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ERROR ANALYSIS FOR PARABOLIC OPTIMAL CONTROL PROBLEMS WITH MEASURE DATA IN A NONCONVEX POLYGONAL DOMAIN*

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Abstract

This paper considers the finite element approximation to parabolic optimal control problems with measure data in a nonconvex polygonal domain. Such problems usually possess low regularity in the state variable due to the presence of measure data and the nonconvex nature of the domain. The low regularity of the solution allows the finite element approximations to converge at lower orders. We prove the existence, uniqueness and regularity results for the solution to the control problem satisfying the first order optimality condition. For our error analysis we have used piecewise linear elements for the approximation of the state and co-state variables, whereas piecewise constant functions are employed to approximate the control variable. The temporal discretization is based on the implicit Euler scheme. We derive both a priori and a posteriori error bounds for the state, control and co-state variables. Numerical experiments are performed to validate the theoretical rates of convergence.

Mathematics subject classification: 49J20, 49K20, 65N15, 65N30.
Key words: A priori and a posteriori error estimates, Finite element method, Measure data, Nonconvex polygonal domain, Optimal control problem.

1. Introduction

The aim of this paper is to study both a priori and a posteriori error analysis of finite element approximations to the following model control problem:

$$\min_{u\in U_{ad}} J(y,u), \tag{1.1}$$

where

$$J(y,u) := \frac{1}{2}\int_0^T \|y-y_d\|^2_{L^2(\Omega)}dt + \frac{\Lambda}{2}\int_0^T \|u\|_{L^2(\Omega)}dt$$

with u represents the control variable and y indicates the associated state variable. The state equation is given by

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y = \sigma\tau + u & \text{in } \Omega_T, \\ y = 0 & \text{on } \Gamma_T, \\ y(\cdot,0) = y_0 & \text{in } \Omega. \end{cases} \tag{1.2}$$

In the above, Ω is a nonconvex polygonal domain in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. Set $\Omega_T = \Omega\times(0,T]$ and $\Gamma_T = \partial\Omega\times(0,T]$. The boundary $\partial\Omega$ can be expressed as $\partial\Omega = \cup_{j=1}^m \Gamma_j$

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with Γ_j, $j = 1, 2, \ldots, m$, are edges of the polygon. The constraints on the control variable are specified through the closed and convex subset of $L^2(0,T;L^2(\Omega))$ as follows:

$$U_{ad} := \{u \in L^2\big(0,T;L^2(\Omega)\big) : u_a \le u(x,t) \le u_b \text{ for a. a. } (x,t) \in \Omega_T\}. \tag{1.3}$$

Assume that the given functions $y_0 \in L^2(\Omega)$, $y_d \in H^1(0,T;L^2(\Omega))$, $\sigma \in \mathcal{C}([0,T];L^2(\Omega))$ and $\tau \in \mathfrak{B}[0,T]$, where $\mathfrak{B}[0,T]$ is the space of real and regular Borel measures in $[0,T]$. Further, the constants $u_a, u_b \in \mathbb{R}$ satisfy $u_a < u_b$, the regularization parameter $\Lambda > 0$ and the final time $T < \infty$.

Optimal control problems are widely used in scientific and engineering applications [26, 36]. The numerical study of such type of problems began in early 1970s [16, 17]. Thereafter there have been several notable contributions to this discipline and it is impossible to list all of them. Nevertheless, for the development of the finite element approach for parabolic optimal control problems (POCPs), see [19, 23, 32, 37, 42] and references therein. The authors of [33, 34] have utilized discontinuous Galerkin technique for temporal discretization and established convergence results for space-time finite element discretizations for POCPs. In [35], the authors have adopted Petrov-Galerkin Crank-Nicolson method for discretization of the control problem and derived related error estimates. The sparse POCPs have been analyzed by the authors of [11], where the control variable is taken to be an element of the measure space. They have provided a priori error estimates for the control problem.

Following the work of Babuška and Rheinboldt [4], the adaptive finite element method has grown popularity in scientific computing. It is well known that a posteriori error estimation is a necessary part of adaptivity for mesh refinement. The pioneer work has been made by Liu and Yan [28] for residual based a posteriori error estimates, Becker *et al.* [7] for dual-weighted goal oriented adaptivity and Li *et al.* [25] for recovery type a posteriori error estimators. A posteriori error analysis for optimal control problems governed by parabolic equation have been extensively investigated by numerous authors in [29, 31, 40, 41, 43].

POCPs are widely encountered in mathematical models representing groundwater contamination transmission, environmental modeling, petroleum reservoir simulation, and a variety of other applications. There are several real-world applications for POCPs when the state variable possesses less regularity due to the support of the source. Essentially, the support for the source function must be relatively tiny in comparison to the real size of the domain Ω. This feature drives us to explore control problems in which the source functions are measure data (elements from $\mathfrak{B}(\Omega)$). The POCPs with measure data encounter environmental concerns such as air pollution and waste-water treatment. Due to the presence of measure data, the solution of the state variable possesses less regularity which makes finite element error analysis more challenging. Therefore, an attempt has been made to study the convergence properties of the finite element method for such problems.

The study of optimal control problems governed by partial differential equations over a nonsmooth domain is a difficult task. The existence of re-entrant corners in the domain causes both theoretical and numerical analysis to be complicated. However, although there is a significant amount of research on the numerical analysis of the elliptic problem with a nonconvex domain [3, 5, 6, 20, 21] and quite a few works on the parabolic problem [12, 13]. For optimal control problems, there was not much work done in the nonconvex polygonal domain. In recently published article Apel *et al.* [2] developed a priori error estimates for the optimal control problem on a nonconvex domain.