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ANALYSIS OF MULTIPHYSICS FINITE ELEMENT METHOD FOR A NONLINEAR POROELASTICITY MODEL*

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Abstract

In this paper, we propose and analyze a multiphysics finite element method for a nonlinear poroelasticity model. To more effectively capture the deformation and diffusion processes, we reformulate the original nonlinear fluid-solid coupling problem into a fluid-fluid coupling problem using a multiphysics approach. We then establish the growth, coercivity, and monotonicity properties of the nonlinear stress-strain relation, derive energy estimates, and use Schaefer's fixed point theorem to prove the existence and uniqueness of the weak solution. Furthermore, we design a fully discrete time-stepping scheme – multiphysics finite element method with $\mathbf{P}_2 - P_1 - P_1$ elements for spatial variables and the backward Euler method for time variable. To handle nonlinearity, we employ the Newton iterative method, establish discrete energy laws and give the optimal convergence order error estimates. Finally, we show some numerical examples to verify the rationality of theoretical analysis and the proposed method has no "locking phenomenon".

Mathematics subject classification: 65N30, 65N12.
Key words: Nonlinear poroelasticity, Stokes equations, Multiphysics finite element method, Schaefer's fixed point theorem, Error estimates, Locking phenomenon.

1. Introduction

The poroelasticity model, a fluid-solid coupled system at the pore scale, is widely used in fields such as geophysics, biomechanics, civil engineering, chemical engineering, and materials science (cf. [3, 13, 15, 20–22, 25, 28, 30, 37]). There are various types of nonlinear poroelasticity, including the models with stress-dependent permeability tensors (cf. [4, 5, 9, 10, 16, 29, 38]) and those with nonlinear constitutive stress-strain relationships for solids (cf. [2, 11, 23]). This paper specifically considers a quasi-static nonlinear poroelasticity model with a nonlinear stress-strain relationship as follows (for the linear poroelasticity model, one can refer to [18, 31]):

$$-\operatorname{div}\tilde{\sigma}(\mathbf{u}) + \alpha\nabla p = \mathbf{f} \quad \text{in } \Omega_T := \Omega \times (0,T) \subset \mathbb{R}^d \times (0,T), \tag{1.1}$$

$$(c_0 p + \alpha \operatorname{div}\mathbf{u})_t + \operatorname{div}\mathbf{v}_f = \phi \quad \text{in } \Omega_T, \tag{1.2}$$

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where

$$\tilde{\sigma}(\mathbf{u}) = \mu\tilde{\varepsilon}(\mathbf{u}) + \lambda tr\big(\tilde{\varepsilon}(\mathbf{u})\big)\mathbf{I}, \quad \tilde{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla^{\top}\mathbf{u} + \nabla^{\top}\mathbf{u}\nabla\mathbf{u}), \tag{1.3}$$

$$\mathbf{v}_f = -\frac{K}{\mu_f}\big(\nabla p - \rho_f\mathbf{g}\big). \tag{1.4}$$

Here $\Omega \subset \mathbb{R}^d\,(d=2,3)$ denotes a bounded polygonal domain with the boundary $\partial\Omega$. $\tilde{\varepsilon}(\mathbf{u})$ is known as the Green strain tensor, $\mathbf{u}$ denotes the displacement vector of the solid and p denotes the pressure of the fluid. $\mathbf{I}$ denotes the $d\times d$ identity matrix. $\mathbf{f}$ is the body force and ϕ is the source term. The permeability tensor $K = K(x)$ is assumed to be symmetric and uniformly positive definite in the sense that there exists positive constants K_1 and K_2 such that

$$K_1|\zeta|^2 \le K(x)\zeta\cdot\zeta \le K_2|\zeta|^2$$

for a.e. $x \in \Omega$ and any $\zeta \in \mathbb{R}^d$; the fluid viscosity, Biot-Willis constant and constrained specific storage coefficient are μ_f, α, c_0, respectively. In addition, $\tilde{\sigma}(\mathbf{u})$ is called the (effective) stress tensor. $\mathbf{v}_f$ is the fluid flux and (1.4) is called the well-known Darcy's law. ρ_f is the density of fluid and $\mathbf{g}$ is the acceleration of gravity. λ and μ are Lamé constants, $\widehat{\sigma}(\mathbf{u},p) := \tilde{\sigma}(\mathbf{u}) - \alpha p\mathbf{I}$ is the total stress tensor.

To close the above system, the following set of boundary and initial conditions will be considered in this paper:

$$\widehat{\sigma}(\mathbf{u},p)\mathbf{n} = \tilde{\sigma}(\mathbf{u})\mathbf{n} - \alpha p\mathbf{n} = \mathbf{f}_1 \qquad \text{on } \partial\Omega_T := \partial\Omega\times(0,T), \tag{1.5}$$

$$\mathbf{v}_f\cdot\mathbf{n} = -\frac{K}{\mu_f}\big(\nabla p - \rho_f\mathbf{g}\big)\cdot\mathbf{n} = \phi_1 \quad \text{on } \partial\Omega_T, \tag{1.6}$$

$$\mathbf{u} = \mathbf{u}_0, \quad p = \mathbf{u}_0 \qquad \text{in } \Omega\times\{t=0\}, \tag{1.7}$$

where $\mathbf{f}_1, \phi_1$ and $\mathbf{u}_0, \mathbf{u}_0$ are given functions.

In some engineering literature, the Lamé constant μ is also called the shear modulus and denoted by G, and $B := \lambda + 2G/3$ is called the bulk modulus. λ, μ and B are computed from the Young's modulus E and the Poisson ratio ν by the following formulas:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = G = \frac{E}{2(1+\nu)}, \quad B = \frac{E}{3(1-2\nu)}.$$

For linear poroelasticity, Showalter [35] analyzed the well-posedness of the weak solution to a linear poroelasticity model. Phillips and Wheeler [31, 32] introduced and analyzed both continuous- and discrete-time mixed finite element methods for poroelasticity, which approximate pressure, its gradient, and the displacement vector field simultaneously. They noted that the continuous Galerkin finite element method is prone to "locking phenomenon". Feng, Ge and Li [18] developed a multiphysics finite element method to eliminate the above mentioned "locking phenomenon" and displacement "locking phenomenon". Following the approach of [18], we firstly reformulate the original model and prove the growth, coercivity and monotonicity of nonlinear operator $\mathcal{N}(\nabla\mathbf{u})$ for the nonlinear poroelasticity model with the stress-strain relation $\tilde{\sigma}(\mathbf{u})$ (see (1.3)). We then derive energy estimates and use Schaefer's fixed-point theorem to prove the existence and uniqueness of a weak solution for the nonlinear model without any assumptions on the nonlinear stress-strain relation. Additionally, we propose a fully discrete multiphysics finite element method for the nonlinear poroelasticity model by using the $\mathbf{P}_2 - P_1 - P_1$ element pairs for space variables and backward Euler method for time variable. And we derive the discrete energy estimates, show that the existence and uniqueness of the numerical solution, give the optimal convergence order error estimates of the time-stepping method. To our knowledge, it is the first time to establish the existence and uniqueness of a weak solution without any assumptions on the nonlinear stress-strain relation, propose a fully discrete multiphysics finite element method, and derive optimal order error estimates for a nonlinear poroelasticity model. In a word, we summarize the contributions of this paper as follows:

(1) Introducing the variables $q = \operatorname{div} \mathbf{u}, \eta = c_0 p + \alpha q, \xi = \alpha p - \lambda q$, we transform the nonlinear fluid-solid coupled problem to a nonlinear fluid-fluid coupled problem – a generalized nonlinear Stokes problem coupled with a diffusion problem. Consequently, any modified stable solver for the Stokes problem and any Lagrange element for the diffusion problem can be applied.

(2) We establish the growth, coercivity, and monotonicity of the nonlinear stress-strain relation, derive energy estimates, and use Schaefer's fixed-point theorem to prove the existence and uniqueness of a weak solution with no any assumptions on the nonlinear stress-strain relation.

(3) We propose a fully discrete multiphysics finite element method for the nonlinear poroelasticity model, using $\mathbf{P}_2 - P_1 - P_1$ element pairs, which inherently overcomes the "locking" phenomena of pressure and displacement. We also perform a stability analysis and derive optimal convergence order estimates.

The remainder of this paper is organized as follows. In Section 2, we use a multiphysics approach to reformulate the original model as a fluid-fluid coupled system and define the weak solutions for both the original and reformulated models. We then establish the growth, coercivity, and monotonicity properties, derive energy estimates, and apply Schaefer's fixed-point theorem to prove the well-posedness of the weak solution for the nonlinear poroelasticity model. In Section 3, we introduce and analyze coupled and decoupled time-stepping schemes based on the multiphysics approach, deriving the discrete energy law. In Section 4, we prove that the time-stepping method achieves the optimal convergence order. In Section 5, we present some numerical examples to verify the theoretical results. Finally, we draw conclusions to summarize the main results of this paper.

