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WEAK CONVERGENCE ANALYSIS OF A SPLITTING-UP METHOD FOR STOCHASTIC DIFFERENTIAL EQUATIONS*

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Abstract

The weak convergence analysis plays an important role in error estimates for stochastic differential equations, which concerns with the approximation of the probability distribution of solutions. In this paper, we investigate the weak convergence order of a splitting-up method for stochastic differential equations. We first construct a splitting-up approximation, based on which we also set up a splitting-up numerical solution. We prove both of these two approximation methods are of first order of weak convergence with the help of Malliavin calculus. Finally, we present several numerical experiments to illustrate our theoretical analysis.

Mathematics subject classification: 60H10, 60H35, 65C30.
Key words: Stochastic differential equation, Splitting-up method, Weak convergence, Malliavin calculus.

1. Introduction

Stochastic differential equations play an important role in many fields such as physics [18], biology [28], finance [5], medicine [20], etc. Most stochastic differential equations (SDEs) arising in practice cannot be solved explicitly. Thus, the construction of efficient numerical methods is of great importance.

The weak error, sometimes more relevant in various fields such as finance and engineering, concerns with the approximation of the probability distribution of solutions [23,31]. It measures the error made by sampling from an approximate probability law of the exact solution at a fixed time, rather than the deviation from trajectory of the exact solution, as for the strong error [2]. There are different strategies on studying weak convergence of numerical schemes for SDEs. Kohatsu-Higa [24] study the weak convergence of Euler-Maruyama scheme for nonlinear SDEs via integration by parts formula in Malliavin calculus. Zygalakis [42] study the weak convergence of numerical schemes for SDEs via weak Taylor expansion. Usually, the weak convergence order is twice of the strong convergence order. Meanwhile, there are also many studies on weak convergence to various types of stochastic differential equations. Buckwar and Shardlow [11] study the weak convergence rate of a forward Euler approximation to stochastic differential delay equations. Kohatsu-Higa *et al.* [25] study weak convergence of Euler-Maruyama scheme for SDEs with irregular drift coefficient. Zhao and Wang [41] study the weak convergence order of numerical schemes for SDEs with super-linearly growing coefficients. Cui *et al.* [14,16] discretize stochastic partial differential equations (SPDEs) with non-globally Lipschitz coefficients by Galerkin finite element method (FEM) spatial semi-discretization and backward Euler

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temporal semi-discretization and analyze the weak convergence order of the numerical scheme. Cai *et al.* [12] discretize stochastic Allen-Cahn equation by spectral Galerkin spatial semi-discretization and tamed exponential Euler temporal semi-discretization and analyze its weak convergence order. Cai *et al.* [13] discretize SPDEs with fractional noise by spectral Galerkin spatial semi-discretization and exponential Euler temporal semi-discretization and analyze its weak convergence order. The temporal weak convergence order of the above numerical schemes is twice of the strong convergence order.

Splitting-up methods [30, 35] can convert a complicated equation into several easily solvable ones and improve computational efficiency. Splitting-up methods have been widely applied to various fields, such as Hamilton system [3], Maxwell equation [26], nonlinear Schrödinger equation [15], nonlinear filtering [40], etc. There are a few achievement on studying the splitting-up method of stochastic differential equations, and many scholars focus on the strong convergence analysis for splitting-up numerical approximation [1, 4, 7–10, 19]. For SDEs with global Lipschitz coefficients, Wang and Li [36] study the mean-square convergence of split-step forward methods for autonomous SDEs and obtain convergence order of $1/2$. Ding *et al.* [17] construct a split-step theta method for SDEs with global Lipschitz coefficients and obtain mean-square convergence order of $1/2$. Singh [34] study the strong convergence of split-step forward Milstein methods for autonomous SDEs and obtain convergence order of 1.

Meanwhile, as for numerical approximation for SDEs with coefficients under more relaxed conditions, splitting-up backward methods are applied. Higham *et al.* [21] study p-th-moments strong convergence of split-step backward methods for autonomous SDEs which drift coefficient satisfies one-sided Lipschtiz condition and obtain convergence order of $1/2$. Huang [22] constructs a split-step θ-method for SDEs with one-sided Lipschtiz drift coefficient and analyze its exponential mean-square stability. Wu and Gan [38] analyze the mean-square convergence rate of the split-step θ-method for SDEs with non-globally Lipschitz diffusion coefficients and obtain convergence order of $1/2$. Liu *et al.* [29] analyze the split-step balanced theta method for autonomous SDEs under a non-globally Lipschtiz condition and obtain the strong convergence order of $1/2$. Yang and Zhao [39] analyze a split-step theta scheme for nonlinear SDEs with jump and obtain the strong convergence order of $1/2$. Beyn *et al.* [6] study the split-step backward Milstein scheme for SDEs with super-linearly growing drift and diffusion coefficients and get the strong convergence of order 1. Wu and Gan [37] study the split-step theta Milstein scheme for SDEs with super-linearly growing drift and diffusion coefficients and get the strong convergence of order 1. The splitting-up method have been successfully applied in solving various SDEs, which results that the analysis of weak convergence rate becomes a hot topic.

The main work of this paper is to analyze the weak convergence properties of a numerical approximation to SDE based on splitting-up technique. We divide an SDE into two equations: one is an SDE without drift term and the other one is a determined ordinary differential equation (ODE). Based on these two equations we construct a splitting-up approximation. Furthermore, applying Euler-Maruyama and Euler method to these two equations, respectively, we construct a splitting-up numerical approximation. With the help of theory of Malliavin calculus, we prove that both of these two approximations converge to the exact solution of SDE with weak convergence order of 1.

The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries and notations used in this paper, and gives a brief introduction of Malliavin calculus. In Section 3, we construct a splitting-up approximation to an SDE and prove its weak convergence order is of 1. In Section 4, we construct a splitting-up numerical solution and obtain its first order of weak convergence. In Section 5, we carry out several numerical experiments to demonstrate the theoretical analysis.

2. Preliminaries

2.1. Notations and main assumptions

Let $(\mathbb{R}^d, \|\cdot\|)$ be an Euclidean space for $d \geq 1$ and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. For any integer $m \geq 1$, by $\mathcal{C}^m$ we denote the space of smooth functions which together its all partial derivatives up to order m are continuous. Let $\mathcal{C}_b^m(\mathbb{R}^d)$ be the space of smooth functions $f : \mathbb{R}^d \to \mathbb{R}$ with continuous and bounded Fréchet derivatives up to order m. $(\mathbb{R}^{m\times n}, \|\cdot\|)$ denotes the space of all $\mathbb{R}^{m\times n}$ matrix endowed with Frobenius norm which is induced by $\mathbb{R}^d$-norm. For $A, B \in \mathbb{R}^{m\times n}$, define an inner product by

$$(A, B) = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{i,j}B_{i,j}.$$

Consider an autonomous SDE in $\mathbb{R}^d$

$$dX(t) = b\big(X(t)\big)dt + \sigma\big(X(t)\big)dW(t), \quad t \in (0, T], \tag{2.1}$$

$$X(0) = X_0. \tag{2.2}$$

We assume that

(H1) $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are $\mathcal{C}^3$ smooth functions and all their derivatives are bounded.

(H2) $W(t)$, $0 \leq t \leq T$, is a standard $\mathbb{R}^m$-valued Brownian motion.

(H3) $X_0 \in \bigcap_{p\geq 2} L^p(\Omega; \mathbb{R}^d)$ is an $\mathbb{R}^d$-valued random variable independent of W.

It directly follows from (H1) that there exists a constant $\beta > 0$ such that

$$\|b(x)\| \leq \beta(1 + \|x\|), \quad \|\sigma(x)\| \leq \beta(1 + \|x\|), \quad \forall\, x \in \mathbb{R}^d. \tag{2.3}$$

Let $\sigma = (\sigma_1, \cdots, \sigma_m)$ and $W = (W^1, \cdots, W^m)^\mathsf{T}$, where σ_i is an $\mathbb{R}^d$-valued function and W^i is a standard one-dimensional Brownian motion for $i = 1, \ldots, m$, respectively. Then (2.1) can be written as

$$dX(t) = b\big(X(t)\big)dt + \sum_{i=1}^{m} \sigma_i\big(X(t)\big)dW^i(t), \quad t \in (0, T]. \tag{2.4}$$

Define a filtration $\mathcal{F}(t) = \sigma\{X_0, W(s), s \in [0, t]\}$ for any $0 \leq t \leq T$. By $\mathbb{L}^2_{d\times m}(0, T)$ we denote the space consisting of $\mathcal{F}(t)$ progressively measurable $\mathbb{R}^{d\times m}$-valued stochastic processes $\xi(\cdot)$ satisfying

$$\mathbb{E}\int_0^T \|\xi(t)\|^2 dt < \infty.$$

According to [21, Theorem 2.4.1] and (H1)-(H3), (2.1)-(2.2) possesses a unique solution progressively adapted to $\mathcal{F}(t)$ satisfying for each integer $p \geq 2$,

$$\mathbb{E}\|X(t)\|^p \leq \big(1 + \mathbb{E}\|X_0\|^p\big)e^{Ct}, \quad t \in [0, T], \tag{2.5}$$

where $C = C(p) > 0$ is a constant.

