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WONG-ZAKAI APPROXIMATIONS FOR STOCHASTIC VOLTERRA EQUATIONS*

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Abstract

In this paper, we shall prove a Wong-Zakai approximation for stochastic Volterra equations under appropriate assumptions. We may apply it to a class of stochastic differential equations with the kernel of fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ and subfractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. As far as we know, this is the first result on stochastic Volterra equations in this topic.

Mathematics subject classification: 60H10, 60H07.
Key words: Stochastic Volterra equations, Wong-Zakai approximations, Fractional Brownian motion, Subfractional Brownian motion, Quadratic mean convergence.

1. Introduction and Main Results

Consider the following stochastic Volterra equations:

$$X_t = \xi + \int_0^t b(t, s, X_s)\mathrm{d}s + \int_0^t \sigma(t, s, X_s)\mathrm{d}W_s, \tag{1.1}$$

where $\xi \in \mathbb{R}^d$ and $b : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d, \sigma : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are Borel measurable functions, and $\{W_t\}_{t\geq 0}$ is an m-dimensional standard Brownian motion defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F})_t, \mathbb{P})$. Here the stochastic integral is the usual Itô's integral.

Stochastic Volterra equations arise in many applications such as mathematical finance, biology, etc. There is a big amount of literature devoted to the study of stochastic Volterra equations. Let us mention a few of them. When the coefficients $\sigma(t, s, x)$ and $b(t, s, x)$ are Lipschitz continuous in x and uniformly with respect to t, s, the existence and uniqueness of the strong solutions to Eq. (1.1) were first studied by Berger and Mizel [6, 7]. Later, the existence and uniqueness as well as the continuity of the solution to stochastic Volterra equations with singular kernels and non-Lipschitz coefficients were considered in [42]. Meanwhile, Euler schemes and large deviations for stochastic Volterra equations with singular kernels were established by Zhang [49].

Note that an important task in applications is to realize stochastic differential equations (abbreviated SDEs) on computers, that is, to construct a discretized approximation. The

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Wong-Zakai approximation of SDEs (a.s. or in mean square) by random differential equations is considered by Wong-Zakai [43,44], Ikeda-Watanabe [23], Karatzas-Shreve [25]. It is well known that if we replace the Brownian motion in SDEs by some smooth approximations (such as linear interpolation, mollifier, etc.), then the solution of the approximating equation converges (a.s. or q.s. or in mean square) to the Stratonovich form of the original equation (e.g. [2,3,5,9,15, 18–22,27,31,34–41,45–48,50]).

However, to the best of our knowledge, the Wong-Zakai approximation for stochastic Volterra equations has not been established. It is natural to ask whether the Wong-Zakai continues to hold for stochastic Volterra equations. We remark that the stochastic Volterra equations is in fact an anticipating SDEs, which is much difficult to study. Since the solution of stochastic Volterra equations is neither Markovian, nor a semimartingale, Itô's formula usually used in the studies of SDEs is not available in this case. In this paper, we shall prove the Wong-Zakai approximation for stochastic Volterra equations, which is first paper to study the problem.

Here and below, C will denote a positive constant that is not depending on n and may have different values from one place to another one. For simplicity, we use $|\cdot|$ to denote both the Euclidean norm for a vector in $\mathbb{R}^d$ and the Hilbert-Schmidt norm for a matrix in $\mathbb{R}^{d\times m}$.

In the present paper, we shall restrict our discussion to time interval $[0,1]$ and make the following assumptions:

(H1) The function $b(t,s,x)$ is differentiable with respect to the first variable, and the function $\sigma(t,s,x)$ is differentiable with respect to the first and the third variable. Also there are binary functions $g_i(t,s)\geq 0, i=1,2,3,4$, and $\lambda_1,\lambda_2\in(0,1/2)$ such that for any $t,s\in[0,1], 0\leq u\leq v\leq 1$ and $x\in\mathbb{R}^d$,

$$|b_1(t,s,x)|+|\sigma_1(t,s,x)|+|\sigma_{13}(t,s,x)|+|\sigma_{31}(t,s,x)|\leq Cg_1(t,s), \tag{1.2}$$

$$|b(t,s,x)|\leq Cg_2(t,s)(1+|x|^{\lambda_1}), \tag{1.3}$$

$$|\sigma(t,s,x)|\leq Cg_3(t,s)(1+|x|^{\lambda_2}), \tag{1.4}$$

$$|\sigma_3(t,s,x)|\leq Cg_4(t,s), \tag{1.5}$$

$$\sigma_1(u,s,x)\leq\sigma_1(v,s,x), \tag{1.6}$$

where

$$\sup_{0\leq t\leq 1}\int_0^1 g_1(t,s)\mathrm{d}s<\infty,\quad \sup_{0\leq t\leq 1}\int_0^1 |g_j(t,s)|^p\mathrm{d}s<\infty,\quad j=2,3,4,\quad \forall p\geq 1,$$

and $g_j(r,s)\leq g_j(t,s), j=2,3,4$ for any $0\leq s\leq r\leq t\leq 1$. $b_1(t,s,x)$ represents the partial derivative of $b(t,s,x)$ with respect to the first variable. $\sigma_1(t,s,x)$ and $\sigma_3(t,s,x)$ represent the partial derivatives of $\sigma(t,s,x)$ with respect to the first variable and the third variable respectively. $\sigma_{ij}(t,s,x)$ means that $\sigma(t,s,x)$ first seeks a partial derivative of the i-th variable, and then seeks a partial derivative of the j-th variable, where $i,j=1,3$, and $i\neq j$.

(H2) For all $t,t',s\in[0,1]$ and $x\in\mathbb{R}^d$,

$$|b(t',s,x)-b(t,s,x)|\leq F_1(t',t,s), \tag{1.7}$$

$$|\sigma(t's,x)-\sigma(t,s,x)|^2\leq F_2(t',t,s), \tag{1.8}$$

$$|\sigma_3(t's,x)-\sigma_3(t,s,x)|^2\leq F_3(t',t,s), \tag{1.9}$$

where $F_i(t',t,s), i=1,2,3$, are nonnegative functions on $[0,1]\times[0,1]\times[0,1]$, and satisfy for some $\gamma>1$,