

STRONG CONVERGENCE OF AN EXPLICIT FULL-DISCRETE SCHEME FOR STOCHASTIC BURGERS-HUXLEY EQUATION*

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Abstract

The strong convergence of an explicit full-discrete scheme is investigated for the stochastic Burgers-Huxley equation driven by additive space-time white noise, which possesses both Burgers-type and cubic nonlinearities. To discretize the continuous problem in space, we utilize a spectral Galerkin method. Subsequently, we introduce a nonlinear-tamed exponential integrator scheme, resulting in a fully discrete scheme. Within the framework of semigroup theory, this study provides precise estimations of the Sobolev regularity, L^∞ regularity in space, and Hölder continuity in time for the mild solution, as well as for its semi-discrete and full-discrete approximations. Building upon these results, we establish moment boundedness for the numerical solution and obtain strong convergence rates in both spatial and temporal dimensions. A numerical example is presented to validate the theoretical findings.

Mathematics subject classification: 60H35, 65C30, 60H15.

Key words: Stochastic Burgers-Huxley equation, Strong convergence rate, Non-globally monotone nonlinearity, Fully discrete scheme, Tamed exponential integrator scheme.

1. Introduction

In this paper, we consider numerical approximations of the following nonlinear stochastic partial differential equation (SPDE) driven by an additive space-time white noise:

$$\begin{cases} \frac{\partial u(t, x)}{\partial t} = \frac{\partial^2 u(t, x)}{\partial x^2} + u(t, x) \frac{\partial u(t, x)}{\partial x} \\ \quad + \nu u(t, x)(1 - u(t, x))(u(t, x) - \theta) + \frac{\partial W(t, x)}{\partial t}, & t \in (0, T], \quad x \in \mathcal{U}, \\ u(t, 0) = u(t, 1) = 0, & t \in [0, T], \\ u(0, x) = u_0(x), & x \in \mathcal{U}, \end{cases} \quad (1.1)$$

in which $\mathcal{U} = (0, 1)$, $\nu > 0$ and $\theta \in (0, 1)$ are parameters. $\{W(t)\}_{t \geq 0}$ is a cylindrical Wiener process. Eq. (1.1) is known as the stochastic Burgers-Huxley equation (SBHE), which shows a prototype model for describing the interaction between the reaction mechanism, the convective effects and diffusion transport [25, 26]. The SBHE also has significant applications in biology

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to describe the nerve propagation in the nerve fiber and electro-physiology [8, 34]. The well-posedness and the regularity properties of the SBHE were established in [25, 26, 35]. The SBHE (1.1) contains two types of nonlinear drift terms: $u \partial u / \partial x$ and $\nu u(1-u)(u-\theta)$, which are recognized as the Burgers-type nonlinearity and the cubic nonlinearity, respectively.

When the Burgers-type nonlinearity in (1.1) disappears, the SBHE degenerates into the stochastic Huxley equation (SHE), also known as the stochastic Allen-Cahn equation for some adequately chosen parameters. For numerical approximations of the SHE, it is known that the explicit Euler scheme and the linear-implicit Euler scheme (only treat the linear operator implicitly, see [32] for more details) do not converge in the strong sense [15, 16]. Lately, several numerical methods for such nonlinear SPDEs take full advantage of the global monotone (more precisely, one-sided Lipschitz) condition of the cubic nonlinearity, i.e.

$$\langle u - v, f(u) - f(v) \rangle \leq C \|u - v\|^2. \quad (1.2)$$

Some notable examples include the drift-implicit Euler-Galerkin scheme [24], the splitting time discretization scheme [5, 6], the truncated exponential Euler scheme [2], and the tamed accelerated exponential integrator scheme [31]. For more numerical studies of monotone SPDEs, one can refer to [9, 12, 14, 23, 28] and references therein.

When the cubic nonlinearity is not considered, i.e. $\nu = 0$, the SBHE (1.1) becomes the stochastic Burgers equation (SBE). The well-posedness of SBEs driven by various types of noises was studied, e.g. the additive white noise [10], the correlated noise [11] and the fractional Brownian motion [30]. Progress on numerical methods for SBEs has also been made, for example, the finite difference method [1, 13] and the Galerkin approximation [3, 4] for spatial semi-discretization, and the full spatial-temporal discretization [3, 21]. Notably, the convergence rates obtained in the aforementioned publications are all in the pathwise sense, not in the strong sense. For strong convergence of numerical solutions of SBEs driven by space-time white noises, the only known work is [20], where the proposed full-discrete numerical approximations were shown to be strongly convergent. However, no convergence rate was achieved in this work.

The primary inquiry regarding a numerical method for solving the SBHE (1.1) driven by space-time white noise is whether it exhibits behavior similar to that of the SBE, where strong convergence is attainable but not the convergence rate, or if it resembles the SHE, where a strong convergence rate can be achieved. This paper aims to address this question by introducing an explicit full-discrete scheme for solving the SBHE (1.1) driven by additive space-time white noise and determining the strong convergence rate of the proposed scheme. Specifically, we employ the spectral Galerkin method to spatially discretize (1.1), resulting in a finite-dimensional approximation represented through a spectral expansion. For the temporal discretization, we implement a tamed accelerated exponential integrator scheme [19, 31], which differs from the truncated method for SBEs (see [17, 18, 20]). By ensuring the stability of the numerical solution through moment boundedness, we are able to establish the strong convergence rate of the proposed scheme, as presented in Theorem 4.1, as follows:

$$\sup_{0 \leq m \leq M} \|u(t_m) - u_m^N\|_{L^p(\Omega; H)} \leq C (N^{-\alpha} + \tau^{\frac{\alpha}{2}}), \quad \forall \alpha \in \left(0, \frac{1}{2}\right), \quad (1.3)$$

where $H := L^2(\mathcal{U})$ and the positive constant C is independent of the temporal step size τ and the dimension of the spectral Galerkin projection space N . Here $u(t_m)$ is the mild solution of the SBHE and u_m^N is the numerical solution produced by the fully discrete scheme (4.1). (1.3) implies that for arbitrarily small $\epsilon > 0$, the convergence rates of our proposed algorithm are $1/2 - \epsilon$ in space and $1/4 - \epsilon$ in time.