Let Φ_s^t $(0 \leq s \leq t \leq T)$ with $\Phi_s^s = I$ be the solution operator of (2.1), then Φ_s^t $(s \leq t)$ is a semigroup due to the autonomous property of (2.1), and hence $\Phi_s^t(X_0) = \Phi_0^{t-s}(X_0)$. Then, $\Phi_s^t(X_0)$ solves a stochastic integral equation

$$\Phi_s^t(X_0) = X_0 + \int_s^t b\big(\Phi_s^r(X_0)\big)dr + \sum_{i=1}^m \int_s^t \sigma_i\big(\Phi_s^r(X_0)\big)dW^i(r), \quad t \in [s,T]. \tag{2.6}$$

The following lemma presents the differentiability of the stochastic flow Φ_s^t with respect to the initial value X_0.

Lemma 2.1 ([24, Lemma 7.4]). *Assume* (H1)-(H3) *hold. Let $X \in \bigcap_{p\geq 2} L^p(\Omega;\mathbb{R}^d)$, then for each integer $p \geq 2$, there exists a constant $C = C(p)$ such that*

$$\sup_{s\in[0,T]} \mathbb{E} \sup_{t\in[s,T]} \big\|\nabla_x^k \Phi_s^t(x)\big|_{x=X}\big\|^p \leq C, \quad k = 1,2,3, \tag{2.7}$$

where ∇_x^k represents the k-th Fréchet derivative with respect to x.

2.2. Malliavin calculus

By $\mathcal{C}_{poly}^\infty(\mathbb{R}^n)$ we denote the space of all infinitely continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f and all of its partial derivatives have at most polynomial growth. Denote by $H = (L^2([0,T];\mathbb{R}^m), \|\cdot\|)$ the Hilbert space endowed with the inner product

$$(g_1, g_2) = \int_0^T \big(g_1(\tau), g_2(\tau)\big)d\tau.$$

For any $h = (h^1, \cdots, h^m)^\mathsf{T} \in H$, define

$$W(h) = \sum_{i=1}^m \int_0^T h^i(\tau)dW^i(\tau),$$

which is a Gaussian process on H. By $\mathcal{S}$ we denote the class of all $\mathbb{R}^d$-valued smooth random variables which is expressed as

$$F = (f_1, \cdots, f_d)^\mathsf{T} = \sum_{j=1}^d f_j\big(W(h_1), \cdots, W(h_n)\big)e_j,$$

where $n \geq 1$ is any integer, $h_1, \cdots, h_n \in H$, $f_1, \cdots, f_d \in \mathcal{C}_{poly}^\infty(\mathbb{R}^n)$ and $\{e_j\}_{j=1}^d$ is the standard basis of $\mathbb{R}^d$. The Malliavin derivative of $f_i(W(h_1), \cdots, W(h_n))$ is an H-valued random variable given by

$$Df_i\big(W(h_1), \cdots, W(h_n)\big) = \sum_{j=1}^n \partial_j f_i\big(W(h_1), \cdots, W(h_n)\big)h_j,$$

and the second order Malliavin derivative of $f_i(W(h_1), \cdots, W(h_n))$ is an $H \otimes H$-valued random variable given by

$$D^2 f_i\big(W(h_1), \cdots, W(h_n)\big) = \sum_{j,k=1}^n \partial_j\partial_k f_i\big(W(h_1), \cdots, W(h_n)\big)h_j \otimes h_k,$$

where the tensor product $h_j \otimes h_k$ denotes a bounded symmetric bilinear map from $H \times H$ to $\mathbb{R}$ defined by $(h_j \otimes h_k)(u_1, u_2) = (h_j, u_1)(h_k, u_2)$ for $(u_1, u_2) \in H \times H$. The Malliavin derivative of vector-valued function $F \in \mathcal{S}$ is an $L^2([0,T];\mathbb{R}^{d\times m})$-valued random variable given by

$$DF = \sum_{i=1}^{d} \sum_{j=1}^{n} \partial_j f_i\big(W(h_1), \cdots, W(h_n)\big) h_j \otimes e_i.$$

Noticing

$$L^2\big(\Omega; L^2([0,T];\mathbb{R}^{d\times m})\big) \simeq L^2([0,T]\times\Omega;\mathbb{R}^{d\times m}),$$

$DF = \{D_\tau F, \tau \in [0,T]\}$ is a measurable function on $[0,T]\times\Omega$, which can be understood to be an $\mathbb{R}^{d\times m}$-valued stochastic process. The second order Malliavin derivative of $F \in \mathcal{S}$ is defined by

$$D^2 F = \sum_{i=1}^{d} \sum_{j,k=1}^{n} \partial_j \partial_k f_i\big(W(h_1), \cdots, W(h_n)\big) h_j \otimes h_k \otimes e_i.$$

More precisely, $D^2F = \{D^2_{\tau_1,\tau_2} F, \tau_1, \tau_2 \in [0,T]\}$ is a measurable function on the product space $[0,T]\times[0,T]\times\Omega$.

For any integer $p \geq 2$, let $\mathbb{D}^{1,p}$ be the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p}^p = \mathbb{E}\|F\|^p + \mathbb{E}\int_0^T \|D_\tau F\|^p_{\mathbb{R}^{d\times m}} d\tau.$$

For any given $0 \leq s < t \leq T, \Phi_s^t(X_0) \in \mathbb{L}^2(0,T)$ and its Malliavin derivative $D_\tau \Phi_s^t(X_0)$ is an $\mathbb{R}^{d\times m}$-valued stochastic process with

$$\begin{aligned} D_\tau \Phi_s^t(X_0) &= \nabla_x \Phi_\tau^t\big(\Phi_s^\tau(X_0)\big)\sigma\big(\Phi_s^\tau(X_0)\big) \quad \text{a.e.} \quad \text{for} \quad s \leq \tau \leq t, \\ D_\tau \Phi_s^t(X_0) &= 0 \quad \text{a.e.} \quad \text{for} \quad \tau < s \quad \text{or} \quad \tau > t, \end{aligned}$$

cf. [27]. Furthermore, its second Malliavin derivative satisfies

$$D^2_{\tau_1,\tau_2} \Phi_0^t(X_0) = 0 \quad \text{a.e.} \quad \text{for} \quad \max(\tau_1,\tau_2) > t \quad \text{and} \quad \min(\tau_1,\tau_2) < s.$$

According to the following lemma, we have $\Phi_0^t(X_0) \in \mathbb{D}^{1,2}$ and $D_\tau \Phi_0^t(X_0) \in \mathbb{D}^{1,2}$.

Lemma 2.2 ([33, Theorem 6.4]). *Assume* (H1)-(H3) *hold. Then for each integer $p \geq 2$, there exists a constant $C = C(p)$ such that*

$$\begin{aligned} &\sup_{\tau\in[0,T]} \sup_{s\in[0,T]} \mathbb{E} \sup_{t\in[s,T]} \big\|D_\tau \Phi_s^t(X_0)\big\|^p \leq C, \\ &\sup_{\tau_1,\tau_2\in[0,T]} \sup_{s\in[0,T]} \mathbb{E} \sup_{t\in[s,T]} \big\|D^2_{\tau_1,\tau_2} \Phi_s^t(X_0)\big\|^p \leq C. \end{aligned} \tag{2.8}$$

Direct computation indicates that $\nabla_x \Phi_s^t(X_0)$ satisfies the integral equation, also see [27]

$$\nabla_x \Phi_s^t(X_0) = I_d + \int_s^t b'\big(\Phi_s^r(X_0)\big)\nabla_x \Phi_s^r(X_0) dr + \int_s^t \sigma'\big(\Phi_s^r(X_0)\big)\nabla_x \Phi_s^r(X_0) dW(r).$$

Following the proof of [27, Theorem 6.1.1], we get the conclusion below.

Lemma 2.3. *Assume* (H1)-(H3) *hold. Then for each integer $p \geq 2$, there exists a constant $C = C(p)$ such that*

$$\begin{aligned} \sup_{\tau\in[0,T]} \sup_{s\in[0,T]} \mathbb{E} \sup_{t\in[s,T]} \left\|D_\tau \nabla_x \Phi_s^t(X_0)\right\|^p \leq C, \\ \sup_{\tau_1,\tau_2\in[0,T]} \sup_{s\in[0,T]} \mathbb{E} \sup_{t\in[s,T]} \left\|D^2_{\tau_1,\tau_2} \nabla_x \Phi_s^t(X_0)\right\|^p \leq C. \end{aligned} \tag{2.9}$$

The following two lemmas introduce the Malliavin integration by parts formula and chain rule, respectively.