2. Multiphysics Approach and PDE Analysis

2.1. Multiphysics approach

Introduce new variables $q := \operatorname{div} \mathbf{u}, \eta := c_0 p + \alpha q, \xi := \alpha p - \lambda q$, and denote

$$\mathcal{N}(\nabla \mathbf{u}) = \tilde{\sigma}(\mathbf{u}) - \lambda \operatorname{div} \mathbf{u} \mathbf{I},$$

then we have

$$\mathcal{N}(\nabla \mathbf{u}) = \mu \varepsilon(\mathbf{u}) + \frac{\mu}{2} \nabla^\top \mathbf{u} \nabla \mathbf{u} + \frac{\lambda}{2} \|\nabla \mathbf{u}\|_F^2 \mathbf{I}, \tag{2.1}$$

where

$$\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla^\top \mathbf{u} + \nabla \mathbf{u}).$$

It is easy to check that

$$p = \kappa_1\xi + \kappa_2\eta, \quad q = \kappa_1\eta - \kappa_3\xi, \tag{2.2}$$

where

$$\kappa_1 = \frac{\alpha}{\alpha^2 + \lambda c_0}, \quad \kappa_2 = \frac{\lambda}{\alpha^2 + \lambda c_0}, \quad \kappa_3 = \frac{c_0}{\alpha^2 + \lambda c_0}.$$

Then the problem (1.1)-(1.4) can be rewritten as

$$-\operatorname{div}\mathcal{N}(\nabla\mathbf{u}) + \nabla\xi = \mathbf{f} \qquad \text{in } \Omega_T, \tag{2.3}$$

$$\kappa_3\xi + \operatorname{div}\mathbf{u} = \kappa_1\eta \qquad \text{in } \Omega_T, \tag{2.4}$$

$$\eta_t - \frac{1}{\mu_f}\operatorname{div}\left[K\big(\nabla(\kappa_1\xi + \kappa_2\eta) - \rho_f\mathbf{g}\big)\right] = \phi \quad \text{in } \Omega_T. \tag{2.5}$$

The boundary and initial conditions (1.5)-(1.7) can be rewritten as

$$\tilde{\sigma}(\mathbf{u})\mathbf{n} - \alpha(\kappa_1\xi + \kappa_2\eta)\mathbf{n} = \mathbf{f}_1 \qquad \text{on } \partial\Omega_T := \partial\Omega \times (0,T), \tag{2.6}$$

$$-\frac{K}{\mu_f}\big(\nabla(\kappa_1\xi + \kappa_2\eta) - \rho_f\mathbf{g}\big)\cdot\mathbf{n} = \phi_1 \quad \text{on } \partial\Omega_T, \tag{2.7}$$

$$\mathbf{u} = \mathbf{u}_0, \quad p = p_0 \qquad \text{in } \Omega \times \{t = 0\}. \tag{2.8}$$

Remark 2.1. From the problem (2.3)-(2.8), it is well known that $(\mathbf{u}, \xi)$ satisfies a generalized nonlinear Stokes problem for a given η, and η satisfies a diffusion problem for a given ξ and $\mathbf{u}$. Thus, this new formulation reveals the underlying diffusion process in the original poroelasticity model.

2.2. PDE analysis

In this paper, the standard function space notation is adopted, their precise definitions can be found in [7, 12, 36]. In particular, $(\cdot\,,\cdot)$ and $\langle\cdot\,,\cdot\rangle$ denote respectively the standard $L^2(\Omega)$ and $L^2(\partial\Omega)$ inner products. For any Banach space B, we let $\mathbf{B} = [B]^d$, and use $\mathbf{B}'$ to denote its dual space. In particular, we use $(\cdot\,,\cdot)_{\text{dual}}$ to denote the dual product on $\mathbf{H}^1(\Omega)' \times \mathbf{H}^1(\Omega)$. We also introduce the function spaces

$$L_0^2(\Omega) := \{q \in L^2(\Omega);\ (q, 1) = 0\}, \quad \mathbf{X} := \mathbf{H}^1(\Omega).$$

From [36], it is well known that the following inf-sup condition holds in the space $\mathbf{X} \times L_0^2(\Omega)$:

$$\sup_{\mathbf{v}\in\mathbf{X}} \frac{(\operatorname{div}\mathbf{v}, \varphi)}{\|\mathbf{v}\|_{H^1(\Omega)}} \geq \alpha_0\|\varphi\|_{L^2(\Omega)}, \quad \forall\,\varphi \in L_0^2(\Omega), \quad \alpha_0 > 0. \tag{2.9}$$

Let

$$\mathbf{RM} := \{\mathbf{r} := \mathbf{a} + \mathbf{b} \times x;\ \mathbf{a}, \mathbf{b}, x \in \mathbb{R}^d\}$$

denote the space of infinitesimal rigid motions. It is well known [7,24,36] that $\mathbf{RM}$ is the kernel of the strain operator ε, that is, $\mathbf{r} \in \mathbf{RM}$ if and only if $\varepsilon(\mathbf{r}) = 0$. Hence, we have

$$\varepsilon(\mathbf{r}) = 0, \quad \operatorname{div}\mathbf{r} = 0, \quad \forall\,\mathbf{r} \in \mathbf{RM}. \tag{2.10}$$

Let $\mathbf{L}_\perp^2(\partial\Omega)$ and $\mathbf{H}_\perp^1(\Omega)$ denote respectively the subspaces of $\mathbf{L}^2(\partial\Omega)$ and $\mathbf{H}^1(\Omega)$ which are orthogonal to $\mathbf{RM}$, that is,

$$\mathbf{H}_\perp^1(\Omega) := \{\mathbf{v} \in \mathbf{H}^1(\Omega);\ (\mathbf{v}, \mathbf{r}) = 0,\ \forall\,\mathbf{r} \in \mathbf{RM}\},$$
$$\mathbf{L}_\perp^2(\partial\Omega) := \{\mathbf{g} \in \mathbf{L}^2(\partial\Omega);\ \langle\mathbf{g}, \mathbf{r}\rangle = 0,\ \forall\,\mathbf{r} \in \mathbf{RM}\}.$$

The well-known Korn's inequality [14] yields that for some $c_2 > 0$,

$$\|\mathbf{v}\|_{H^1(\Omega)} \le c_2 \|\varepsilon(\mathbf{v})\|_{L^2(\Omega)}, \quad \forall\, \mathbf{v} \in \mathbf{H}^1_\perp(\Omega). \tag{2.11}$$

From [6, 18], we know that for each $\mathbf{v} \in \mathbf{H}^1_\perp(\Omega)$ there holds the following inf-sup condition:

$$\sup_{\mathbf{v}\in\mathbf{H}^1_\perp(\Omega)} \frac{(\operatorname{div}\mathbf{v},\varphi)}{\|\mathbf{v}\|_{H^1(\Omega)}} \ge \alpha_1 \|\varphi\|_{L^2(\Omega)}, \quad \forall\, \varphi \in L^2_0(\Omega), \quad \alpha_1 > 0. \tag{2.12}$$

For convenience, we assume that $\mathbf{f}, \mathbf{f}_1, \phi$ and ϕ_1 all are independent of t in the remaining of the paper. We note that all the results of this paper can be easily extended to the case of time-dependent source functions. Next, we introduce the definitions of the weak solution to the problem (1.1)-(1.7) and (2.3)-(2.8) as follows.

Definition 2.1. *Let* $\mathbf{u}_0 \in \mathbf{H}^1(\Omega), \mathbf{f} \in \mathbf{L}^2(\Omega), \mathbf{f}_1 \in \mathbf{L}^2(\partial\Omega), p_0 \in L^2(\Omega), \phi \in L^2(\Omega)$*, and* $\phi_1 \in L^2(\partial\Omega)$*. Assume that* $c_0 > 0$ *and* $(\mathbf{f}, \mathbf{v}) + \langle \mathbf{f}_1, \mathbf{v}\rangle = 0$ *for any* $\mathbf{v} \in \mathbf{RM}$*. Given* $T > 0$*, a tuple* $(\mathbf{u}, p)$ *with*

$$\begin{aligned}
&\mathbf{u} \in L^\infty\big(0,T;\mathbf{H}^1_\perp(\Omega)\big),\\
&p \in L^\infty\big(0,T;L^2(\Omega)\big) \cap L^2\big(0,T;H^1(\Omega)\big),\\
&(c_0 p + \alpha \operatorname{div}\mathbf{u})_t \in L^2\big(0,T;H^1(\Omega)'\big)
\end{aligned}$$

is called a weak solution to the problem (1.1)-(1.7)*, if there hold for almost every* $t \in (0, T]$*,*

$$\big(\mathcal{N}(\nabla\mathbf{u}),\varepsilon(\mathbf{v})\big) + \lambda(\operatorname{div}\mathbf{u},\operatorname{div}\mathbf{v}) - \alpha(p,\operatorname{div}\mathbf{v}) = (\mathbf{f},\mathbf{v}) + \langle \mathbf{f}_1,\mathbf{v}\rangle, \qquad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega), \tag{2.13}$$

$$\big((c_0 p + \alpha\operatorname{div}\mathbf{u})_t,\varphi\big)_{\rm dual} + \frac{1}{\mu_f}\big(K(\nabla p - \rho_f \mathbf{g}),\nabla\varphi\big) = (\phi,\varphi) + \langle\phi_1,\varphi\rangle, \quad \forall\, \varphi \in H^1(\Omega), \tag{2.14}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad p(0) = p_0. \tag{2.15}$$

