Journal of Computational Mathematics
Vol.42, No.6, 2024, 1502–1525.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2305-m2023-0014

NUMERICAL ANALYSIS FOR STOCHASTIC TIME-SPACE FRACTIONAL DIFFUSION EQUATION DRIVEN BY FRACTIONAL GAUSSIAN NOISE*

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Abstract

In this paper, we consider the strong convergence of the time-space fractional diffusion equation driven by fractional Gaussian noise with Hurst index $H \in (1/2, 1)$. A sharp regularity estimate of the mild solution and the numerical scheme constructed by finite element method for integral fractional Laplacian and backward Euler convolution quadrature for Riemann-Liouville time fractional derivative are proposed. With the help of inverse Laplace transform and fractional Ritz projection, we obtain the accurate error estimates in time and space. Finally, our theoretical results are accompanied by numerical experiments.

Mathematics subject classification: 65M12, 65M60, 35R11, 35R60.
Key words: Fractional Laplacian, Stochastic fractional diffusion equation, Fractional Gaussian noise, Finite element, Convolution quadrature, Error analysis.

1. Introduction

In the framework of uncoupled continuous time random walk, if both the second moment of the jump length and the mean waiting time diverge, the model describes competition between subdiffusion and Lévy flights [32]. The equivalent microscopic model is based on the subordinated Langevin equation with stable noise. The probability density function of the position of the particle motion is governed by the fractional Fokker-Planck equation with temporal and spatial fractional derivatives [8]. If the system is influenced by external fluctuating source term, e.g. fractional Gaussian noise, it has the form (1.1), which is the equation we focus on in this paper.

Let $\mathbb{D} \subset \mathbb{R}^d, d = 1, 2, 3$, be a bounded domain with smooth boundary and $\psi(x, t)$ the solution of

$$\begin{cases} \partial_t \psi(x,t) + {}_0\partial_t^{1-\alpha} \mathcal{A}^s \psi(x,t) = \dot{W}_Q^H(x,t), & (x,t) \in \mathbb{D} \times (0,T], \\ \psi(x,0) = \psi_0(x), & x \in \mathbb{D}, \\ \psi(x,t) = 0, & (x,t) \in \mathbb{D}^c \times [0,T], \end{cases} \tag{1.1}$$

where $\mathcal{A}^s (= (-\Delta)^s)$ is defined by [10]

$$\mathcal{A}^s \psi = c_{d,s}\, \mathrm{P.V.} \int_{\mathbb{R}^d} \frac{\psi(x) - \psi(y)}{|x-y|^{d+2s}} dy, \quad s \in (0,1)$$

* Received January 17, 2023 / Revised version received April 7, 2023 / Accepted May 6, 2023 / Published online November 16, 2023 /
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with

$$c_{d,s} = \frac{2^{2s} s \Gamma(d/2+s)}{\pi^{d/2}\Gamma(1-s)},$$

$\mathbb{D}^c$ means the complement of $\mathbb{D}$, T denotes a fixed terminal time, ∂_t is the first-order derivative in t, ${}_0\partial_t^{1-\alpha}$ is the Riemann-Liouville fractional derivative, defined by [36]

$$ {}_0\partial_t^{1-\alpha}\psi = \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t (t-\xi)^{\alpha-1}\psi(\xi)d\xi, \quad \alpha \in (0,1), \tag{1.2}$$

$\dot{W}_Q^H$ denotes fractional Gaussian noise, W_Q^H is fractional Gaussian process with Hurst index $H\in(1/2,1)$ and covariance operator Q on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ and it can be written as

$$W_Q^H(x,t) = \sum_{k=1}^{\infty} \sqrt{\Lambda_k}\phi_k(x)W_k^H(t),$$

where $\{(\Lambda_k, \phi_k)\}_{k=1}^{\infty}$ are eigenvalues and orthonormal eigenfunctions of the self-adjoint, non-negative linear operator Q on $\mathbb{H} = L^2(\mathbb{D}), W_k^H, k = 1, 2, \ldots,$ are independent one-dimensional fractional Brownian motion (fBm) process with Hurst index H, which are determined by covariance function [37]

$$\mathrm{Cov}(t,r) = \mathbb{E}\left[W_k^H(t)W_k^H(r)\right] = \frac{1}{2}\left(t^{2H} + r^{2H} - |t-r|^{2H}\right), \quad t, r \geq 0,$$

where $\mathbb{E}$ means the expectation operator. In this paper, we assume that $A^{-\rho}Q^{1/2}$ is a Hilbert-Schmidt operator on $\mathbb{H}$, where ρ is a real number, A denotes the classical Laplace operator $-\Delta$ with a zero Dirichlet boundary condition, and its domain $\mathcal{D}(A) = H_0^1(\mathbb{D}) \cap H^2(\mathbb{D})$.

Obviously, the problem (1.1) can be divided into the following two problems, i.e. a deterministic problem

$$\begin{cases} \partial_t v + {}_0\partial_t^{1-\alpha}\mathcal{A}^s v = 0, & (x,t) \in \mathbb{D}\times(0,T], \\ v(\cdot,0) = \psi_0, & x \in \mathbb{D}, \\ v = 0, & (x,t) \in \mathbb{D}^c \times [0,T], \end{cases} \tag{1.3}$$

and a stochastic problem

$$\begin{cases} \partial_t u + {}_0\partial_t^{1-\alpha}\mathcal{A}^s u = \dot{W}_Q^H, & (x,t) \in \mathbb{D}\times(0,T], \\ u(\cdot,0) = 0, & x \in \mathbb{D}, \\ u = 0, & (x,t) \in \mathbb{D}^c \times [0,T]. \end{cases} \tag{1.4}$$

Extensive numerical schemes for the deterministic fractional diffusion equation (1.3) have been proposed in [2, 5, 34]. Also, there have been many works for numerically solving stochastic partial differential equations (PDEs) involving Laplace and spectral fractional Laplacian, one can refer to [7, 18, 19, 25, 26, 28, 35, 39]. But for stochastic PDEs involving integral fractional Laplacian, the related researches are still few. In this paper, we provide a numerical scheme for stochastic PDE (1.4) based on backward Euler convolution quadrature for Riemann-Liouville fractional derivative and finite element method for integral fractional Laplacian.

Different from the Laplace and spectral fractional Laplacian, the eigenfunctions of $\mathcal{A}^s$ are unknown, so how to well characterize the influence of the noise on the regularity of the solution for (1.4) is a challenge. Here with the help of the representation of the mild solution, we provide the sharp regularity of the mild solution of (1.4) by building the resolvent estimate of