Lemma 2.4 ([32, Lemma 1.2.1]). *Suppose that $F \in \mathbb{D}^{1,2}$ and $G \in \mathbb{L}^2_{d\times m}(0,T)$, then*

$$\mathbb{E}\int_0^T \big(D_\tau F, G(\tau)\big)d\tau = \mathbb{E}\bigg(F, \int_0^T G(\tau)dW(\tau)\bigg). \tag{2.10}$$

Lemma 2.5 ([32, Proposition 1.2.3]). *Let $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ be a continuously differentiable function. Suppose that $F = (F^1, \cdots, F^d) \in \mathbb{D}^{1,2}$. Then $\varphi(F) \in \mathbb{D}^{1,2}$ and*

$$D\big(\varphi(F)\big) = \nabla_x\varphi(F)DF.$$

3. A Splitting-up Approximation and Weak Convergence Analysis

In this section, we construct a splitting-up approximation for (2.1)-(2.2) and analyze its weak convergence order.

3.1. Splitting-up approximation

In this section, we construct a splitting-up approximation for stochastic differential equations. The basic idea of the splitting-up method is to decomposed (2.1) into an SDE without drift term and a determined ODE. The prototype of these equations are defined as follow: For $t \in (0,T)$, find X_1 and X_2 such that

$$dX_1(t) = \sigma\big(X_1(t)\big)dW(t), \tag{3.1}$$

$$dX_2(t) = b\big(X_2(t)\big)dt. \tag{3.2}$$

By Q_s^t $(0 \leq s \leq t \leq T)$ with $Q_s^s = I$ we denote the solution operator of (3.1). By R_s^t $(0 \leq s \leq t \leq T)$ with $R_s^s = I$ we denote the solution operator of (3.2).

Noticing that $b(\cdot) = 0$ implies Φ_s^t coincides with Q_s^t, and $\sigma(\cdot) = 0$ leads to Φ_s^t coincides with R_s^t. Therefore, for any $X \in \bigcap_{p\geq 2} L^p(\Omega,\mathbb{R}^d)$, the two solution processes $Q_s^t(X)$ and $R_s^t(X)$ also satisfy estimates in Lemmas 2.1 and 2.2, which is sumerized as below.

Lemma 3.1. *Assume* (H1)-(H3) *hold. Let $X \in \bigcap_{p\geq 2} L^p(\Omega;\mathbb{R}^d)$, then for each integer $p \geq 2$ and $k = 1,2,3$, there exists a constant $C = C(k,p)$ such that for $0 \leq s < t \leq T$,*

$$\begin{aligned} \sup_{s\in[0,T]} \mathbb{E} \sup_{t\in[s,T]} \left\|\nabla_x^k Q_s^t(x)|_{x=X}\right\|^p \leq C, \\ \sup_{s\in[0,T]} \mathbb{E} \sup_{t\in[s,T]} \left\|\nabla_x^k R_s^t(x)|_{x=X}\right\|^p \leq C. \end{aligned}$$

For a positive integer $N > 0$, let $\kappa = T/N$ be a time stepsize and $t_i = i\kappa, i = 0, 1, \ldots, N$, be the uniform partition of interval $[0, T]$. Then, we define the splitting-up approximation solution $\tilde{X}_i$ at t_i by

$$\tilde{X}_0 = X_0, \quad \tilde{X}_i = R_{t_{i-1}}^{t_i}\big(Q_{t_{i-1}}^{t_i}(\tilde{X}_{i-1})\big), \quad i = 1, \ldots, N. \tag{3.3}$$

According to (H1)-(H3), we have

$$\tilde{X}_i \in \bigcap_{p\geq 2} L^p(\Omega; \mathbb{R}^d), \quad i = 0, 1, \ldots, N. \tag{3.4}$$

We introduce two stochastic processes $X_{1\kappa}$ and $X_{2\kappa}$ on $[0, T]$ with $X_{1\kappa}(0) = X_0$ and $X_{2\kappa}(0) = Q_0^\kappa(X_0)$. For any $t \in (0, T]$, there exists an integer $0 \leq i \leq N - 1$ such that $t \in (t_{i-1}, t_i]$, then define

$$X_{1\kappa}(t) = Q_{t_{i-1}}^t(\tilde{X}_{i-1}), \quad X_{2\kappa}(t) = R_{t_{i-1}}^t\big(Q_{t_{i-1}}^{t_i}(\tilde{X}_{i-1})\big). \tag{3.5}$$

Then $X_{1\kappa}(t)$ is $\mathcal{F}(t)$-measurable and $X_{2\kappa}(t)$ is $\mathcal{F}(t_i)$-measurable for $t \in (t_{i-1}, t_i]$. The process $X_{2\kappa}(t)$ interpolates the splitting-up approximation solution in the sense $X_{2\kappa}(t_i) = \tilde{X}_i$. Therefore, we have

$$\begin{aligned}
\tilde{X}_i &= R_{t_{i-1}}^{t_i}\big(Q_{t_{i-1}}^{t_i}(\tilde{X}_{i-1})\big) \\
&= X_{1\kappa}(t_i) + \int_{t_{i-1}}^{t_i} b\big(X_{2\kappa}(t)\big)dt \\
&= \tilde{X}_{i-1} + \int_{t_{i-1}}^{t_i} \sigma\big(X_{1\kappa}(t)\big)dW(t) + \int_{t_{i-1}}^{t_i} b\big(X_{2\kappa}(t)\big)dt \\
&= \tilde{X}_{i-1} + \int_{t_{i-1}}^{t_i} \sigma\big(Q_{t_{i-1}}^t \tilde{X}_{i-1}\big)dW(t) + \int_{t_{i-1}}^{t_i} b\big(R_{t_{i-1}}^t Q_{t_{i-1}}^{t_i} \tilde{X}_{i-1}\big)dt \\
&= \tilde{X}_{i-1} + \int_{t_{i-1}}^{t_i} \sigma\Big(\tilde{X}_{i-1} + \int_{t_{i-1}}^t \sigma\big(X_{1\kappa}(s)\big)dW(s)\Big)dW(t) \\
&\quad + \int_{t_{i-1}}^{t_i} b\Big(\tilde{X}_{i-1} + \int_{t_{i-1}}^t b\big(X_{2\kappa}(s)\big)ds + \int_{t_{i-1}}^{t_i} \sigma\big(X_{2\kappa}(s)\big)dW(s)\Big)dt.
\end{aligned} \tag{3.6}$$

3.2. Weak convergence order of the splitting-up approximation

In this subsection, we analyze the weak convergence order of the splitting-up approximation. Firstly, we transform the global weak error to the summation of local weak errors. Secondly, we derive some prior estimates and obtain the estimate of the local weak error. Finally, we accumulate local weak errors to obtain the global weak convergence order.

Lemma 3.2. *Assume* (H1)-(H3) *hold. Then for any test function* $\phi(\cdot) \in \mathcal{C}_b^3(\mathbb{R}^d)$*, there holds that*

$$\mathbb{E}\phi\big(X(T)\big) - \mathbb{E}\phi(\tilde{X}_N) = \sum_{i=1}^N \mathbb{E}\big[\Theta_i^\top\big(\Phi_{t_{i-1}}^{t_i}(\tilde{X}_{i-1}) - \tilde{X}_i\big)\big], \tag{3.7}$$

where Θ_i *is an* $\mathbb{R}^d$*-valued random variable given by*

$$\begin{aligned}
\Theta_i &= \int_0^1 \nabla_x \Phi_{t_i}^T\big(\xi_i(\theta)\big)^\top \nabla_x \phi\big(\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big)^\top d\theta, \\
\xi_i(\theta) &= \theta \Phi_{t_{i-1}}^{t_i}(\tilde{X}_{i-1}) + (1 - \theta)\tilde{X}_i,
\end{aligned} \tag{3.8}$$

and $\nabla_x \phi \in \mathbb{R}^{1\times d}$ *is the Fréchet derivative of* ϕ.