Definition 2.2. *Let* $\mathbf{u}_0 \in \mathbf{H}^1(\Omega), \mathbf{f} \in \mathbf{L}^2(\Omega), \mathbf{f}_1 \in \mathbf{L}^2(\partial\Omega), p_0 \in L^2(\Omega), \phi \in L^2(\Omega)$*, and* $\phi_1 \in L^2(\partial\Omega)$*. Assume that* $c_0 > 0$ *and* $(\mathbf{f}, \mathbf{v}) + \langle \mathbf{f}_1, \mathbf{v}\rangle = 0$ *for any* $\mathbf{v} \in \mathbf{RM}$*. Given* $T > 0$*, a tuple* $(\mathbf{u}, \xi, \eta)$ *with*

$$\begin{aligned}
&\mathbf{u} \in L^\infty\big(0,T;\mathbf{H}^1_\perp(\Omega)\big),\\
&\xi \in L^\infty\big(0,T;L^2(\Omega)\big),\\
&\eta \in L^\infty\big(0,T;L^2(\Omega)\big) \cap H^1\big(0,T;H^1(\Omega)'\big),\\
&q \in L^\infty\big(0,T;L^2(\Omega)\big),\\
&p \in L^\infty\big(0,T;L^2(\Omega)\big) \cap L^2\big(0,T;H^1(\Omega)\big)
\end{aligned}$$

is called a weak solution to the problem (2.3)-(2.8)*, if there hold for almost every* $t \in (0, T]$*,*

$$\big(\mathcal{N}(\nabla\mathbf{u}),\varepsilon(\mathbf{v})\big) - (\xi,\operatorname{div}\mathbf{v}) = (\mathbf{f},\mathbf{v}) + \langle\mathbf{f}_1,\mathbf{v}\rangle, \qquad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega), \tag{2.16}$$

$$\kappa_3(\xi,\varphi) + (\operatorname{div}\mathbf{u},\varphi) = \kappa_1(\eta,\varphi), \qquad \forall\, \varphi \in L^2(\Omega), \tag{2.17}$$

$$(\eta_t,\psi)_{\rm dual} + \frac{1}{\mu_f}\big(K\big(\nabla(\kappa_1\xi + \kappa_2\eta) - \rho_f\mathbf{g}\big),\nabla\psi\big) = (\phi,\varphi) + \langle\phi_1,\varphi\rangle, \quad \forall\, \varphi \in H^1(\Omega), \tag{2.18}$$

$$p := \kappa_1\xi + \kappa_2\eta, \quad q := \kappa_1\eta - \kappa_3\xi, \tag{2.19}$$

$$\eta(0) = \eta_0 := c_0 p_0 + \alpha q_0. \tag{2.20}$$

Let

$$\tilde{\sigma}(\mathbf{w}) = \mu\tilde{\varepsilon}(\mathbf{w}) + \lambda tr\big(\tilde{\varepsilon}(\mathbf{w})\big)\mathbf{I}, \quad \tilde{\sigma}(\mathbf{z}) = \mu\tilde{\varepsilon}(\mathbf{z}) + \lambda tr\big(\tilde{\varepsilon}(\mathbf{z})\big)\mathbf{I}, \quad \forall\, \mathbf{w}, \mathbf{z} \in \mathbf{H}^1(\Omega),$$

we establish the growth, coercivity, and monotonicity of the nonlinear stress-strain relation as follows.

Lemma 2.1. *There exist positive constants C_1, C_2, C_4 such that*

$$\|\mathcal{N}(\nabla\mathbf{w})\|_{L^2(\Omega)} \le C_1\|\varepsilon(\mathbf{w})\|_{L^2(\Omega)}, \tag{2.21}$$

$$\big(\mathcal{N}\big(\nabla(\mathbf{w})\big), \varepsilon(\mathbf{w})\big) \ge C_2\|\varepsilon(\mathbf{w})\|^2_{L^2(\Omega)}, \tag{2.22}$$

$$\big(\mathcal{N}\big(\nabla(\mathbf{w})\big) - \mathcal{N}\big(\nabla(\mathbf{z})\big), \varepsilon(\mathbf{w}) - \varepsilon(\mathbf{z})\big) \ge C_4\|\varepsilon(\mathbf{w}) - \varepsilon(\mathbf{z})\|^2_{L^2(\Omega)}. \tag{2.23}$$

Proof. From [39], we know that $\|\nabla\mathbf{w}\|_F$ is bounded and the bound is positive, that is, there exist two positive constants $\mathcal{M}$ and $\mathcal{D}$ such that $\mathcal{M} \le \|\nabla\mathbf{w}\|_F \le \mathcal{D}$.

Using the Cauchy-Schwarz inequality, the definition of norm, Korn's inequality, we have

$$\big\|\mathcal{N}\big(\nabla(\mathbf{w})\big)\big\|_{L^2(\Omega)} \le c_2\Big(\mu + \frac{\mu\mathcal{D}}{2} + \frac{\lambda\sqrt{d}\mathcal{D}}{2}\Big)\|\varepsilon(\mathbf{w})\|_{L^2(\Omega)}. \tag{2.24}$$

Taking $C_1 = c_2(\mu + \mu\mathcal{D}/2 + \lambda\sqrt{d}\mathcal{D}/2)$ in (2.24), we see that (2.21) holds.

Using (2.24), we have

$$\Big\|\frac{\mu}{2}\nabla^\top\mathbf{w}\nabla\mathbf{w} + \frac{\lambda}{2}\|\nabla\mathbf{w}\|^2_F\mathbf{I}\Big\|_{L^2(\Omega)} \le c_2\Big(\frac{\mu\mathcal{D}}{2} + \frac{\lambda\sqrt{d}\mathcal{D}}{2}\Big)\|\varepsilon(\mathbf{w})\|_{L^2(\Omega)}. \tag{2.25}$$

It is easy to check

$$\frac{1}{2}[(a+b, a+b) - (a,a) - (b,b)] = (a,b), \tag{2.26}$$

$$\begin{aligned}\|x+y\|^2_{L^2(\Omega)} &\ge \big(\|x\|_{L^2(\Omega)} - \|y\|_{L^2(\Omega)}\big)^2 \\ &\ge \|x\|^2_{L^2(\Omega)} - \|y\|^2_{L^2(\Omega)} \quad \text{if and only if} \quad \|x\|_{L^2(\Omega)} \le \|y\|_{L^2(\Omega)}.\end{aligned} \tag{2.27}$$

Using (2.25)-(2.27), we get

$$\big(\mathcal{N}\big(\nabla(\mathbf{w})\big), \varepsilon(\mathbf{w})\big) \ge \Big(\mu - c_2\Big(\frac{\mu\mathcal{D}}{2} + \frac{\lambda\sqrt{d}\mathcal{D}}{2}\Big)\Big)\|\varepsilon(\mathbf{w})\|^2_{L^2(\Omega)}. \tag{2.28}$$

Taking $C_2 = \mu - c_2(\mu\mathcal{D}/2 + \lambda\sqrt{d}\mathcal{D}/2)$ by choosing a suitable $\mathcal{D}$ such that $C_2 > 0$, we see that (2.22) holds.

Using the triangle inequality and (2.11), we have

$$\begin{aligned}&\Big\|\frac{\mu}{2}\nabla^\top\mathbf{w}\nabla\mathbf{w} - \frac{\mu}{2}\nabla^\top\mathbf{z}\nabla\mathbf{z} + \frac{\lambda}{2}\|\nabla\mathbf{w}\|^2_F\mathbf{I} - \frac{\lambda}{2}\|\nabla\mathbf{z}\|^2_F\mathbf{I}\Big\|_{L^2(\Omega)} \\ \le &\Big\|\frac{\mu}{2}\nabla^\top\mathbf{w}\nabla\mathbf{w} - \frac{\mu}{2}\nabla^\top\mathbf{z}\nabla\mathbf{z}\Big\|_{L^2(\Omega)} + \Big\|\frac{\lambda}{2}\|\nabla\mathbf{w}\|^2_F\mathbf{I} - \frac{\lambda}{2}\|\nabla\mathbf{z}\|^2_F\mathbf{I}\Big\|_{L^2(\Omega)} \\ = &\Big\|\frac{\mu}{2}\nabla^\top\mathbf{w}\nabla\mathbf{w} - \frac{\mu}{2}\nabla^\top\mathbf{w}\nabla\mathbf{z} + \frac{\mu}{2}\nabla^\top\mathbf{w}\nabla\mathbf{z} - \frac{\mu}{2}\nabla^\top\mathbf{z}\nabla\mathbf{z}\Big\|_{L^2(\Omega)} + \Big\|\frac{\lambda}{2}\|\nabla\mathbf{w}\|^2_F\mathbf{I} - \frac{\lambda}{2}\|\nabla\mathbf{z}\|^2_F\mathbf{I}\Big\|_{L^2(\Omega)} \\ \le &\frac{\mu\mathcal{D}}{2}\|\nabla\mathbf{w} - \nabla\mathbf{z}\|_{L^2(\Omega)} + \frac{\mu\mathcal{D}}{2}\|\nabla\mathbf{w} - \nabla\mathbf{z}\|_{L^2(\Omega)} + \lambda\sqrt{d}\mathcal{D}\|\nabla\mathbf{w} - \nabla\mathbf{z}\|_{L^2(\Omega)} \\ \le &c_2(\mu\mathcal{D} + \lambda\sqrt{d}\mathcal{D})\|\varepsilon(\mathbf{w}) - \varepsilon(\mathbf{z})\|_{L^2(\Omega)}.\end{aligned} \tag{2.29}$$

