fitted mesh methods [2, 5, 11, 20, 39]. This kind of method features by the computational mesh fits the interface, it means that an element of the underlying mesh is required to intersect with the interface only through its boundaries. This approach is beneficial for the numerical scheme to reach optimal convergence. However, when the geometry is complex, this usually leads to a nontrivial interface meshing problem. Another disadvantage of the fitted mesh is encountered when solving moving interface problems. Since the interface is moving, a new fitted mesh has to be generated at each time step and an interpolation is required to transfer the numerical solutions solved on different meshes.

(2) The interface unfitted mesh method. In this second approach, the interface is allowed to cut computational cells. One difficulty is that special treatment needs to be introduced on these elements in which the interface pass through. And the conditioning of the resulting linear system has a strong dependence on how the interface cuts the mesh cells [4]. There are many interface unfitted mesh methods, such as, the immersed interface method [8, 19, 21, 26, 33], the immersed finite volume method [3,6,23,35,40], and the immersed finite element method [7, 12, 15, 22, 37].

To summarize, elliptic interface problem has been well-studied and can be well solved using finite difference, finite element and finite volume type methods. However, there are significant differences at the connection conditions imposed along the interface. Usually the continuity of the temperature in addition to the conservation of the conductive heat flux are imposed on the interfaces and are referred to as the continuity interface conditions [12, 22] (known as perfect contact or the homogeneous jump interface conditions, $[u] = 0$, $[\beta\partial u/\partial\vec{n}] = 0$). Besides, the imperfect contact condition with nonhomogeneous jump conditions are frequently imposed along the interface. On this type of interface, the jumps of the temperature as well as the conductive heat flux along the interface are known explicitly [15, 21, 37], (say $[u] = g_1$, $[\beta\partial u/\partial\vec{n}] = g_2$, with known g_1 and g_2). There is another imperfect interface condition [5, 17, 36], which has