Proof. $X(T)$ and $\tilde{X}_N$ can be represented by the solution operator of (2.1) as follows:

$$X(T) = \Phi_0^T(\tilde{X}_0), \quad \tilde{X}_N = \Phi_{t_N}^T(\tilde{X}_N).$$

By the properties of semigroup Φ_s^t $(0 \le s \le t \le T)$, there holds

$$\Phi_{t_{i-1}}^T = \Phi_{t_i}^T \Phi_{t_{i-1}}^{t_i}, \quad i = 1, \dots, N.$$

Then by the mean value theorem and properties of conditional expectation, we have

$$\begin{aligned}
&\mathbb{E}\big[\phi\big(X(T)\big)\big] - \mathbb{E}[\phi(\tilde{X}_N)] \\
&= \mathbb{E}\big[\phi\big(\Phi_0^T(\tilde{X}_0)\big) - \phi\big(\Phi_{t_N}^T(\tilde{X}_N)\big)\big] \\
&= \mathbb{E}\big[\phi\big(\Phi_0^T(\tilde{X}_0)\big) \mp \phi\big(\Phi_{t_1}^T(\tilde{X}_1)\big) \mp \cdots \mp \phi\big(\Phi_{t_{N-1}}^T(\tilde{X}_{N-1})\big) - \phi\big(\Phi_{t_N}^T(\tilde{X}_N)\big)\big] \\
&= \sum_{i=1}^N \mathbb{E}\big[\phi\big(\Phi_{t_i}^T \Phi_{t_{i-1}}^{t_i}(\tilde{X}_{i-1})\big) - \phi\big(\Phi_{t_i}^T(\tilde{X}_i)\big)\big] \\
&= \sum_{i=1}^N \mathbb{E}\Big\{\mathbb{E}\big[\phi\big(\Phi_{t_i}^T\big(\Phi_{t_{i-1}}^{t_i}(\tilde{X}_{i-1})\big)\big) - \phi\big(\Phi_{t_i}^T(\tilde{X}_i)\big)\big]\big|\mathcal{F}(t_i)\Big\} \\
&= \sum_{i=1}^N \mathbb{E}\Big\{\mathbb{E}\big[\phi\big(\Phi_{t_i}^T(x_i)\big) - \phi\big(\Phi_{t_i}^T(y_i)\big)\big]\big|_{x_i=\Phi_{t_{i-1}}^{t_i}(\tilde{X}_{i-1}), y_i=\tilde{X}_i}\Big\} \\
&= \sum_{i=1}^N \mathbb{E}\bigg\{\mathbb{E}\bigg[\int_0^1 \nabla_x\phi\big(\Phi_{t_i}^T\big(\theta x_i + (1-\theta)y_i\big)\big)\nabla_x\Phi_{t_i}^T\big(\theta x_i + (1-\theta)y_i\big)d\theta \\
&\qquad\qquad \times (x_i - y_i)\bigg]\bigg|_{x_i=\Phi_{t_{i-1}}^{t_i}(\tilde{X}_{i-1}), y_i=\tilde{X}_i}\bigg\} \\
&:= \sum_{i=1}^N \mathbb{E}\big[\Theta_i^\mathsf{T}\big(\Phi_{t_{i-1}}^{t_i}(\tilde{X}_{i-1}) - \tilde{X}_i\big)\big],
\end{aligned}$$

where Θ_i is an $\mathbb{R}^d$-valued random variable given by (3.8). □

To derive the local weak error estimates, we need the following boundedness property for Θ_i.

Lemma 3.3. *Assume* (H1)-(H3) *hold. Then for any* $\phi(\cdot) \in \mathcal{C}_b^3(\mathbb{R}^d)$*, there exists a positive constant* C *independent of* κ *such that for* $i = 0, 1, \dots, N$*,*

$$\mathbb{E}\|\Theta_i\|^4 + \sup_{\tau\in[0,T]} \mathbb{E}\|D_\tau\Theta_i\|^4 + \sup_{\tau_1,\tau_2\in[0,T]} \mathbb{E}\big\|D_{\tau_1,\tau_2}^2\Theta_i\big\|^4 \le C. \tag{3.9}$$

Proof. From (3.4), it follows that $\xi_i(\theta) \in \bigcap_{p\ge2} L^p(\Omega, \mathbb{R}^d)$ for all $\theta \in [0,1]$ and $i = 0, 1, \dots, N$. For any given $\phi(\cdot) \in \mathcal{C}_b^3(\mathbb{R}^d)$, there exists a constant $C = C(\phi) > 0$ such that $\|\nabla_x^i\phi(\cdot)\| \le C$, $i = 1, 2, 3$. Then, by Hölder inequality and Lemma 2.1, we have

$$\begin{aligned}
\mathbb{E}\|\Theta_i\|^4 &= \mathbb{E}\bigg\|\int_0^1 \nabla_x\Phi_{t_i}^T\big(\xi_i(\theta)\big)^\mathsf{T}\nabla_x\phi\big(\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big)^\mathsf{T}d\theta\bigg\|^4 \\
&\le \mathbb{E}\bigg(\int_0^1 \big\|\nabla_x\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big\| \cdot \big\|\nabla_x\phi\big(\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big)^\mathsf{T}\big\|d\theta\bigg)^4
\end{aligned}$$

$$
\begin{aligned}
&\le C\mathbb{E}\bigg(\int_0^1 \big\|\nabla_x\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big\|d\theta\bigg)^4\\
&\le C\mathbb{E}\bigg(\int_0^1 \big\|\nabla_x\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big\|^2d\theta\bigg)^2\\
&\le C\mathbb{E}\int_0^1 \big\|\nabla_x\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big\|^4d\theta \le C.
\end{aligned}
$$

By Lemma 2.5, we have for $\tau \in [t_i, T]$,

$$
\begin{aligned}
D_\tau\Theta_i = &\int_0^1 D_\tau\nabla_x\Phi_{t_i}^T\big(\xi_i(\theta)\big)^\top\nabla_x\phi\big(\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big)^\top d\theta\\
&+\int_0^1 \nabla_x\Phi_{t_i}^T\big(\xi_i(\theta)\big)^\top\nabla_x^2\phi\big(\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big)^\top D_\tau\Phi_{t_i}^T\big(\xi_i(\theta)\big)d\theta,
\end{aligned}
$$

and $D_\tau\Theta_i = 0$ for $\tau \in [0, t_i)$. Obviously, we have

$$
\begin{aligned}
\mathbb{E}\|D_\tau\Theta_i\|^4 \le\ & 8\mathbb{E}\bigg\|\int_0^1 D_\tau\nabla_x\Phi_{t_i}^T\big(\xi_i(\theta)\big)^\top\nabla_x\phi\big(\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big)^\top d\theta\bigg\|^4\\
&+8\mathbb{E}\bigg\|\int_0^1 \nabla_x\Phi_{t_i}^T\big(\xi_i(\theta)\big)^\top\nabla_x^2\phi\big(\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big)^\top D_\tau\Phi_{t_i}^T\big(\xi_i(\theta)\big)d\theta\bigg\|^4\\
:=\ & I + II.
\end{aligned}
$$

By Hölder inequality, (3.4) and Lemmas 2.2, 2.3, we obtain

$$
\begin{aligned}
I &\le 8\mathbb{E}\bigg(\int_0^1 \big\|D_\tau\nabla_x\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big\|\cdot\big\|\nabla_x\phi\big(\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big)^\top\big\|d\theta\bigg)^4\\
&\le 8C\mathbb{E}\bigg(\int_0^1 \big\|D_\tau\nabla_x\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big\|d\theta\bigg)^4\\
&\le 8C\mathbb{E}\int_0^1 \big\|D_\tau\nabla_x\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big\|^4d\theta \le C.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
II &\le 8\mathbb{E}\bigg(\int_0^1 \big\|\nabla_x\Phi_{t_i}^T\big(\xi_i(\theta)\big)^\top\nabla_x^2\phi\big(\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big)^\top D_\tau\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big\|d\theta\bigg)^4\\
&\le 8\mathbb{E}\bigg(\int_0^1 \big\|\nabla_x\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big\|\cdot\big\|\nabla_x^2\phi\big(\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big)^\top\big\|\cdot\big\|D_\tau\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big\|d\theta\bigg)^4\\
&\le 8C\mathbb{E}\bigg(\int_0^1 \big\|\nabla_x\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big\|\cdot\big\|D_\tau\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big\|d\theta\bigg)^4\\
&\le 8C\mathbb{E}\int_0^1 \big\|\nabla_x\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big\|^4\cdot\big\|D_\tau\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big\|^4d\theta\\
&= 4C\mathbb{E}\int_0^1 \Big(\big\|\nabla_x\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big\|^8+\big\|D_\tau\Phi_{t_i}^T\big(\xi_i(\theta)\big)\big\|^8\Big)d\theta \le C.
\end{aligned}
$$

Thus, we have proved

$$
\sup_{\tau\in[0,T]}\mathbb{E}\|D_\tau\Theta_i\|^4 \le C.
$$

In a similar way, we can prove

$$\sup_{\tau_1,\tau_2\in[0,T]} \mathbb{E}\|D^2_{\tau_1,\tau_2}\Theta_i\|^4 \le C.$$

The proof is complete. □

Theorem 3.1. *Assume* (H1)-(H3) *hold. Then for any* $\phi(\cdot)\in\mathcal{C}_b^3(\mathbb{R}^d)$*, there exists a positive constant* C *independent of* κ *such that*

$$\big|\mathbb{E}\phi\big(X(T)\big)-\mathbb{E}\phi(\tilde X_N)\big| \le C\kappa. \tag{3.10}$$

Proof. From Lemma 3.2, it follows that

$$\begin{aligned}
&\big|\mathbb{E}\phi\big(X(T)\big)-\mathbb{E}\phi(\tilde X_N)\big|\\
\le&\sum_{i=1}^N \big|\mathbb{E}\big(\Theta_i^{\mathsf T}\big(\Phi_{t_{i-1}}^{t_i}(\tilde X_{i-1})-\tilde X_i\big)\big)\big|\\
=&\sum_{i=1}^N \big|\mathbb{E}\big(\Theta_i,\Phi_{t_{i-1}}^{t_i}(\tilde X_{i-1})-\tilde X_i\big)\big|,
\end{aligned}$$

here we use the fact that $x^{\mathsf T}y=(x,y)$.