Using (2.26), (2.27) and (2.29), we have

$$\big(\mathcal{N}\big(\nabla(\mathbf{w})\big) - \mathcal{N}\big(\nabla(\mathbf{z})\big), \varepsilon(\mathbf{w}) - \varepsilon(\mathbf{z})\big) \geq \big(\mu - c_2(\mu\mathcal{D} + \lambda\sqrt{d}\mathcal{D})\big)\|\varepsilon(\mathbf{w}) - \varepsilon(\mathbf{z})\|^2_{L^2(\Omega)}. \tag{2.30}$$

Taking $C_4 = \mu - c_2(\mu\mathcal{D} + \lambda\sqrt{d}\mathcal{D})$ by choosing a suitable $\mathcal{D}$ such that $C_4 > 0$, we see that (2.23) holds. The proof is complete. □

Lemma 2.2. *There exists positive constant C_3 such that*

$$\|\mathcal{N}(\nabla\mathbf{w}) - \mathcal{N}(\nabla\mathbf{z})\|_{L^2(\Omega)} \leq C_3\|\varepsilon(\mathbf{w}) - \varepsilon(\mathbf{z})\|_{L^2(\Omega)}. \tag{2.31}$$

Proof. Using the Cauchy-Schwarz inequality and (2.11), one has

$$\begin{aligned}
&\|\mathcal{N}(\nabla\mathbf{w}) - \mathcal{N}(\nabla\mathbf{z})\|_{L^2(\Omega)} \\
&= \left\|\frac{\mu}{2}\nabla^\top\mathbf{w}\nabla\mathbf{w} - \frac{\mu}{2}\nabla^\top\mathbf{z}\nabla\mathbf{z} + \frac{\lambda}{2}\|\nabla\mathbf{w}\|_F^2\mathbf{I} - \frac{\lambda}{2}\|\nabla\mathbf{z}\|_F^2\mathbf{I} + \frac{\mu}{2}\varepsilon(\mathbf{w}) - \frac{\mu}{2}\varepsilon(\mathbf{z})\right\|_{L^2(\Omega)} \\
&\leq \big(\mu + c_2\big(\mu\mathcal{D} + \lambda\sqrt{d}\mathcal{D}\big)\big)\|\varepsilon(\mathbf{w}) - \varepsilon(\mathbf{z})\|_{L^2(\Omega)}.
\end{aligned}$$

Taking $C_3 = \mu + c_2(\mu\mathcal{D} + \lambda\sqrt{d}\mathcal{D})$, then we deduce that (2.31) holds. □

Next, we present that the weak solution of the problem (2.13)-(2.15) satisfies a energy law.

Lemma 2.3. *Every weak solution $(\mathbf{u}, p)$ of the problem (2.13)-(2.15) satisfies the following energy law:*

$$E(t) + \frac{1}{\mu_f}\int_0^t \big(K(\nabla p - \rho_f\mathbf{g}), \nabla p\big)\,ds - \int_0^t (\phi, p)\,ds - \int_0^t \langle\phi_1, p\rangle\,ds = E(0) \tag{2.32}$$

for all $t \in [0, T]$, where

$$\begin{aligned}
E(t) := \frac{1}{2}\Big[&\tilde{C}\big\|\varepsilon\big(\mathbf{u}(t)\big)\big\|^2_{L^2(\Omega)} + \lambda\|\mathrm{div}\,\mathbf{u}(t)\|^2_{L^2(\Omega)} + c_0\|p(t)\|^2_{L^2(\Omega)} \\
&- 2\big(\mathbf{f}, \mathbf{u}(t)\big) - 2\langle\mathbf{f}_1, \mathbf{u}(t)\rangle\Big],
\end{aligned} \tag{2.33}$$

and $\tilde{C}$ is a positive constant satisfying $C_1 \leq \tilde{C} \leq C_2$.

Moreover, there holds

$$\begin{aligned}
&\|(c_0 p + \alpha\mathrm{div}\,\mathbf{u})_t\|_{L^2((0,T);H^1(\Omega)')} \\
&\leq \frac{1}{\mu_f}\|K\nabla p - \rho_f\mathbf{g}\|_{L^2((0,T);L^2(\Omega))} + \|\phi\|_{L^2(\Omega)} + \|\phi_1\|_{L^2(\partial\Omega)} < \infty.
\end{aligned} \tag{2.34}$$

Proof. It is enough to prove the case of $\mathbf{u}_t \in \mathbf{L}^2((0, T); \mathbf{L}^2(\Omega))$ for that the general case can be converted into this case using the Steklov average technique (cf. [26, Chapter 2]). From (2.21) and (2.22), we deduce that there exists $\tilde{C}$ satisfying $C_1 \leq \tilde{C} \leq C_2$ such that

$$\big(\mathcal{N}(\nabla\mathbf{u}), \nabla\mathbf{u}\big) = \big(\mathcal{N}(\nabla\mathbf{u}), \varepsilon(\mathbf{u})\big) = \tilde{C}\big(\varepsilon(\mathbf{u}), \varepsilon(\mathbf{u})\big). \tag{2.35}$$

Setting $\mathbf{v} = \mathbf{u}_t$ in (2.13) and $\varphi = p$ in (2.14), we get

$$\big(\mathcal{N}(\nabla\mathbf{u}), \varepsilon(\mathbf{u}_t)\big) + \lambda(\mathrm{div}\,\mathbf{u}, \mathrm{div}\,\mathbf{u}_t) - \alpha(p, \mathrm{div}\,\mathbf{u}_t) = (\mathbf{f}, \mathbf{u}_t) + \langle\mathbf{f}_1, \mathbf{u}_t\rangle, \tag{2.36}$$

$$\big((c_0 p + \alpha\mathrm{div}\,\mathbf{u})_t, p(t)\big) + \frac{1}{\mu_f}\big(K(\nabla p - \rho_f\mathbf{g}), \nabla p\big) = (\phi, p) + \langle\phi_1, p\rangle. \tag{2.37}$$

Adding the above two equations, using (2.35) and integrating over the interval $(0,t)$, we have

$$E(t)+\frac{1}{\mu_f}\int_0^t \big(K(\nabla p-\rho_f\mathbf{g}),\nabla p\big)\,ds-\int_0^t(\phi,p)\,ds-\int_0^t\langle\phi_1,p\rangle\,ds=E(0), \tag{2.38}$$

which implies that (2.32) holds. Eq. (2.34) follows immediately from (2.32) and (2.14). The proof is complete. □

Likewise, using a similar argument to Lemma 2.3, we obtain that the weak solution of the problem (2.16)-(2.20) satisfies a similar energy law as follows, here we omit the details of its proof.

Lemma 2.4. *Every weak solution* $(\mathbf{u},\xi,\eta)$ *of the problem* (2.16)-(2.20) *satisfies the following energy law:*

$$\begin{aligned}J(t)&+\frac{1}{\mu_f}\int_0^t\Big(K\big(\nabla(\kappa_1\xi+\kappa_2\eta)-\rho_f\mathbf{g}\big),\nabla\kappa_1\xi+\kappa_2\eta\Big)\,ds\\&-\int_0^t(\phi,\kappa_1\xi+\kappa_2\eta)\,ds-\int_0^t\langle\phi_1,\kappa_1\xi+\kappa_2\eta\rangle\,ds=J(0)\end{aligned} \tag{2.39}$$

for all $t\in[0,T]$, *where*

$$J(t):=\frac{1}{2}\Big[\tilde{C}\big\|\varepsilon\big(\mathbf{u}(t)\big)\big\|^2_{L^2(\Omega)}+\kappa_2\|\eta(t)\|^2_{L^2(\Omega)}+\kappa_3\|\xi(t)\|^2_{L^2(\Omega)}-2\big(\mathbf{f},\mathbf{u}(t)\big)-2\langle\mathbf{f}_1,\mathbf{u}(t)\rangle\Big]. \tag{2.40}$$

Moreover, there holds

$$\begin{aligned}&\|\eta_t\|_{L^2((0,T);H^1(\Omega)')}\\&\le\frac{1}{\mu_f}\|K\nabla(\kappa_1\xi+\kappa_2\eta)-\rho_f\mathbf{g}\|_{L^2((0,T);L^2(\Omega))}\\&\quad+\|\phi\|_{L^2(\Omega)}+\|\phi_1\|_{L^2(\partial\Omega)}<\infty.\end{aligned} \tag{2.41}$$

From Lemma 2.4, we immediately obtain the following stability estimates.

