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ERROR ANALYSIS OF THE SECOND-ORDER SERENDIPITY VIRTUAL ELEMENT METHOD FOR SEMILINEAR PSEUDO-PARABOLIC EQUATIONS ON CURVED DOMAINS*

Yang Xu, Zhenguo Zhou and Jingjun Zhao[1)]
School of Mathematics, Harbin Institute of Technology, Harbin 150001, China
Email: hit_zjj@hit.edu.cn

Abstract

The second-order serendipity virtual element method is studied for the semilinear pseudo-parabolic equations on curved domains in this paper. Nonhomogeneous Dirichlet boundary conditions are taken into account, the existence and uniqueness are investigated for the weak solution of the nonhomogeneous initial-boundary value problem. The Nitsche-based projection method is adopted to impose the boundary conditions in a weak way. The interpolation operator is used to deal with the nonlinear term. The Crank-Nicolson scheme is employed to discretize the temporal variable. There are two main features of the proposed scheme: (i) the internal degrees of freedom are avoided no matter what type of mesh is utilized, and (ii) the Jacobian is simple to calculate when Newton's iteration method is applied to solve the fully discrete scheme. The error estimates are established for the discrete schemes and the theoretical results are illustrated through some numerical examples.

Mathematics subject classification: 65M15, 65M60.
Key words: Semilinear pseudo-parabolic equation, Serendipity virtual element method, Projection method, Curved domain.

1. Introduction

Pseudo-parabolic equations are a vital class of mathematical physics equations and can describe a huge amount of physical evolution processes, including non-steady infiltration in fissured rocks [5], the two-temperature theory in thermodynamics [31], phase separation by spinodal decomposition [26] and so forth. In [1], a more detailed survey on the applications of pseudo-parabolic equations is provided.

The focus of this work is the following semilinear pseudo-parabolic equation with nonhomogeneous initial-boundary value conditions:

$$a(\boldsymbol{x})u_t - \nabla\cdot\big(b(\boldsymbol{x})\nabla(u_t+u)\big) + c(u) = f(\boldsymbol{x},t), \quad (\boldsymbol{x},t)\in\Omega\times(0,T], \tag{1.1a}$$

$$u = g(\boldsymbol{x},t), \quad (\boldsymbol{x},t)\in\partial\Omega\times(0,T], \tag{1.1b}$$

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \quad \boldsymbol{x}\in\Omega, \tag{1.1c}$$

where Ω is a convex bounded open subset of $\mathbb{R}^2$ and a finite number of curves $\{(\partial\Omega)_i\}_{i=1}^{N_\Omega}$ constitute its boundary $\partial\Omega$. Each curve $(\partial\Omega)_i$ is assumed to be sufficiently smooth and the boundary $\partial\Omega$ is supposed to be Lipschitz. T denotes the finite terminal time. The source

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1) Corresponding author

function f, boundary value g and initial value u_0 are given data. The coefficients a and b depend only on $\boldsymbol{x}, c(u)$ is the nonlinear term.

Due to the complexity of the shape of the domain Ω as well as the presence of nonlinear term $c(u)$, it is difficult or even impossible to give the exact solution to (1.1) in an explicit way. Therefore, efficient and accurate numerical methods should be considered. Some typical numerical schemes for linear pseudo-parabolic equations include finite difference schemes [16], finite volume element methods [38], finite element methods [36] and mixed finite element methods [23]. For nonlinear pseudo-parabolic equations, various numerical methods, including characteristic finite element methods [19], conforming and nonconforming finite element methods [27,33] and discontinuous Galerkin methods [29, 37], have been developed. A more comprehensive survey of numerical methods for various types of pseudo-parabolic equations can be found in [1]. In recent years, numerical methods that can deal with polygonal or polyhedral meshes have become important issues in the field of scientific computing, and such methods have been used to numerically solve pseudo-parabolic equations, including weak Galerkin methods [17], hybrid high-order methods [34] and virtual element methods [35].

The above-mentioned virtual element method (VEM) can be deemed as an extension of the finite element method towards meshes with general polygonal or polyhedral elements, and it has been used to numerically approximate a wide range of nonlinear initial-boundary value problems. In the framework of VEM, the common strategy to deal with nonlinear terms is to use L^2-projection operator, and this idea has already been applied to semilinear parabolic problem [2], Swift-Hohenberg equation [15], nonlocal model [4], nonlinear Schrödinger equation [22] and so forth. Recently, the idea of using interpolation operator to deal with nonlinear terms was proposed in [18]. In this idea, the features of the serendipity virtual element method (SVEM) are fully utilized, and a new way of numerically solving nonlinear evolution equations is provided in VEM framework.

SVEM [6] is a novel variant of VEM and its aim is to decrease the amount of internal-moment degrees of freedom. We focus on the second-order SVEM in this paper. The first motivation of our interest is that internal degrees of freedom are completely avoided without the need to consider the relationship between the degree of polynomials adopted in SVEM and the shape of the mesh elements. Indeed, as stated in [6, 18], for higher order SVEM and certain types of meshes, some additional internal degrees of freedom may be needed. The second motivation is that when we adopt the idea in [18] to approximate nonlinear term $c(u)$ in (1.1) by interpolation operator and solve the nonlinear system by Newton's iteration, the calculation of Jacobian is convenient. It is well known that Newton's iteration is of second-order convergence, so it is often used to solve nonlinear problems. However, if we use the L^2-projection operator to deal with nonlinear terms as in [2], the calculation of Jacobian is complex and time-consuming. The main reason is that the L^2-projection operator involves the integral on the mesh elements, which makes the form of Jacobian complicated.

Iso-parametric finite elements [21] are popular for partial differential equations on curved domains. This kind of methods relies on the reference element technique, which is not available in VEM or SVEM due to the use of general polygonal or polyhedral meshes [30]. Thus, the iso-parametric idea cannot be easily generalized to VEM or SVEM. Here, we will use the Nitsche-based projection method, which was first proposed in [9] and then extended into the framework of VEM in [7,8]. The boundary conditions are imposed by the Nitsche-based projection method which was first proposed in [9] and then extended into the framework of VEM in [7, 8]. To implement the Nitsche-based projection method, we need to know the gradient of functions