First, we calculate the expression of the local weak error $\mathbb{E}(\Theta_i,\Phi_{t_{i-1}}^{t_i}(\tilde X_{i-1})-\tilde X_i)$ for $i=1,\dots,N$. Similarly to (3.6), we have

$$\begin{aligned}
\Phi_{t_{i-1}}^{t_i}(\tilde X_{i-1}) &= \tilde X_{i-1}+\int_{t_{i-1}}^{t_i} b\big(\Phi_{t_{i-1}}^{t}(\tilde X_{i-1})\big)dt+\int_{t_{i-1}}^{t_i}\sigma\big(\Phi_{t_{i-1}}^{t}(\tilde X_{i-1})\big)dW(t)\\
&= \tilde X_{i-1}+\int_{t_{i-1}}^{t_i} b\bigg(\tilde X_{i-1}+\int_{t_{i-1}}^{t} b\big(X(s)\big)ds+\int_{t_{i-1}}^{t}\sigma\big(X(s)\big)dW(s)\bigg)dt\\
&\quad+\int_{t_{i-1}}^{t_i}\sigma\bigg(\tilde X_{i-1}+\int_{t_{i-1}}^{t} b\big(X(s)\big)ds+\int_{t_{i-1}}^{t}\sigma\big(X(s)\big)dW(s)\bigg)dW(t).
\end{aligned}$$

Subtract the above equation and (3.6) and by mean value theorem, we get

$$\begin{aligned}
&\Phi_{t_{i-1}}^{t_i}(\tilde X_{i-1})-\tilde X_i\\
=&\int_{t_{i-1}}^{t_i}\int_0^1 \nabla_x b\big(\theta\Phi_{t_{i-1}}^{t}(\tilde X_{i-1})+(1-\theta)X_{2\kappa}(t)\big)d\theta\big(\Phi_{t_{i-1}}^{t}(\tilde X_{i-1})-X_{2\kappa}(t)\big)dt\\
&+\int_{t_{i-1}}^{t_i}\int_0^1 \nabla_x \sigma\big(\theta\Phi_{t_{i-1}}^{t}(\tilde X_{i-1})+(1-\theta)X_{1\kappa}(t)\big)d\theta\big(\Phi_{t_{i-1}}^{t}(\tilde X_{i-1})-X_{1\kappa}(t)\big)dW(t)\\
=&\int_{t_{i-1}}^{t_i}\Theta_i^1(t)\big(\Phi_{t_{i-1}}^{t}(\tilde X_{i-1})-X_{2\kappa}(t)\big)dt+\int_{t_{i-1}}^{t_i}\Theta_i^2(t)\big(\Phi_{t_{i-1}}^{t}(\tilde X_{i-1})-X_{1\kappa}(t)\big)dW(t)\\
=&\int_{t_{i-1}}^{t_i}\Theta_i^1(t)\bigg(\int_{t_{i-1}}^{t} b\big(X(s)\big)-b\big(X_{2\kappa}(s)\big)ds+\int_{t_{i-1}}^{t}\sigma\big(X(s)\big)dW(s)-\int_{t_{i-1}}^{t_i}\sigma\big(X_{1\kappa}(s)\big)dW(s)\bigg)dt\\
&+\int_{t_{i-1}}^{t_i}\Theta_i^2(t)\bigg(\int_{t_{i-1}}^{t} b\big(X(s)\big)ds+\int_{t_{i-1}}^{t}\sigma\big(X(s)\big)-\sigma\big(X_{1\kappa}(s)\big)dW(s)\bigg)dW(t)\\
=:&\ I+II,
\end{aligned}$$

where for $t \in [t_{i-1}, t_i]$,

$$\Theta_i^1(t) = \int_0^1 \nabla_x b\big(\theta \Phi_{t_{i-1}}^t(\tilde{X}_{i-1}) + (1-\theta) X_{2\kappa}(t)\big) d\theta \in \mathbb{R}^{d\times d},$$
$$\Theta_i^2(t) = \int_0^1 \nabla_x \sigma\big(\theta \Phi_{t_{i-1}}^t(\tilde{X}_{i-1}) + (1-\theta) X_{1\kappa}(t)\big) d\theta \in \mathcal{L}(\mathbb{R}^{d\times d}, \mathbb{R}^m).$$

Following the proof for Lemma 3.3, there exists a positive constant C independent of κ such that

$$\mathbb{E} \sup_{t\in[t_{i-1},t_i]} \|\Theta^j(t)\|^4 + \sup_{\tau\in[0,T]} \mathbb{E} \sup_{t\in[t_{i-1},t_i]} \big\|D_\tau \Theta^j(t)\big\|^4 \le C, \quad j = 1, 2. \tag{3.11}$$

Thus $\Theta_i^1(t), \Theta_i^2(t) \in \mathbb{D}^{1,4}$, and there holds that

$$\mathbb{E}\big(\Theta_i, \Phi_{t_{i-1}}^{t_i}(\tilde{X}_{i-1}) - \tilde{X}_i\big) = \mathbb{E}(\Theta_i, I) + \mathbb{E}(\Theta_i, II). \tag{3.12}$$

Next, we estimate the local weak error. Now we study the first item in above equation. By triangle inequality, we have

$$\begin{aligned}
|\mathbb{E}(\Theta_i, I)| \le& \left|\mathbb{E}\left(\Theta_i, \int_{t_{i-1}}^{t_i} \Theta_i^1(t) \int_{t_{i-1}}^{t} b\big(X(s)\big) - b\big(X_{2\kappa}(s)\big) ds\, dt\right)\right| \\
&+ \left|\mathbb{E}\left(\Theta_i, \int_{t_{i-1}}^{t_i} \Theta_i^1(t) \int_{t_{i-1}}^{t} \sigma\big(X(s)\big) dW(s) dt\right)\right| \\
&+ \left|\mathbb{E}\left(\Theta_i, \int_{t_{i-1}}^{t_i} \Theta_i^1(t) \int_{t_{i-1}}^{t_i} \sigma\big(X_{1\kappa}(s)\big) dW(s) dt\right)\right|.
\end{aligned}$$

By Hölder inequality, Lemma 3.3 and mean value theorem, we have

$$\begin{aligned}
&\left|\mathbb{E}\left(\Theta_i, \int_{t_{i-1}}^{t_i} \Theta_i^1(t) \int_{t_{i-1}}^{t} b\big(X(s)\big) - b\big(X_{2\kappa}(s)\big) ds\, dt\right)\right|^2 \\
\le& \mathbb{E}\|\Theta_i\|^2 \cdot \mathbb{E}\left\| \int_{t_{i-1}}^{t_i} \Theta_i^1(t) \int_{t_{i-1}}^{t} b\big(X(s)\big) - b\big(X_{2\kappa}(s)\big) ds\, dt\right\|^2 \\
\le& C\mathbb{E}\left(\int_{t_{i-1}}^{t_i} \big\|\Theta_i^1(t)\big\| \int_{t_{i-1}}^{t} \left\| \int_0^1 \nabla_x b\big(\theta X(s) + (1-\theta) X_{2\kappa}(s)\big) d\theta \big(X(s) - X_{2\kappa}(s)\big)\right\| ds\, dt\right)^2 \\
\le& C\kappa^4.
\end{aligned}$$