Lemma 2.5. *There exists a positive constant*

$$\acute{C}_1=\acute{C}_1\big(\|\mathbf{u}_0\|_{H^1(\Omega)},\|p_0\|_{L^2(\Omega)},\|\mathbf{f}\|_{L^2(\Omega)},\|\mathbf{f}_1\|_{L^2(\partial\Omega)},\|\phi\|_{L^2(\Omega)},\|\phi_1\|_{L^2(\partial\Omega)}\big)$$

such that

$$\begin{aligned}&\sqrt{\tilde{C}}\|\varepsilon(\mathbf{u})\|_{L^\infty((0,T);L^2(\Omega))}+\sqrt{\kappa_2}\|\eta\|_{L^\infty((0,T);L^2(\Omega))}\\&\quad+\sqrt{\kappa_3}\|\xi\|_{L^\infty((0,T);L^2(\Omega))}+\sqrt{\frac{K_1}{\mu_f}}\|\nabla p\|_{L^2((0,T);L^2(\Omega))}\le\acute{C}_1,\end{aligned} \tag{2.42}$$

$$\|\mathbf{u}\|_{L^\infty((0,T);L^2(\Omega))}\le\acute{C}_1,\quad\|p\|_{L^\infty((0,T);L^2(\Omega))}\le\acute{C}_1\big(\kappa_2^{\frac{1}{2}}+\kappa_1\kappa_3^{-\frac{1}{2}}\big), \tag{2.43}$$

$$\|p\|_{L^2((0,T);L^2(\Omega))}\le\acute{C}_1,\quad\|\xi\|_{L^2((0,T);L^2(\Omega))}\le\acute{C}_1\kappa_1^{-1}\big(1+\kappa_2^{\frac{1}{2}}\big). \tag{2.44}$$

Furthermore, by leveraging the nonlinearity of the PDE system, we obtain the following priori estimates for the weak solution of the problem (2.16)-(2.20).

Lemma 2.6. *Suppose that* $\mathbf{u}_0$ *and* p_0 *are sufficiently smooth, then there exist positive constants*

$$\acute{C}_2 = \acute{C}_2\big(\acute{C}_1, \|\nabla p_0\|_{L^2(\Omega)}\big), \quad \acute{C}_3 = \acute{C}_3\big(\acute{C}_1, \acute{C}_2, \|\mathbf{u}_0\|_{H^2(\Omega)}, \|p_0\|_{H^2(\Omega)}\big)$$

such that

$$\begin{aligned}&\sqrt{\bar{C}}\|\varepsilon(\mathbf{u}_t)\|_{L^2((0,T);L^2(\Omega))} + \sqrt{\kappa_2}\|\eta_t\|_{L^2((0,T);L^2(\Omega))}\\ &\quad + \sqrt{\kappa_3}\|\xi_t\|_{L^2((0,T);L^2(\Omega))} + \sqrt{\frac{K_1}{\mu_f}}\|\nabla p\|_{L^\infty((0,T);L^2(\Omega))} \le \acute{C}_2,\end{aligned} \tag{2.45}$$

$$\begin{aligned}&\sqrt{\bar{C}}\|\varepsilon(\mathbf{u}_t)\|_{L^\infty((0,T);L^2(\Omega))} + \sqrt{\kappa_2}\|\eta_t\|_{L^\infty((0,T);L^2(\Omega))}\\ &\quad + \sqrt{\kappa_3}\|\xi_t\|_{L^\infty((0,T);L^2(\Omega))} + \sqrt{\frac{K_1}{\mu_f}}\|\nabla p_t\|_{L^2((0,T);L^2(\Omega))} \le \acute{C}_3,\end{aligned} \tag{2.46}$$

$$\|\eta_{tt}\|_{L^2((0,T);H^1(\Omega)')} \le \sqrt{\frac{K_2}{\mu_f}}\acute{C}_3. \tag{2.47}$$

Proof. Following the proof of Lemma 2.1, we know that there exist $\bar{C}_1$ and $\bar{C}_2$ such that

$$\|\mathcal{N}_t(\nabla\mathbf{u})\|_{L^2(\Omega)} \le \bar{C}_1\|\varepsilon(\mathbf{u}_t)\|_{L^2(\Omega)}, \tag{2.48}$$
$$\big(\mathcal{N}_t(\nabla\mathbf{u}), \varepsilon(\mathbf{u}_t)\big) \ge \bar{C}_2\|\varepsilon(\mathbf{u}_t)\|^2_{L^2(\Omega)}. \tag{2.49}$$

Using (2.48) and (2.49), we deduce that there exists a positive constant $\bar{C}$ satisfying $\bar{C}_1 \le \bar{C} \le \bar{C}_2$ such that

$$\big(\mathcal{N}_t(\nabla\mathbf{u}), \nabla\mathbf{u}_t\big) = \big(\mathcal{N}_t(\nabla\mathbf{u}), \varepsilon(\mathbf{u}_t)\big) = \bar{C}\big(\varepsilon(\mathbf{u}_t), \varepsilon(\mathbf{u}_t)\big). \tag{2.50}$$

Differentiating (2.16) and (2.17) with respect to t, taking $\mathbf{v} = \mathbf{u}_t$ and $\varphi = \xi_t$ in (2.16) and (2.17) respectively, and adding the resulting equations, we have

$$\big(\mathcal{N}_t(\nabla\mathbf{u}), \varepsilon(\mathbf{u}_t)\big) = (q_t, \xi_t) = \kappa_1(\eta_t, \xi_t) - \kappa_3\|\xi_t\|^2_{L^2(\Omega)}. \tag{2.51}$$

Setting $\psi = p_t = \kappa_1\xi_t + \kappa_2\eta_t$ in (2.18), we get

$$\kappa_1(\eta_t, \xi_t) + \kappa_2\|\eta_t\|^2_{L^2(\Omega)} + \frac{K}{2\mu_f}\frac{d}{dt}\|\nabla p - \rho_f\mathbf{g}\|^2_{L^2(\Omega)} = \frac{d}{dt}[(\phi, p) + \langle\phi_1, p\rangle]. \tag{2.52}$$

Adding (2.51) and (2.52), using (2.50) and integrating from 0 to t, we have

$$\begin{aligned}&\frac{K}{2\mu_f}\|\nabla p(t) - \rho_f\mathbf{g}\|^2_{L^2(\Omega)} + \int_0^t \big[\bar{C}\big(\varepsilon(\mathbf{u}), \varepsilon(\mathbf{u}_t)\big) + \kappa_2\|\eta_t\|^2_{L^2(\Omega)} + \kappa_3\|\xi_t\|^2_{L^2(\Omega)}\big]\,ds\\ &= \frac{K}{2\mu_f}\|\nabla p_0 - \rho_f\mathbf{g}\|^2_{L^2(\Omega)} + (\phi, p(t) - p_0) + \langle\phi_1, p(t) - p_0\rangle,\end{aligned}$$

which implies that (2.45) holds.

Differentiating (2.16) one time with respect to t and setting $\mathbf{v} = \mathbf{u}_{tt}$, differentiating (2.17) twice with respect to t and setting $\varphi = \xi_t$, and adding the resulting equations, we get

$$\big(\mathcal{N}_t(\nabla\mathbf{u}), \varepsilon(\mathbf{u}_{tt})\big) = (q_{tt}, \xi_t) = \kappa_1(\eta_{tt}, \xi_t) - \frac{\kappa_3}{2}\frac{d}{dt}\|\xi_t\|^2_{L^2(\Omega)}. \tag{2.53}$$

Differentiating (2.18) with respect t one time and taking $\psi = p_t = \kappa_1\xi_t + \kappa_2\eta_t$, we get

$$\kappa_1(\eta_{tt}, \xi_t) + \frac{\kappa_2}{2}\frac{d}{dt}\|\eta_t\|^2_{L^2(\Omega)} + \frac{1}{\mu_f}(K\nabla p_t, \nabla p_t) = 0. \tag{2.54}$$

