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AN OVERLAPPING DOMAIN DECOMPOSITION SPLITTING ALGORITHM FOR STOCHASTIC NONLINEAR SCHRÖDINGER EQUATION*

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Abstract

A novel overlapping domain decomposition splitting algorithm based on a Crank-Nicolson method is developed for the stochastic nonlinear Schrödinger equation driven by a multiplicative noise with non-periodic boundary conditions. The proposed algorithm can significantly reduce the computational cost while maintaining the similar conservation laws. Numerical experiments are dedicated to illustrating the capability of the algorithm for different spatial dimensions, as well as the various initial conditions. In particular, we compare the performance of the overlapping domain decomposition splitting algorithm with the stochastic multi-symplectic method in [S. Jiang *et al.*, Commun. Comput. Phys., 14 (2013), 393–411] and the finite difference splitting scheme in [J. Cui *et al.*, J. Differ. Equ., 266 (2019), 5625–5663]. We observe that our proposed algorithm has excellent computational efficiency and is highly competitive. It provides a useful tool for solving stochastic partial differential equations.

Mathematics subject classification: 60H35, 35Q55, 60H15.
Key words: Stochastic nonlinear Schrödinger equation, Domain decomposition method, Operator splitting, Overlapping domain decomposition splitting algorithm.

1. Introduction

The main purpose of this work is to propose an innovative overlapping domain decomposition splitting (ODDS for short) algorithm for the stochastic nonlinear Schrödinger (NLS) equation driven by a multiplicative noise

$$\begin{cases} \mathrm{i}du = [\Delta u + \lambda |u|^2 u]dt + \varepsilon u \circ dW(t), & t \in (0,T], \\ u(0,x) = u_0(x), & x \in D \subset \mathbb{R}^d, \quad d \geq 1 \end{cases} \tag{1.1}$$

with Dirichlet boundary conditions, $\varepsilon > 0$, $\lambda \in \mathbb{R}$ and W an $L^2(D;\mathbb{R})$-valued Q-Wiener process defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$. More precisely, $W(t)$ has the following Karhunen-Loève expansion:

$$W(t) = \sum_{k\in\mathbb{N}^d} Q^{\frac{1}{2}} e_k \beta_k(t), \quad t \in [0,T]$$

with $\{e_k\}_{k\in\mathbb{N}^d}$ being an orthonormal basis of $L^2(D;\mathbb{R})$, $\{\beta_k\}_{k\in\mathbb{N}^d}$ being a sequence of real-valued mutually independent and identically distributed Brownian motions, and $Qe_k = \eta_k e_k$ for $\eta_k \geq 0$,

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$k \in \mathbb{N}^d$. For convenience, we always consider the equivalent Itô form of (1.1)

$$\mathrm{i}du = \left[\Delta u + \lambda |u|^2 u - \frac{\mathrm{i}}{2}\varepsilon^2 F_Q u\right] dt + \varepsilon u dW(t) \tag{1.2}$$

with the initial value $u(0) = u_0$ and

$$F_Q(x) := \sum_{k \in \mathbb{N}^d} \left(Q^{\frac{1}{2}} e_k(x)\right)^2.$$

In the last two decades, much progress has been made in theoretical analysis and numerical approximation for the stochastic NLS equation. To numerically inherit the charge conservation law and the geometric structure of (1.1), [8, 9, 13, 23] propose the stochastic symplectic and multi-symplectic algorithms. Particularly, the authors in [11] applies the large deviation principle to investigate the probabilistic superiority of the stochastic symplectic algorithms. Very recently, the authors in [17] discover the stochastic symplectic structure of the stochastic Schrödinger equation on Wasserstein manifold at the first time. To preserve the ergodicity of the numerical solution of (1.1), [20, 21] study the ergodic numerical approximations. To reduce the computational cost of (1.1), [22] designs a parareal algorithm and [12, 14, 25, 26] propose the splitting algorithm, respectively. For more details about other kinds of numerical approximations of the stochastic NLS equation, we refer to [2, 3, 5, 6, 10, 15, 16, 18] and references therein. These existing semi-discretizations and full discretizations for the stochastic NLS equation mentioned above are all investigated under the assumption of homogeneous or periodic boundary conditions. It is worth to point out that the soliton solution of the nonlinear dispersive wave propagation problems in a very large or unbounded domain for the stochastic NLS equation is an interesting and important subject in applications (see, e.g. [1]). However, solving high dimensional stochastic partial differential equations can be a computationally intensive task, particularly when the computational domain is very large. To simulate such systems in a moderate amount of time, we must employ high-performance computing. This motivates us to construct highly efficient and numerically stable algorithms for the d-dimensional stochastic NLS equation (1.1) in a large spatial domain with inhomogeneous or non-periodic boundary conditions.

To this end, we first apply the splitting technique of [25] to the Eq. (1.1) and get a deterministic linear PDE with random initial datum and a nonlinear stochastic PDE

$$\mathrm{i}du^{[1]} = \Delta u^{[1]} dt, \tag{1.3}$$

$$\mathrm{i}du^{[2]} = \lambda \left|u^{[2]}\right|^2 u^{[2]} dt + \varepsilon u^{[2]} \circ dW(t). \tag{1.4}$$

Then, for the subsystem (1.4) we can get the analytic solution due to the point-wise conservation law $|u(t,x)| = |u_0(x)|$. The key issues lie in the numerical approximation for the subsystem (1.3), we first discretize it based on the Crank-Nicolson scheme in the temporal direction and get a temporal semi-discretization.

To overcome the difficulties introduced by the non-periodic boundary conditions, we use the Chebyshev pseudo-spectral interpolation idea in space. To efficiently exploit modern high performance computing platforms, it is essential to design high performance algorithms. Domain decomposition method provides a useful tool to develop fast and efficient solvers for stochastic PDEs with a large number of random inputs. The non-overlapping domain decomposition method for PDEs with random coefficients is first proposed in [27] and then extended by [28] to