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NUMERICAL APPROXIMATION OF THE INVARIANT DISTRIBUTION FOR A CLASS OF STOCHASTIC DAMPED WAVE EQUATIONS*

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Abstract

We study a class of stochastic semilinear damped wave equations driven by additive Wiener noise. Owing to the damping term, under appropriate conditions on the nonlinearity, the solution admits a unique invariant distribution. We apply semi-discrete and fully-discrete methods in order to approximate this invariant distribution, using a spectral Galerkin method and an exponential Euler integrator for spatial and temporal discretization respectively. We prove that the considered numerical schemes also admit unique invariant distributions, and we prove error estimates between the approximate and exact invariant distributions, with identification of the orders of convergence. To the best of our knowledge this is the first result in the literature concerning numerical approximation of invariant distributions for stochastic damped wave equations.

Mathematics subject classification: 60H15, 60H35, 65C30.
Key words: Stochastic damped wave equation, Invariant distribution, Exponential integrator, Spectral Galerkin method, Weak error estimates, Infinite dimensional Kolmogorov equations.

1. Introduction

In the last decades, stochastic partial differential equations (SPDEs) have become the subject of intensive research, with various possible perspectives, from modelling and applications, to theoretical analysis using advanced stochastic and PDE analysis techniques. Due to the need of simulating efficiently the SPDE models, there has been a huge interest in the design and analysis of numerical methods, see for instance the monograph [35]. Like for all deterministic and stochastic, finite and infinite dimensional, systems, understanding the long-time behavior of the solutions and of the numerical methods is a crucial and challenging problem. In this manuscript, we provide a contribution in this direction for a class of ergodic stochastic damped semilinear wave equations. Precisely, we apply semi-discrete and fully-discrete schemes and we

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prove that the approximations preserve ergodicity of the exact system, and we obtain upper bounds with rates for the error between the approximate and exact invariant distributions.

In this work, we study stochastic damped semilinear wave equations driven by additive Wiener noise, which can formally be written as

$$\begin{cases} \partial_t^2 u(t,x) = -2\gamma\partial_t u(t,x) + \Delta u(t,x) + f\big(u(t,x)\big) + \dot{W}^Q(t,x), \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = v_0(x), \quad x \in \mathcal{D}, \\ u(t,x) = 0, \quad x \in \partial\mathcal{D}, \end{cases} \tag{1.1}$$

where $\mathcal{D}$ is a d-dimensional domain, $\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_d}^2$ is the Laplace operator, and homogeneous Dirichlet boundary conditions are imposed. The positive real number $\gamma \in (0,\infty)$ is the damping parameter. In addition, $u_0 : \mathcal{D} \to \mathbb{R}$ and $v_0 : \mathcal{D} \to \mathbb{R}$ denote the initial values of u and $\partial_t u$. Finally, $f : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz continuous function, and $\dot{W}^Q$ denotes space-time noise, which is white in time and has covariance operator Q in space. We refer to Section 2 below for precise formulations and assumptions. In this manuscript, the stochastic damped wave equation (1.1) is interpreted as a stochastic evolution system with unknowns $u(t) \in H$ and $v(t) \in H^{-1}$

$$\begin{cases} du(t) = v(t)dt, \\ dv(t) = \big(-\Lambda u(t) - 2\gamma v(t)\big)dt + f\big(u(t)\big)dt + dW^Q(t), \\ u(0) = u_0, \quad v(0) = v_0, \end{cases} \tag{1.2}$$

or as a stochastic evolution equation with unknown $X(t) = (u(t), v(t))$ with values in $\mathcal{H} = H \times H^{-1}$, written as

$$\begin{cases} dX(t) = A_\gamma X(t)dt + F\big(X(t)\big)dt + d\mathcal{W}^Q(t), \\ X(0) = x_0, \end{cases} \tag{1.3}$$

(see Section 2 for details on the notation). We refer to the monograph [18] for the theory of stochastic evolution equations and to [19] for a chapter devoted to the analysis of the stochastic wave equation. In the setting introduced below, the stochastic evolution system (1.3) admits a unique global mild solution, for any initial value.

Assuming that the Lipschitz constant of the function f is sufficiently small (see condition (2.15) from Assumption 2.1 below), the process $(X(t))_{t\geq 0}$ admits a unique invariant distribution denoted by μ_∞, see Proposition 3.2 below, and the monograph [17] for general results on ergodicity for stochastic evolution equations. The objective of this work is to prove that μ_∞ can be approximated using fully-discrete numerical methods, and to provide rates of convergence with respect to the spatial and temporal discretization parameters. Precisely, the spatial discretization is performed using a spectral Galerkin method with parameter denoted by $N \in \mathbb{N}$, and the temporal discretization is performed using an exponential Euler integrator with time-step size denoted by $\tau \in (0,1)$. We refer to Section 4 for precise notation. The fully-discrete scheme (see Section 4.2) is written as

$$X_{m+1}^N = e^{\tau A_\gamma}\big(X_m^N + \tau F_N\big(X_m^N\big) + \mathcal{P}_N \Delta\mathcal{W}_m^Q\big), \tag{1.4}$$

where $(e^{tA_\gamma})_{t\geq 0}$ is the semigroup associated with the linear damped wave equation with no forcing. We also consider a semi-discrete scheme (see Section 4.1)

$$dX^N(t) = A_\gamma X^N(t)dt + F_N\big(X^N(t)\big)dt + \mathcal{P}_N d\mathcal{W}^Q(t), \tag{1.5}$$

where only spatial discretization is performed.