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ANALYSIS OF TWO ANY ORDER SPECTRAL VOLUME METHODS FOR 1-D LINEAR HYPERBOLIC EQUATIONS WITH DEGENERATE VARIABLE COEFFICIENTS*

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Abstract

In this paper, we analyze two classes of spectral volume (SV) methods for one-dimensional hyperbolic equations with degenerate variable coefficients. Two classes of SV methods are constructed by letting a piecewise k-th order ($k \geq 1$ is an integer) polynomial to satisfy the conservation law in each control volume, which is obtained by refining spectral volumes (SV) of the underlying mesh with k Gauss-Legendre points (LSV) or Radaus points (RSV) in each SV. The L^2-norm stability and optimal order convergence properties for both methods are rigorously proved for general non-uniform meshes. Surprisingly, we discover some very interesting superconvergence phenomena: At some special points, the SV flux function approximates the exact flux with $(k+2)$-th order and the SV solution itself approximates the exact solution with $(k+3/2)$-th order, some superconvergence behaviors for element averages errors have been also discovered. Moreover, these superconvergence phenomena are rigorously proved by using the so-called correction function method. Our theoretical findings are verified by several numerical experiments.

Mathematics subject classification: 65N30, 65N25, 65N15.
Key words: Spectral Volume Methods, L^2 stability, Error estimates, Superconvergence.

1. Introduction

Hyperbolic equations have wide applications in chemical reactions, combustion, explosions, and multi-phase flow problems, transmission of electrical signal in the animal nervous system and so on. Numerical simulation becomes more and more important in the study of hyperbolic equations. Recently, high-order (or high resolution) numerical schemes attracted a lot of attention in the study of numerical simulation for hyperbolic equations. An incomplete list of

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high-order schemes include the high-order k-exact FV method [1,9], the essentially nonoscillatory (ENO) method [4,10], the weighted ENO (WENO) method [14,19], and the discontinuous Galerkin (DG) method [6–8,11,17], and the spectral volume (SV) method [26–30]. Among all the above methods, the SV and the DG are two comparable high-order methods which share many common advantages such as: Both are capable of achieving arbitrary high-orders, both can be established on nonuniform and/or unstructured grids, both have compact stencils, both only require the information of the immediate cell neighbours to evaluate the residuals of one target cell and thus are easily parallelizable. Compared with the DG method, the SV method enjoys some advantages such as sub-element-level local conservation property, high-resolution for the discontinuity. We refer to [20,31] for more comparisons between the SV and DG methods.

It is known that the mathematical theory for the DG method, including the stability and convergence properties (see, e.g. [5,12,13,16,18]), has been heavily studied. To our knowledge, the theory for the SV method is far less developed, however. The SV method was first introduced and studied by Wang and his colleagues [15,21,22,26–30]. Van den Abeele *et al.* [23,25] investigated the influence of the partitioning approaches which divide a spectral volume into control volumes and the wave propagation properties of the SV method for 1D and 2D hyperbolic equations. By studying the wave propagation properties, they showed that the third- and fourth-order SV schemes based on the Gauss-Lobatto distributions are weakly unstable. Later, Van den Abeele *et al.* [24] proved that the second-order SV scheme on 3D tetrahedral grids was stable while a two-parameters family of third-order scheme on 3D tetrahedral grids was unstable. Zhang and Shu [31] used Fourier type analysis to show the stability of p-th order ($p \leq 3$) SV schemes derived by uniform subdividing of spectral volumes in 1D uniform grids. All the above analysis are for lower-order SV schemes upon uniform grids. All the above analysis are case-by-case and are based on lower-order SV schemes over uniform meshes. To our best knowledge, there is no stability analysis of any high-order SV schemes on non-uniform meshes in the literature. Moreover, it seems that no theoretic analysis for the convergence order and superconvergence phenomenon has been reported in the literature yet, even for 1D constant-coefficient scalar equations.

In this paper, we will extend the analysis in [3] to the following variable-coefficients problem:

$$\begin{cases} u_t + (\alpha u)_x = g(x,t), & (x,t) \in [0,2\pi] \times [0,T], \\ u(x,0) = u_0(x), & x \in [0,2\pi], \\ u(0,t) = u(2\pi,t), & t \in [0,T], \end{cases} \tag{1.1}$$

where $u_0(x), \alpha(x)$ and $g(x,t)$ are given smooth functions. We emphasize that α might be degenerate in the sense that it has a finite number of zero points in $[0,2\pi]$. Instead of case-by-case studies for low-order SV schemes, we will propose a unified approach to analyze two classes of any order SV schemes, which are constructed by dividing each SV (an interval element) with Gauss-Legendre points (LSV) or Radaus points (RSV) into control volumes (CVs).

Essentially, the SV method is a Petrov-Galerkin method. Its trial space is the standard discontinuous finite element space with respect to SVs, while its test space is the piecewise constant space with respect to CVs. Therefore, standard analysis tools for a Galerkin method can not be applied directly to a SV method, novel tools need to be developed for the unified analysis of the SV method. To overcome this difficulty, we first introduce a special from-trial-to-test-space mapping, and then with the help of this mapping, we represent the SV method as a special Galerkin method, of which the SV bilinear form can be regarded as a perturbation or numerical quadrature of the corresponding DG bilinear form. Based on the analysis approach