Adding (2.53)-(2.54), using (2.50) and integrating from 0 to t, we obtain

$$\begin{aligned}
&\frac{\bar{C}}{2}\big\|\varepsilon\big(\mathbf{u}_t(t)\big)\big\|^2_{L^2(\Omega)} + \frac{\kappa_2}{2}\|\eta_t(t)\|^2_{L^2(\Omega)} + \frac{\kappa_3}{2}\|\xi_t(t)\|^2_{L^2(\Omega)} + \frac{1}{\mu_f}\int_0^t (K\nabla p_t, \nabla p_t)\, ds \\
&= \frac{\bar{C}}{2}\big\|\varepsilon\big(\mathbf{u}_t(t)\big)\big\|^2_{L^2(\Omega)} + \kappa_2\|\eta_t(0)\|^2_{L^2(\Omega)} + \kappa_3\|\xi_t(0)\|^2_{L^2(\Omega)},
\end{aligned} \tag{2.55}$$

which implies that (2.46) holds. Eq. (2.47) follows immediately from the following inequality:

$$(\eta_{tt}, \psi) = -\frac{1}{\mu_f}(K\nabla p_t, \nabla\psi) \le \frac{K_2}{\mu_f}\|\nabla p_t\|_{L^2(\Omega)}\|\nabla\psi\|_{L^2(\Omega)},$$

(2.46) and the definition of the $H^1(\Omega)'$-norm. The proof is complete. □

Remark 2.2. The above estimates require $p_0 \in H^1(\Omega), \mathbf{u}_t(0) \in \mathbf{L}^2(\Omega), \eta_t(0) \in L^2(\Omega)$ and $\xi_t(0) \in L^2(\Omega)$. The values of $\mathbf{u}_t(0), \eta_t(0)$ and $\xi_t(0)$ can be computed using the PDEs as follows. It follows from (2.18) that $\eta_t(0)$ satisfies

$$\eta_t(0) = \phi + \frac{1}{\mu_f}\mathrm{div}\,[K(\nabla p_0 - \rho_f\mathbf{g})].$$

Hence, $\eta_t(0) \in L^2(\Omega)$ provided that $p_0 \in H^2(\Omega)$. To find $\mathbf{u}_t(0)$ and $\xi_t(0)$, differentiating (2.16) and (2.17) with respect to t and setting $t = 0$, we get

$$\begin{aligned}
-\mathrm{div}\,\mathcal{N}\big(\nabla\mathbf{u}_t(0)\big) + \nabla\xi_t(0) = 0 \quad &\text{in } \Omega, \\
\kappa_3\xi_t(0) + \mathrm{div}\,\mathbf{u}_t(0) = \kappa_1\eta_t(0) \quad &\text{in } \Omega.
\end{aligned}$$

Hence, $\mathbf{u}_t(0)$ and $\xi_t(0)$ can be determined by solving the above generalized Stokes problem.

The next lemma shows that the weak solution of the problem (2.16)-(2.20) preserves some "invariant" quantities, it turns out that these "invariant" quantities play a vital role in the proof of existence and uniqueness of the weak solution to the reformulated problem.

Lemma 2.7. *Every weak solution* $(\mathbf{u}, \xi, \eta)$ *of problem* (2.16)-(2.20) *satisfies the following energy laws:*

$$C_\eta(t) := \big(\eta(\cdot, t), 1\big) = (\eta_0, 1) + [(\phi, 1) + \langle\phi_1, 1\rangle]t, \quad t \ge 0, \tag{2.56}$$

$$C_\xi(t) := \big(\xi(\cdot, t), 1\big) = \frac{1}{d + \kappa_3}\big[\big(\mathcal{N}(\nabla\mathbf{u}), \mathbf{I}\big) + \kappa_1 C_\eta(t) - (\mathrm{div}\,\mathbf{u}, 1) - (\mathbf{f}, \mathbf{x}) - \langle\mathbf{f}_1, \mathbf{x}\rangle\big], \tag{2.57}$$

$$C_q(t) := \big(q(\cdot, t), 1\big) = \kappa_1 C_\eta(t) - \kappa_3 C_\xi(t), \tag{2.58}$$

$$C_p(t) := \big(p(\cdot, t), 1\big) = \kappa_1 C_\xi(t) + \kappa_2 C_\eta(t), \tag{2.59}$$

$$C_{\mathbf{u}}(t) := \langle\mathbf{u}(\cdot, t) \cdot \mathbf{n}, 1\rangle = C_q(t). \tag{2.60}$$

Proof. We first notice that (2.56) follows immediately from taking $\psi \equiv 1$ in (2.18). To prove (2.57), taking $\mathbf{v} = \mathbf{x}$ in (2.16) and $\varphi = 1$ in (2.17), which are valid test functions, and using the identities $\nabla \mathbf{x} = \mathbf{I}$, $\operatorname{div} \mathbf{x} = d$, and $\varepsilon(\mathbf{x}) = \mathbf{I}$, we get

$$\begin{aligned}
&\big(\mathcal{N}(\nabla \mathbf{u}), \mathbf{I}\big) = d(\xi, 1) + (\mathbf{f}, \mathbf{x}) + \langle \mathbf{f}_1, \mathbf{x}\rangle,\\
&(\operatorname{div} \mathbf{u}, 1) = \kappa_1(\eta, 1) - \kappa_3(\xi, 1).
\end{aligned}$$

It is easy to check that

$$C_\xi(t) := \big(\xi(\cdot, t), 1\big) = \frac{1}{d+\kappa_3}\big[\big(\mathcal{N}(\nabla \mathbf{u}), \mathbf{I}\big) + \kappa_1 C_\eta(t) - (\operatorname{div} \mathbf{u}, 1) - (\mathbf{f}, \mathbf{x}) - \langle \mathbf{f}_1, \mathbf{x}\rangle\big],$$

which implies that (2.57) holds.

Finally, since $q = \kappa_1\eta - \kappa_3\xi, p = \kappa_1\xi + \kappa_2\eta$, (2.58) and (2.59) follow from (2.56) and (2.57). Eq. (2.60) is an immediate consequence of $q = \operatorname{div} \mathbf{u}$ and the Gauss divergence theorem. The proof is complete. □

With the help of the above lemmas, we can show the solvability of the problem (1.1)-(1.7).

Theorem 2.1. *Let* $\mathbf{u}_0 \in \mathbf{H}^1(\Omega), \mathbf{f} \in \mathbf{L}^2(\Omega), \mathbf{f}_1 \in \mathbf{L}^2(\partial\Omega), p_0 \in L^2(\Omega), \phi \in L^2(\Omega)$, *and* $\phi_1 \in L^2(\partial\Omega)$. *Suppose* $c_0 > 0$ *and* $(\mathbf{f}, \mathbf{v}) + \langle \mathbf{f}_1, \mathbf{v}\rangle = 0$ *for any* $\mathbf{v} \in \mathbf{RM}$. *Then there exists a unique weak solution to the problem* (1.1)-(1.7) *in the sense of Definition* 2.1. *Likewise, there exists a unique weak solution to the problem* (2.3)-(2.8) *in the sense of Definition* 2.2.

Proof. Firstly, using the Schaefer's fixed point theorem [17, Theorem 4] and the existence of the weak solution of linear poroelasticity model, we can prove the existence of a weak solution of the problem (2.16)-(2.20). Here we omit the more details.

Secondly, we prove that the problem (2.16)-(2.20) has a unique solution. Lemmas 2.5 and 2.6 give the priori estimates for the weak solution. Since

$$C_\eta(t) = \big(\eta(\cdot, t), 1\big) = (\eta_0, 1) + [(\phi, 1) + \langle \phi_1, 1\rangle].$$

It is easy to check that η is unique. We assume that $(\mathbf{u}_1, \xi_1, \eta_1)$ and $(\mathbf{u}_2, \xi_2, \eta_2)$ are the different solutions. Using (2.16) and (2.17), we obtain

$$\big(\mathcal{N}(\nabla \mathbf{u}_1) - \mathcal{N}(\nabla \mathbf{u}_2), \varepsilon(\mathbf{v})\big) - (\xi_1 - \xi_2, \operatorname{div} \mathbf{v}) = 0, \quad \forall\, \mathbf{v}_h \in H^1(\Omega), \tag{2.61}$$

$$\kappa_3(\xi_1 - \xi_2, \varphi) + (\operatorname{div} \mathbf{u}_1 - \operatorname{div} \mathbf{u}_2, \varphi) = 0, \quad \forall\, \varphi_h \in L^2(\Omega). \tag{2.62}$$

Adding (2.61) and (2.62), letting $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2, \varphi = \xi_1 - \xi_2$, using (2.23), we have

$$0 \le C_4\|\varepsilon(\mathbf{u}_1) - \varepsilon(\mathbf{u}_2)\|^2_{L^2(\Omega)} + \kappa_3\|\xi_1 - \xi_2\|^2_{L^2(\Omega)} = 0. \tag{2.63}$$

Using (2.63) and the initial value $\mathbf{u}_0$, we obtain

$$\mathbf{u}_1 = \mathbf{u}_2, \quad \xi_1 = \xi_2.$$

Since $p = \kappa_1\xi + \kappa_2\eta, q = \kappa_1\eta - \kappa_3\xi$, so we have

$$p_1 = p_2, \quad q_1 = q_2.$$

Hence, the solution of the problem (2.16)-(2.20) is unique. The proof is complete. □

3. Fully Discrete Multiphysics Finite Element Method

3.1. Formulation of fully discrete finite element method

Let $\mathcal{T}_h$ be a quasi-uniform triangulation or rectangular partition of Ω with maximum mesh size h, and $\bar{\Omega} = \bigcup_{\mathcal{K}\in\mathcal{T}_h} \bar{\mathcal{K}}$. The time interval $[0,T]$ is divided into N equal intervals, denoted by $[t_{n-1}, t_n], n = 1, 2, \ldots, N$, and $\Delta t = T/N$, then $t_n = n\Delta t$. In this work, we use backward Euler method and denote $d_t v^n := (v^n - v^{n-1})/\Delta t$.