Obviously, $1_{[t_{i-1},t]}(\tau)\sigma(X(\tau)) \in \mathbb{L}^2_{d\times m}(0,T)$. Applying Malliavin integration by parts in Lemma 2.4, we obtain

$$\begin{aligned}
&\left|\mathbb{E}\left(\Theta_i, \int_{t_{i-1}}^{t_i} \Theta_i^1(t) \int_{t_{i-1}}^{t} \sigma\big(X(s)\big) dW(s) dt\right)\right| \\
=& \left|\mathbb{E} \int_{t_{i-1}}^{t_i} \left(\Theta_i, \Theta_i^1(t) \int_{t_{i-1}}^{t} \sigma\big(X(s)\big) dW(s)\right) dt\right| \\
=& \left| \int_{t_{i-1}}^{t_i} \mathbb{E}\left(\Theta_i^1(t)^\mathsf{T} \Theta_i, \int_0^T 1_{[t_{i-1},t]}(\tau) \sigma\big(X(\tau)\big) dW(\tau)\right) dt\right| \\
=& \left| \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t} \mathbb{E}\big(D_\tau\big(\Theta_i^1(t)^\mathsf{T} \Theta_i\big), \sigma\big(X(\tau)\big)\big) d\tau dt\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t}\mathbb{E}\Big(\big\|D_\tau\big(\Theta_i^1(t)^\mathsf{T}\Theta_i\big)\big\|^2+\big\|\sigma\big(X(\tau)\big)\big\|^2\Big)d\tau dt\\
&\leq \frac{1}{2}\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t}\mathbb{E}\Big(2\big\|D_\tau\Theta_i^1(t)^\mathsf{T}\Theta_i\big\|^2+2\big\|\Theta_i^1(t)^\mathsf{T}D_\tau\Theta_i\big\|^2+\beta\big(1+\big\|X(\tau)\big\|^2\big)\Big)d\tau dt\\
&\leq \frac{1}{2}\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t}\mathbb{E}\Big(\big\|D_\tau\Theta_i^1(t)\big\|^4+\|\Theta_i\|^4+\big\|\Theta_i^1(t)\big\|^4+\|D_\tau\Theta_i\|^4\Big)d\tau dt\\
&\quad+\frac{1}{2}\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t}\beta\big(1+\big(1+\mathbb{E}\|X_0\|^2\big)e^{2CT}\big)d\tau dt\\
&\leq C\kappa^2.
\end{aligned}
$$

Similarly, we get

$$
\left|\mathbb{E}\bigg(\Theta_i,\int_{t_{i-1}}^{t_i}\Theta_i^1(t)\int_{t_{i-1}}^{t_i}\sigma\big(X_{1\kappa}(s)\big)dW(s)dt\bigg)\right|\leq C\kappa^2.
$$

Therefore, we have proved $|\mathbb{E}(\Theta_i,I)|\leq C\kappa^2$.

Applying Malliavin integration by parts in Lemma 2.4 twice to $\mathbb{E}(\Theta_i,II)$, we get

$$
\begin{aligned}
\mathbb{E}(\Theta_i,II)&=\mathbb{E}\bigg(\Theta_i,\int_0^T\bigg(1_{[t_{i-1},t_i]}\Theta_i^2(t)\int_{t_{i-1}}^{t}b\big(X(s)\big)ds\bigg)dW(t)\bigg)\\
&\quad+\mathbb{E}\bigg(\Theta_i,\int_0^T\bigg(1_{[t_{i-1},t_i]}\Theta_i^2(t)\int_{t_{i-1}}^{t}\sigma\big(X(s)\big)-\sigma\big(X_{1\kappa}(s)\big)dW(s)\bigg)dW(t)\bigg)\\
&=\mathbb{E}\int_{t_{i-1}}^{t_i}\bigg(D_\tau\Theta_i,\Theta_i^2(\tau)\int_{t_{i-1}}^{\tau}b\big(X(s)\big)ds\bigg)d\tau\\
&\quad+\mathbb{E}\int_{t_{i-1}}^{t_i}\bigg(D_\tau\Theta_i,\Theta_i^2(\tau)\int_0^T 1_{[t_{i-1},\tau]}\sigma\big(X(s)\big)-\sigma\big(X_{1\kappa}(s)\big)dW(s)\bigg)d\tau\\
&=\mathbb{E}\int_{t_{i-1}}^{t_i}\bigg(D_\tau\Theta_i,\Theta_i^2(\tau)\int_{t_{i-1}}^{\tau}b\big(X(s)\big)ds\bigg)d\tau\\
&\quad+\mathbb{E}\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{\tau}\Big(D_\rho\big(\Theta_i^2(\tau)^\mathsf{T}D_\tau\Theta_i\big),\sigma\big(X(\rho)\big)-\sigma\big(X_{1\kappa}(\rho)\big)\Big)d\rho d\tau.
\end{aligned}
$$

In a similar way, we can prove $|\mathbb{E}(\Theta_i,II)|\leq C\kappa^2$. Therefore, we have

$$
\big|\mathbb{E}\big(\Theta_i,\Phi_{t_{i-1}}^{t_i}(\tilde{X}_{i-1})-\tilde{X}_i\big)\big|\leq C\kappa^2.
$$

Finally, we obtain the estimate for weak error

$$
\big|\mathbb{E}\phi\big(X(T)\big)-\mathbb{E}\phi(\tilde{X}_N)\big|\leq\sum_{i=1}^N\big|\mathbb{E}\big(\Theta_i,\Phi_{t_{i-1}}^{t_i}(\tilde{X}_{i-1})-\tilde{X}_i\big)\big|\leq\sum_{i=1}^N C\kappa^2=C\kappa.
$$

This completes the proof. □

4. A Splitting-up Numerical Solution and Weak Convergence Analysis

In this section, we construct a splitting-up numerical scheme for (2.1)-(2.2) and analyze its weak convergence order.

4.1. Splitting-up numerical scheme

We discretize (3.1) and (3.2) by the Euler-Maruyama and explicit Euler scheme, respectively, to obtain for $i = 1, \ldots, N$,

$$X_{1\kappa,i} = X_{1\kappa,i-1} + \sigma(X_{1\kappa,i-1})\Delta W_i := \bar{Q}^{t_i}_{t_{i-1}} X_{1\kappa,i-1}, \tag{4.1}$$

$$X_{2\kappa,i} = X_{2\kappa,i-1} + b(X_{2\kappa,i-1})\kappa := \bar{R}^{t_i}_{t_{i-1}} X_{2\kappa,i-1}, \tag{4.2}$$

where $\Delta W_i = W(t_i) - W(t_{i-1})$, and $\bar{Q}^{t_i}_{t_{i-1}}$ and $\bar{R}^{t_i}_{t_{i-1}}$ denote the corresponding iterative mappings from t_{i-1} to t_i, respectively. Then, we define an auxiliary function $X_{1\kappa,i}$ and the splitting-up numerical solution X_i at t_i by

$$X_{1\kappa,i} = \bar{Q}^{t_i}_{t_{i-1}} X_{i-1}, \quad X_i = \bar{R}^{t_i}_{t_{i-1}} \bar{Q}^{t_i}_{t_{i-1}} X_{i-1}, \quad i = 1, \ldots, N, \tag{4.3}$$

where X_0 is given in (H3). Furthermore, $X_{1\kappa,i}$ and X_i are $\mathcal{F}(t_i)$-measurable, and we have

$$\begin{aligned} X_i &= X_{1\kappa,i} + \int_{t_{i-1}}^{t_i} b(X_{1\kappa,i})dt \\ &= X_{i-1} + \int_{t_{i-1}}^{t_i} \sigma(X_{i-1})dW(t) + \int_{t_{i-1}}^{t_i} b\Big(X_{i-1} + \int_{t_{i-1}}^{t_i} \sigma(X_{i-1})dW(s)\Big)dt. \end{aligned} \tag{4.4}$$

The following lemma gives the boundedness property for X_i.

Lemma 4.1. *Assume* (H1)-(H3) *hold. Then* $X_i \in \bigcap_{p\geq 2} L^p(\Omega;\mathbb{R}^d), 0 \leq i \leq N$, *and for any given* $p \geq 2$, *there exists a constant* $C = C(p)$ *independent of* κ *such that*

$$\|X_i\|_{L^p(\Omega;\mathbb{R}^d)} \leq C. \tag{4.5}$$

Higham *et al.* [21, Lemma 3.7] proved a similar property, where the implicit Euler scheme is used. The conclusion of this lemma can be obtained by following a similar proof.

