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AUGMENTED SUBSPACE SCHEME FOR EIGENVALUE PROBLEM BY WEAK GALERKIN FINITE ELEMENT METHOD*

Yue Feng, Zhijin Guan and Hehu Xie
LSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China;
School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China
Emails: fengyue@amss.ac.cn, guanzhijin@lsec.cc.ac.cn, hhxie@lsec.cc.ac.cn
Chenguang Zhou[1)]
Department of Mathematics, Beijing University of Technology, Beijing 100124, China
Email: zhoucg@bjut.edu.cn

Abstract

This study proposes a class of augmented subspace schemes for the weak Galerkin (WG) finite element method used to solve eigenvalue problems. The augmented subspace is built with the conforming linear finite element space defined on the coarse mesh and the eigenfunction approximations in the WG finite element space defined on the fine mesh. Based on this augmented subspace, solving the eigenvalue problem in the fine WG finite element space can be reduced to the solution of the linear boundary value problem in the same WG finite element space and a low dimensional eigenvalue problem in the augmented subspace. The proposed augmented subspace techniques have the second order convergence rate with respect to the coarse mesh size, as demonstrated by the accompanying error estimates. Finally, a few numerical examples are provided to validate the proposed numerical techniques.

Mathematics subject classification: 65N30, 65N25, 65L15, 65B99.
Key words: Eigenvalue problem, Augmented subspace scheme, Weak Galerkin finite element method, Second order convergence rate.

1. Introduction

One of the most important tasks in contemporary scientific and engineering society is solving eigenvalue problems. The difficulty of solving eigenvalue problems is invariably higher than that of solving similar linear boundary value problems due to the increased computing and memory requirements. Large-scale eigenvalue problem solving in particular will provide formidable obstacles to scientific computing. Numerous eigensolvers have been developed so far, including the Jacobi-Davidson type technique [4], the preconditioned inverse iteration (PINVIT) method [5, 11, 14], the Krylov subspace type method (implicitly restarted Lanczos/Arnoldi method (IRLM/IRAM) [25]), and the generalized conjugate gradient eigensolver (GCGE) [16, 17, 38]. The orthogonalization processes involved in solving Rayleigh-Ritz problems are the common bottleneck in the design of effective parallel techniques for identifying a large number of eigenpairs, and they are included in all of these widely used approaches.

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1) Corresponding author

A class of augmented subspace methods and their multilevel correction methods has been proposed recently in [8,13,18,28–32] for the solution of eigenvalue problems. This kind of technique creates an augmented subspace using the low dimensional finite element space generated on the coarse grid, which is employed in each correction step. The notion of an augmented subspace gives rise to a class of augmented subspace techniques that need just the final finite element space on the finest mesh and the low dimension finite element space on the coarse mesh. Using the augmented subspace methods, the solution of the eigenvalue problem on the final level of mesh can be transformed to the solution of linear boundary value problems on the final level of mesh and the solution of the eigenvalue problem on the low dimensional augmented subspace. Even the coarse and finest meshes lack nested properties, these kinds of algorithms can still work [10]. The multilevel correction methods, which are based on the augmented subspace methods, provide ways to construct multigrid methods for eigenvalue problems [8,13,28,29,31]. In addition, the authors design an eigenpair-wise parallel eigensolver for the eigenvalue problems in [32]. A significant amount of the wall time in the parallel computation is saved by using this kind of parallel approach, which avoids performing orthogonalization and inner-products in the high dimensional space. However, the aforementioned references are mostly investigated using conforming finite element methods. There are few results on the augmented subspace approaches based on nonstandard finite element methods for solving eigenvalue problems.

The WG method, which is initially introduced and explored in [27], concerns the finite element methods utilized to solve partial differential equations in which the differential operators, such as gradient operator, divergence operator and curl operator, are approximated as distributions by weak forms. The WG approach employs generalized discrete weak derivatives and parameter-free stabilizers to weakly enforce continuity in the approximation space, in contrast to the standard finite element technique. Consequently, it ought to be more convenient to create high order precision discretization than the conforming finite element approach. Additionally, the WG approach can be easily implemented on polygonal meshes thanks to the relaxation of the continuity constraint, which also gives additional freedom for h- and p-adaptation. So far, the WG method has been applied to various partial differential equations, such as the parabolic equation [15,40], the biharmonic equation [21,26,39], the Brinkman equation [20,37], the Helmholtz equation [22,24] and the Maxwell equation [23]. The convergence analysis and several lower bound findings are produced in [34,35], where the WG approximation to the eigenvalue problems is studied. In [36], the authors adopt the WG method to obtain lower bounds of the Laplacian eigenvalue problem and the post-processing method based on interpolation is used to obtain upper bounds. In [6], the authors show that under certain conditions the WG method yields a guaranteed lower bound for the Laplacian eigenvalue, which is larger than a guaranteed lower bound for the Crouzeix-Raviart finite element space. Then, considering computational efficiency, the authors create a kind of two-grid or two-level schemes using the WG approach [35], and in [33], the shifted-inverse power technique is taken into consideration under the two-grid schemes. Based on the theoretical analysis presented in [35], it can be inferred that there is no independent relationship between the coarse and fine mesh sizes. As a result, the approaches cannot be used to develop an eigensolver for algebraic eigenvalue problems resulting from differential operator eigenvalue problems discretized by WG method.

This paper's contribution is to design an augmented subspace method for eigenvalue problems based on WG approximation. To the best of our knowledge, this is the first work aimed at the numerical analysis of the WG finite element discretization-based augmented subspace approach for eigenvalue problems. In contrast to the findings in [35], our approaches' selections