Also, let $(\mathbf{X}_h, M_h)$ be a stable mixed finite element pair, that is, $\mathbf{X}_h \subset \mathbf{H}^1(\Omega)$ and $M_h \subset L^2(\Omega)$ satisfy the inf-sup condition

$$\sup_{\mathbf{v}_h\in\mathbf{X}_h} \frac{(\mathrm{div}\mathbf{v}_h, \varphi_h)}{\|\mathbf{v}_h\|_{H^1(\Omega)}} \geq \beta_0 \|\varphi_h\|_{L^2(\Omega)}, \quad \forall \varphi_h \in M_{0h} := M_h \cap L_0^2(\Omega), \quad \beta_0 > 0. \tag{3.1}$$

A well-known example is the following so-called Taylor-Hood element (cf. [1,8]):

$$\mathbf{X}_h = \{\mathbf{v}_h \in \mathbf{C}^0(\overline{\Omega}); \mathbf{v}_h|_{\mathcal{K}} \in \mathbf{P}_2(\mathcal{K}), \ \forall \mathcal{K} \in \mathcal{T}_h\},$$
$$M_h = \{\varphi_h \in C^0(\overline{\Omega}); \varphi_h|_{\mathcal{K}} \in P_1(\mathcal{K}), \ \forall \mathcal{K} \in \mathcal{T}_h\}.$$

Finite element approximation space W_h for η variable can be chosen independently, any piecewise polynomial space is acceptable provided that $W_h \supset M_h$, the most convenient choice is $W_h = M_h$.

Define

$$\mathbf{V}_h := \{\mathbf{v}_h \in \mathbf{X}_h; (\mathbf{v}_h, \mathbf{r}) = 0, \ \forall \mathbf{r} \in \mathbf{RM}\},$$

it is easy to check that $\mathbf{X}_h = \mathbf{V}_h \bigoplus \mathbf{RM}$. It was proved in [19] that there holds the following inf-sup condition:

$$\sup_{\mathbf{v}_h\in\mathbf{V}_h} \frac{(\mathrm{div}\,\mathbf{v}_h, \varphi_h)}{\|\mathbf{v}_h\|_{H^1(\Omega)}} \geq \beta_1 \|\varphi_h\|_{L^2(\Omega)}, \quad \forall \varphi_h \in M_{0h}, \quad \beta_1 > 0. \tag{3.2}$$

Also, we recall the following inverse inequality for polynomial functions [7,8,12]:

$$\|\nabla\varphi_h\|_{L^2(\mathcal{K})} \leq c_1 h^{-1} \|\varphi_h\|_{L^2(\mathcal{K})}, \quad \forall \varphi_h \in P_r(\mathcal{K}), \quad \mathcal{K} \in T_h. \tag{3.3}$$

Now, we give the fully discrete multiphysics finite element method for the problem (2.3)-(2.8).

Multiphysics Finite Element Method (MFEM)

(i) Compute $\mathbf{u}_h^0 \in \mathbf{V}_h$ and $q_h^0 \in W_h$ by

$$\mathbf{u}_h^0 = \mathcal{R}_h \mathbf{u}_0, \quad p_h^0 = \mathcal{Q}_h p_0,$$

where $\mathcal{R}_h$ and $\mathcal{Q}_h$, respectively, denote the elliptic and L^2 projection operator (see (4.1) and (4.4)).

(ii) For $n = 0, 1, 2, \ldots$, do the following two steps:

Step 1. Solve for $(\mathbf{u}_h^{n+1}, \xi_h^{n+1}, \eta_h^{n+1}) \in \mathbf{V}_h \times M_h \times W_h$ such that

$$\begin{aligned}
&\big(\mathcal{N}\big(\nabla \mathbf{u}_h^{n+1}\big), \varepsilon(\mathbf{v}_h)\big) - \big(\xi_h^{n+1}, \operatorname{div} \mathbf{v}_h\big) \\
&= (\mathbf{f}, \mathbf{v}_h) + \langle \mathbf{f}_1, \mathbf{v}_h \rangle, \qquad \forall\, \mathbf{v}_h \in \mathbf{V}_h, && (3.4)\\
&\kappa_3\big(\xi_h^{n+1}, \varphi_h\big) + \big(\operatorname{div} \mathbf{u}_h^{n+1}, \varphi_h\big) \\
&= \kappa_1\big(\eta_h^{n+\theta}, \varphi_h\big), \qquad \forall\, \varphi_h \in M_h, && (3.5)\\
&\big(d_t \eta_h^{n+1}, \psi_h\big) + \frac{1}{\mu_f}\big(K\nabla\big(\kappa_1 \xi_h^{n+1} + \kappa_2 \eta_h^{n+1}\big) - \rho_f \mathbf{g}, \nabla \psi_h\big) \\
&= (\phi, \psi_h) + \langle \phi_1, \psi_h \rangle, \quad \forall\, \psi_h \in W_h, && (3.6)
\end{aligned}$$

where $\theta = 0$ or 1.

Step 2. Update p_h^{n+1} and q_h^{n+1} by

$$p_h^{n+1} = \kappa_1 \xi_h^{n+1} + \kappa_2 \eta_h^{n+\theta}, \quad q_h^{n+1} = \kappa_1 \eta_h^{n+1} - \kappa_3 \xi_h^{n+1}. \tag{3.7}$$

Remark 3.1. The Newton's iterative method to solve the nonlinear Stokes problem (3.4)-(3.5) when $\theta = 0$ is

$$\begin{aligned}
&- \big(\lambda \nabla \mathbf{u}_h^n : \nabla \mathbf{u}_h^n \mathbf{I}, \varepsilon(\mathbf{v}_h)\big) + (\mathbf{f}, \mathbf{v}_h) + \langle \mathbf{f}_1, \mathbf{v}_h \rangle + \kappa_1\big(\eta_h^{n+\theta}, \varphi_h\big) \\
&= \mu\big(\varepsilon\big(\mathbf{u}_h^{n+1}\big), \varepsilon(\mathbf{v}_h)\big) + \mu\big(\nabla^\top \mathbf{u}_h^{n+1} \nabla \mathbf{u}_h^n, \varepsilon(\mathbf{v}_h)\big) + \mu\big(\nabla^\top \mathbf{u}_h^n \nabla \mathbf{u}_h^{n+1}, \varepsilon(\mathbf{v}_h)\big) \\
&\quad - \mu\big(\nabla^\top \mathbf{u}_h^n \nabla \mathbf{u}_h^n, \varepsilon(\mathbf{v}_h)\big) + \big(2\lambda \nabla \mathbf{u}_h^n : \big(\nabla \mathbf{u}_h^{n+1} - \nabla \mathbf{u}_h^n\big)\mathbf{I}, \varepsilon(\mathbf{v}_h)\big) \\
&\quad - \big(\xi_h^{n+1}, \operatorname{div} \mathbf{v}_h\big) + \kappa_3\big(\xi_h^{n+1}, \varphi_h\big) + \big(\operatorname{div} \mathbf{u}_h^{n+1}, \varphi_h\big).
\end{aligned} \tag{3.8}$$

Remark 3.2. The MFEM gives a built-in mechanism which can adopt any stable solver for Stokes problem and any Lagrange element for the diffusion problem, and there is no "locking phenomenon". The numerical tests of Section 5 show that MFEM has no "locking phenomenon".

3.2. Stability analysis

The primary goal of this subsection is to derive a discrete energy law which mimics the PDE energy law. Before discussing the stability of MFEM, we first show that the numerical solution satisfies some constraints and discrete energy law.