Noticing that X_{i-1} is independent of ΔW_i, by (2.3), we have for $p \geq 1$,

$$\begin{aligned} \mathbb{E}\|X_{1\kappa,i}\|^{2p} &\leq 2^{2p-1}\mathbb{E}\|X_{i-1}\|^{2p} + 2^{2p-1}\mathbb{E}\|\sigma(X_{i-1})\Delta W_i\|^{2p} \\ &\leq 2^{2p-1}\mathbb{E}\|X_{i-1}\|^{2p} + 2^{2p-1}\mathbb{E}\|\sigma(X_{i-1})\|^{2p}\mathbb{E}\|\Delta W_i\|^{2p} \\ &= 2^{2p-1}\mathbb{E}\|X_{i-1}\|^{2p} + 2^{2p-1}2^{2p-1}\beta^{2p}(1 + \mathbb{E}\|X_{i-1}\|^{2p})\kappa^p, \end{aligned}$$

thus, there exists a constant $C = C(p)$ independent of κ such that

$$\|X_{1\kappa,i}\|_{L^{2p}(\Omega,\mathbb{R}^d)} \leq C. \tag{4.6}$$

4.2. Weak convergence order of the splitting-up numerical solution

In this section, we analyze the weak convergence order of the splitting-up numerical solution X_i. Firstly, we transform the global weak error to the summation of local weak errors. Similarly to Lemma 3.2, we obtain that

$$\mathbb{E}\phi\big(X(T)\big) - \mathbb{E}\phi(X_N) = \sum_{i=1}^{N} \mathbb{E}\big[\Lambda_i^{\mathsf{T}}\big(\Phi^{t_i}_{t_{i-1}}(X_{i-1}) - X_i\big)\big], \tag{4.7}$$

where

$$\Lambda_i = \int_0^1 \nabla_x \Phi_{t_i}^T\big(\eta_i(\theta)\big)^{\mathsf{T}} \nabla_x \phi\big(\Phi_{t_i}^T\big(\eta_i(\theta)\big)\big)^{\mathsf{T}} d\theta,$$
$$\eta_i(\theta) = \theta \Phi_{t_{i-1}}^{t_i}(X_{i-1}) + (1-\theta) X_i.$$

According to Lemmas 3.3 and 4.1, there exists a positive constant C independent of κ such that for $i = 0, 1, \ldots, N$,

$$\mathbb{E}\|\Lambda_i\|^4 + \sup_{\tau\in[0,T]} \mathbb{E}\|D_\tau \Lambda_i\|^4 + \sup_{\tau_1,\tau_2\in[0,T]} \mathbb{E}\big\|D^2_{\tau_1,\tau_2}\Lambda_i\big\|^4 \le C. \tag{4.8}$$

Theorem 4.1. *Assume* (H1)-(H3) *hold. Let $X(t)$ be the solution to* (2.1)-(2.2) *and X_i be its splitting-up numerical solution. Then for any $\phi(\cdot) \in \mathcal{C}_b^3(\mathbb{R}^d)$, there exists a positive constant C independent of κ such that*

$$\big|\mathbb{E}\phi\big(X(T)\big) - \mathbb{E}\phi(X_N)\big| \le C\kappa. \tag{4.9}$$

Proof. From (4.7), it follows that

$$\big|\mathbb{E}\phi\big(X(T)\big) - \mathbb{E}\phi(X_N)\big| \le \sum_{i=1}^N \big|\mathbb{E}\big(\Lambda_i, \Phi_{t_{i-1}}^{t_i}(X_{i-1}) - X_i\big)\big|.$$

First, we calculate an expression of the local weak error $\mathbb{E}(\Lambda_i, \Phi_{t_{i-1}}^{t_i}(X_{i-1}) - X_i)$ for $i = 1, \ldots, N$. By the property of solution operator Φ_s^t, (4.4) and the mean value theorem, direct computation gives

$$\begin{aligned}
&\Phi_{t_{i-1}}^{t_i}(X_{i-1}) - X_i \\
&= \int_{t_{i-1}}^{t_i}\int_0^1 \nabla_x b\big(\theta\Phi_{t_{i-1}}^t(X_{i-1}) + (1-\theta)X_{1\kappa,i}\big)d\theta\big(\Phi_{t_{i-1}}^t(X_{i-1}) - X_{1\kappa,i}\big)dt \\
&\quad + \int_{t_{i-1}}^{t_i}\int_0^1 \nabla_x \sigma\big(\theta\Phi_{t_{i-1}}^t(X_{i-1}) + (1-\theta)X_{i-1}\big)d\theta\big(\Phi_{t_{i-1}}^t(X_{i-1}) - X_{i-1}\big)dW(t) \\
&= \int_{t_{i-1}}^{t_i} \Lambda_i^1(t)\bigg(\int_{t_{i-1}}^t b\big(X(s)\big)ds + \int_{t_{i-1}}^t \sigma\big(X(s)\big)dW(s) - \int_{t_{i-1}}^{t_i}\sigma(X_{i-1})dW(s)\bigg)dt \\
&\quad + \int_{t_{i-1}}^{t_i} \Lambda_i^2(t)\bigg(\int_{t_{i-1}}^t b\big(X(s)\big)ds + \int_{t_{i-1}}^t \sigma\big(X(s)\big)dW(s)\bigg)dW(t) \\
&=: I + II,
\end{aligned}$$

where for $t \in [t_{i-1}, t_i]$,

$$\Lambda_i^1(t) = \int_0^1 \nabla_x b\big(\theta\Phi_{t_{i-1}}^t(X_{i-1}) + (1-\theta)X_{1\kappa,i}\big)d\theta \in \mathbb{R}^{d\times d},$$
$$\Lambda_i^2(t) = \int_0^1 \nabla_x \sigma\big(\theta\Phi_{t_{i-1}}^t(X_{i-1}) + (1-\theta)X_{i-1}\big)d\theta \in \mathcal{L}(\mathbb{R}^{d\times d}, \mathbb{R}^m).$$

According to Lemmas 3.3, 4.1 and (4.6), there exists a positive constant C independent of κ such that

$$\mathbb{E}\sup_{t\in[t_{i-1},t_i]}\|\Lambda^j(t)\|^4 + \sup_{\tau\in[0,T]}\mathbb{E}\sup_{t\in[t_{i-1},t_i]}\big\|D_\tau\Lambda^j(t)\big\|^4 \le C, \quad j = 1, 2. \tag{4.10}$$

Thus, $\Lambda_i^1(t), \Lambda_i^2(t) \in \mathbb{D}^{1,4}$, and there holds that

$$\mathbb{E}\big(\Lambda_i, \Phi_{t_{i-1}}^{t_i}(X_{i-1}) - X_i\big) = \mathbb{E}(\Lambda_i, I) + \mathbb{E}(\Lambda_i, II). \tag{4.11}$$

Next, we estimate the local weak error. Now we study the first item in above equation. By triangular inequality, we have

$$\begin{aligned}|\mathbb{E}(\Lambda_i, I)| \le &\left|\mathbb{E}\left(\Lambda_i, \int_{t_{i-1}}^{t_i} \Lambda_i^1(t) \int_{t_{i-1}}^{t} b\big(X(s)\big) ds\, dt\right)\right| \\ &+ \left|\mathbb{E}\left(\Lambda_i, \int_{t_{i-1}}^{t_i} \Lambda_i^1(t) \int_{t_{i-1}}^{t} \sigma\big(X(s)\big) dW(s) dt\right)\right| \\ &+ \left|\mathbb{E}\left(\Lambda_i, \int_{t_{i-1}}^{t_i} \Lambda_i^1(t) \int_{t_{i-1}}^{t_i} \sigma(X_i) dW(s) dt\right)\right|.\end{aligned}$$

By Hölder inequality and (4.10), we have

$$\begin{aligned}&\left|\mathbb{E}\left(\Lambda_i, \int_{t_{i-1}}^{t_i} \Lambda_i^1(t) \int_{t_{i-1}}^{t} b\big(X(s)\big) ds\, dt\right)\right|^2 \\ &\le \mathbb{E}\|\Lambda_i\|^2 \cdot \mathbb{E}\left\| \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t} \Lambda_i^1(t) b\big(X(s)\big) ds dt \right\|^2 \\ &\le C\mathbb{E}\left(\int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t} \big\|\Lambda_i^1(t)\big\| \cdot \big\|b\big(X(s)\big)\big\| ds\, dt\right)^2 \le C\kappa^4.\end{aligned}$$

Obviously, $1_{[t_{i-1},t]}(\tau)\sigma(X(\tau)) \in \mathbb{L}^2_{d\times m}(0,T)$. Applying Malliavin integration by parts in Lemma 2.4, we obtain

$$\begin{aligned}&\left|\mathbb{E}\left(\Lambda_i, \int_{t_{i-1}}^{t_i} \Lambda_i^1(t) \int_{t_{i-1}}^{t} \sigma\big(X(s)\big) dW(s) dt\right)\right| \\ &= \left| \int_{t_{i-1}}^{t_i} \mathbb{E}\left(\Lambda_i^1(t)^\top \Lambda_i, \int_0^T 1_{[t_{i-1},t]}(\tau)\sigma\big(X(\tau)\big) dW(\tau)\right) dt\right| \\ &= \left| \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t} \mathbb{E}\big(D_\tau\big(\Lambda_i^1(t)^\top \Lambda_i\big), \sigma\big(X(\tau)\big)\big) d\tau dt\right| \\ &\le \frac{1}{2} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t} \mathbb{E}\big\|D_\tau\big(\Lambda_i^1(t)^\top \Lambda_i\big)\big\|^2 + \mathbb{E}\big\|\sigma\big(X(\tau)\big)\big\|^2 d\tau dt \\ &\le \frac{1}{2} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t} \mathbb{E}\Big(2\big\|D_\tau \Lambda_i^1(t)^\top \Lambda_i\big\|^2 + 2\big\|\Lambda_i^1(t)^\top D_\tau \Lambda_i\big\|^2 + \beta\big(1 + \|X(\tau)\|^2\big)\Big) d\tau dt \\ &\le \frac{1}{2} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t} \mathbb{E}\Big(\big\|D_\tau \Lambda_i^1(t)\big\|^4 + \|\Lambda_i\|^4 + \big\|\Lambda_i^1(t)\big\|^4 + \|D_\tau \Lambda_i\|^4\Big) d\tau dt \\ &\quad + \frac{1}{2} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t} \beta\big(1 + \big(1 + \mathbb{E}\|X_0\|^2\big) e^{2CT}\big) d\tau dt \\ &\le C\kappa^2.\end{aligned}$$

Similarly, we can prove

$$\left|\mathbb{E}\left(\Lambda_i, \int_{t_{i-1}}^{t_i} \Lambda_i^1(t) \int_{t_{i-1}}^{t_i} \sigma\big(X_1(s)\big) dW(s) dt\right)\right| \le C\kappa^2.$$

Therefore, we have proved $|\mathbb{E}(\Lambda_i, I)| \le C\kappa^2$.

Applying Malliavin integration by parts in Lemma 2.4 twice to $\mathbb{E}(\Lambda_i, II)$, we get

$$\begin{aligned}\mathbb{E}(\Lambda_i, II) &= \mathbb{E}\Big(\Lambda_i, \int_0^T \Big(1_{[t_{i-1},t_i]}\Lambda_i^2(t)\int_{t_{i-1}}^t b\big(X(s)\big)ds\Big)dW(t)\Big)\\ &\quad + \mathbb{E}\Big(\Lambda_i, \int_0^T \Big(1_{[t_{i-1},t_i]}\Lambda_i^2(t)\int_{t_{i-1}}^t \sigma\big(X(s)\big)dW(s)\Big)dW(t)\Big)\\ &= \mathbb{E}\int_{t_{i-1}}^{t_i}\Big(D_\tau\Lambda_i, \Lambda_i^2(\tau)\int_{t_{i-1}}^{\tau} b\big(X(s)\big)ds\Big)d\tau\\ &\quad + \mathbb{E}\int_{t_{i-1}}^{t_i}\Big(D_\tau\Lambda_i, \Lambda_i^2(\tau)\int_0^T 1_{[t_{i-1},\tau]}\sigma\big(X(s)\big)dW(s)\Big)d\tau\\ &= \mathbb{E}\int_{t_{i-1}}^{t_i}\Big(D_\tau\Lambda_i, \Lambda_i^2(\tau)\int_{t_{i-1}}^{\tau} b\big(X(s)\big)ds\Big)d\tau\\ &\quad + \mathbb{E}\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{\tau}\Big(D_\rho\big(\Lambda_i^2(\tau)^{\mathsf{T}}D_\tau\Lambda_i\big), \sigma\big(X(\rho)\big)\Big)d\rho d\tau.\end{aligned}$$

In a similar way, we can prove $|\mathbb{E}(\Lambda_i, II)| \le C\kappa^2$. Therefore, we have

$$\big|\mathbb{E}\big(\Lambda_i, \Phi_{t_{i-1}}^{t_i}(\tilde{X}_{i-1}) - \tilde{X}_i\big)\big| \le C\kappa^2.$$

Finally, we obtain the weak error estimate

$$\big|\mathbb{E}\phi\big(X(T)\big) - \mathbb{E}\phi(X_N)\big| \le \sum_{i=1}^N \big|\mathbb{E}\big(\Lambda_i, \Phi_{t_{i-1}}^{t_i}(X_{i-1}) - X_i\big)\big| \le \sum_{i=1}^N C\kappa^2 = C\kappa.$$

This completes the proof. □

5. Numerical Experiments

In this section, we present several numerical experiments to illustrate theoretical analysis. In order to grasp weak convergence order, we take several step size $\tau_i = 2^{-i}, i = 5, \dots, 10$, to calculate the splitting-up numerical solution $X_N^{\tau_i}$. And we regard $X_N^{\tau_{10}}$ as the reference "real" solution. Taking $\phi(x) = \sin x$, we define a weak error function

$$e_{\mathrm{w}}^i = \big|\mathbb{E}\big[\phi\big(X(T)\big)\big] - \mathbb{E}\big[\phi\big(X_N^{\tau_i}\big)\big]\big|.$$

Based on Monte-Carlo technique, we choose $M = 10000$ independent sample trajectories $X_N^{\tau_i}(\omega)$ to approximate

$$e_{\mathrm{w}}^i \approx \sum_{\omega}\big|\phi\big(X_N^{\tau_{10}}\big) - \phi\big(X_N^{\tau_i}(\omega)\big)\big|/10000.$$

We approach the weak convergence order by $(\log(e_{\mathrm{w}}^i) - \log(e_{\mathrm{w}}^{i-1}))/(\log\tau^i - \log\tau^{i-1})$.

Example 5.1. Consider the one-dimensional stochastic differential equation on $[0, 1]$

$$dX(t) = -X(t)dt + \cos\big(X(t)\big)dW(t), \quad X(0) = 1,$$

where W is a standard Brownian motion. For each fixed step size τ_i, we compute its 10000 independent sample trajectories to approximate weak error e_{w}^i and derive the convergence order, cf. Table 5.1 and Fig. 5.1.

Table 5.1: Weak error and convergence order.

τ	Weak error	Order
2^{-5}	$1.317\ \times 10^{-3}$	
2^{-6}	$6.436\ \times 10^{-4}$	1.0333
2^{-7}	$3.165\ \times 10^{-4}$	1.0241
2^{-8}	$1.515\ \times 10^{-4}$	1.0627
2^{-9}	$6.834\ \times 10^{-5}$	1.1485

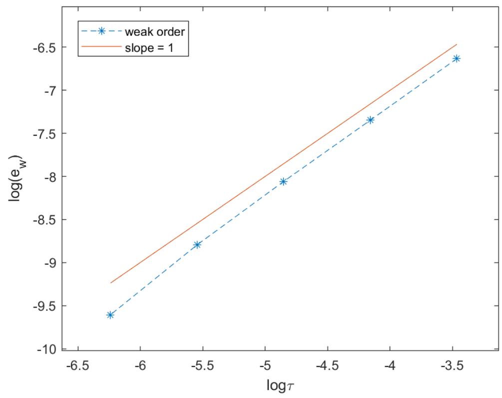

Fig. 5.1. Weak convergence order in log-log scale.

Example 5.2. Consider the two-dimensional stochastic differential equation on $[0, 1]$

$$d\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} \sin(X_1) & 0 \\ 0 & \cos(X_2) \end{bmatrix} \begin{bmatrix} dW^1 \\ dW^2 \end{bmatrix}, \quad \begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where W^1 and W^2 are two independent Brownian motions on $[0, 1]$. For each fixed step size τ_i, we compute its 10000 independent sample trajectories to approximate weak error e_{w}^i and derive the convergence order, cf. Table 5.2 and Fig. 5.2.

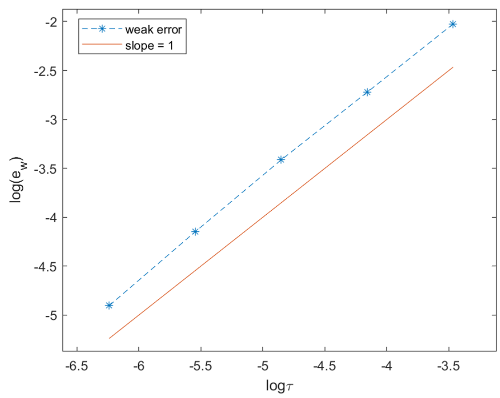

Fig. 5.2. Weak convergence order in log-log scale.

Table 5.2: Weak error and convergence order.

τ	Weak error	Order
2^{-5}	1.321×10^{-1}	
2^{-6}	6.568×10^{-2}	1.0072
2^{-7}	3.282×10^{-2}	1.0138
2^{-8}	1.579×10^{-2}	1.0600
2^{-9}	7.424×10^{-3}	1.0886

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