Lemma 3.1. *Let $\{(\mathbf{u}_h^n, \xi_h^n, \eta_h^n)\}_{n\geq 0}$ be defined by the MFEM, then there hold*

$$\begin{aligned}
&\big(\eta_h^n, 1\big) = C_\eta(t_n), && n = 0, 1, 2, \ldots, && (3.9)\\
&\big(\xi_h^n, 1\big) = C_\xi(t_{n-1+\theta}), && n = 1-\theta, 1, 2, \ldots, && (3.10)\\
&\big\langle \mathbf{u}_h^n \cdot \mathbf{n}, 1\big\rangle = C_{\mathbf{u}}(t_{n-1+\theta}), && n = 1-\theta, 1, 2, \ldots. && (3.11)
\end{aligned}$$

Lemma 3.2. *Let $\{(\mathbf{u}_h^n, \xi_h^n, \eta_h^n)\}_{n\geq 0}$ be defined by the MFEM, then there holds the following equality:*

$$J_{h,0}^{l+1} + S_{h,0}^{l+1} = J_{h,0}^0, \quad l \geq 0, \quad \theta = 0, 1, \tag{3.12}$$

where

$$J_{h,\theta}^{l+1} := \frac{1}{2}\Big[\tilde{C}\big\|\varepsilon\big(\mathbf{u}_h^{l+1}\big)\big\|_{L^2(\Omega)}^2 + \kappa_2\big\|\eta_h^{l+\theta}\big\|_{L^2(\Omega)}^2 + \kappa_3\big\|\xi_h^{l+1}\big\|_{L^2(\Omega)}^2 - 2\big(\mathbf{f},\mathbf{u}_h^{l+1}\big) - 2\big\langle\mathbf{f}_1,\mathbf{u}_h^{l+1}\big\rangle\Big],$$

$$\begin{aligned} S_{h,\theta}^{l+1} := \Delta t\sum_{n=0}^{l}\Big[&\frac{\Delta t}{2}\tilde{C}\big\|d_t\varepsilon\big(\mathbf{u}_h^{n+1}\big)\big\|_{L^2(\Omega)}^2 + \frac{\kappa_2\Delta t}{2}\big\|d_t\eta_h^{n+\theta}\big\|_{L^2(\Omega)}^2 + \frac{\kappa_3\Delta t}{2}\big\|d_t\xi_h^{n+1}\big\|_{L^2(\Omega)}^2 \\ &- \big(\phi, p_h^{n+1}\big) - \big\langle\phi_1, p_h^{n+1}\big\rangle + \frac{1}{\mu_f}\big(K\nabla p_h^{n+1} - K\rho_f\mathbf{g}, \nabla p_h^{n+1}\big) \\ &- (1-\theta)\frac{\kappa_1\Delta t}{\mu_f}\big(Kd_t\nabla\xi_h^{n+1}, \nabla p_h^{n+1}\big)\Big]. \end{aligned}$$

Proof. (i) When $\theta = 0$, based on (3.5), we can define η_h^{-1} by

$$\kappa_1\big(\eta_h^{-1}, \varphi_h\big) = \kappa_3\big(\xi_h^0, \varphi_h\big) + \big(\operatorname{div}\mathbf{u}_h^0, \varphi_h\big). \tag{3.13}$$

Setting $\mathbf{v}_h = d_t\mathbf{u}_h^{n+1}$ in (3.4), we have

$$\big(\mathcal{N}\big(\nabla\mathbf{u}_h^{n+1}\big), \varepsilon\big(d_t\mathbf{u}_h^{n+1}\big)\big) - \big(\xi_h^{n+1}, \operatorname{div} d_t\mathbf{u}_h^{n+1}\big) = \big(\mathbf{f}, d_t\mathbf{u}_h^{n+1}\big) + \big\langle\mathbf{f}_1, d_t\mathbf{u}_h^{n+1}\big\rangle. \tag{3.14}$$

Setting $\varphi_h = \xi_h^{n+1}$ in (3.5), we get

$$\kappa_3\big(d_t\xi_h^{n+1}, \xi_h^{n+1}\big) + \big(\operatorname{div} d_t\mathbf{u}_h^{n+1}, \xi_h^{n+1}\big) = \kappa_1\big(d_t\eta_h^n, \xi_h^{n+1}\big). \tag{3.15}$$

Setting $\psi_h = p_h^{n+1}$ in (3.6) after lowing the super-index from $n+1$ to n on both sides of (3.6), we get

$$\big(d_t\eta_h^n, p_h^{n+1}\big) + \frac{1}{\mu_f}\big(K\big(\nabla\big(\kappa_1\xi_h^n + \kappa_2\eta_h^n\big) - \rho_f\mathbf{g}\big), \nabla p_h^{n+1}\big) = \big(\phi, p_h^{n+1}\big) + \big\langle\phi_1, p_h^{n+1}\big\rangle. \tag{3.16}$$

Using the fact of

$$\big(d_t\xi_h^{n+1}, \xi_h^{n+1}\big) = \frac{\Delta t}{2}\big\|d_t\xi_h^{n+1}\big\|_{L^2(\Omega)}^2 + \frac{1}{2}d_t\big\|\xi_h^{n+1}\big\|_{L^2(\Omega)}^2,$$

we can rewrite (3.15) as

$$\frac{\kappa_3\Delta t}{2}\big\|d_t\xi_h^{n+1}\big\|_{L^2(\Omega)}^2 + \frac{\kappa_3}{2}d_t\big\|\xi_h^{n+1}\big\|_{L^2(\Omega)}^2 + \big(\operatorname{div} d_t\mathbf{u}_h^{n+1}, \xi_h^{n+1}\big) = \kappa_1\big(d_t\eta_h^n, \xi_h^{n+1}\big). \tag{3.17}$$

Similarly, we get

$$\begin{aligned} &\frac{\kappa_2\Delta t}{2}\big\|d_t\eta_h^n\big\|_{L^2(\Omega)}^2 + \frac{\kappa_2}{2}d_t\big\|\eta_h^n\big\|_{L^2(\Omega)}^2 + \kappa_1\big(d_t\eta_h^n, \xi_h^{n+1}\big) \\ &\quad + \frac{1}{\mu_f}\big(K\big(\nabla p_h^{n+1} - \rho_f\mathbf{g}\big), \nabla p_h^{n+1}\big) - \frac{\kappa_1\Delta t}{\mu_f}\big(Kd_t\nabla\xi_h^{n+1}, \nabla p_h^{n+1}\big) \\ &= \big(\phi, p_h^{n+1}\big) + \big\langle\phi_1, p_h^{n+1}\big\rangle. \end{aligned} \tag{3.18}$$

Adding (3.14), (3.17) and (3.18), using (2.35) and applying the summation operator $\Delta t\sum_{n=0}^{l}$ to the both sides of the resulting equation, we see that (3.12) holds.

(ii) When $\theta = 1$, setting $\varphi_h = \xi_h^{n+1}$ in (3.5) and $\psi_h = p_h^{n+1}$ in (3.6), we get

$$\begin{aligned}&\frac{\kappa_3\Delta t}{2}\big\|d_t\xi_h^{n+1}\big\|^2_{L^2(\Omega)} + \frac{\kappa_3}{2}d_t\big\|\xi_h^{n+1}\big\|^2_{L^2(\Omega)} + \big(\operatorname{div} d_t\mathbf{u}_h^{n+1}, \xi_h^{n+1}\big)\\ &= \kappa_1\big(d_t\eta_h^{n+1}, \xi_h^{n+1}\big),\end{aligned} \tag{3.19}$$

$$\begin{aligned}&\frac{\kappa_2\Delta t}{2}\big\|d_t\eta_h^{n+1}\big\|^2_{L^2(\Omega)} + \frac{\kappa_2}{2}d_t\big\|\eta_h^{n+1}\big\|^2_{L^2(\Omega)}\\ &\quad + \kappa_1\big(d_t\eta_h^{n+1}, \xi_h^{n+1}\big) + \frac{1}{\mu_f}\big(K\big(\nabla p_h^{n+1} - \rho_f\mathbf{g}\big), \nabla p_h^{n+1}\big)\\ &= \big(\phi, p_h^{n+1}\big) + \big\langle\phi_1, p_h^{n+1}\big\rangle.\end{aligned} \tag{3.20}$$

Adding (3.14), (3.19) and (3.20), using (2.35) and applying the summation operator $\Delta t\sum_{n=0}^{l}$ to the both sides of the resulting equation, we deduce that (3.12) holds. □

Lemma 3.3. *Let $\{(\mathbf{u}_h^n, \xi_h^n, \eta_h^n)\}_{n\ge 0}$ be defined by the MFEM with $\theta = 0$, then there holds the following inequality:*

$$J_{h,0}^{l+1} + \hat{S}_{h,0}^{l+1} \le J_{h,0}^0, \quad l \ge 0 \tag{3.21}$$

provided that $\Delta t = \mathcal{O}(h^2)$. Here

$$\begin{aligned}\hat{S}_{h,0}^{l+1} := \Delta t\sum_{n=0}^{l}\Big[&\frac{\Delta t}{4}\tilde{C}\big\|d_t\varepsilon\big(\mathbf{u}_h^{n+1}\big)\big\|^2_{L^2(\Omega)} + \frac{K_1}{2\mu_f}\big\|\nabla p_h^{n+1}\big\|^2_{L^2(\Omega)} + \frac{\kappa_2\Delta t}{2}\big\|d_t\eta_h^{n+\theta}\big\|^2_{L^2(\Omega)}\\ &+ \frac{\kappa_3\Delta t}{2}\big\|d_t\xi_h^{n+1}\big\|^2_{L^2(\Omega)} - \frac{1}{\mu_f}\big(K\rho_f\mathbf{g}, \nabla p_h^{n+1}\big) - \big(\phi, p_h^{n+1}\big) - \big\langle\phi_1, p_h^{n+1}\big\rangle\Big].\end{aligned}$$

Proof. Using the Cauchy-Schwarz inequality, Young inequality and inverse inequality (3.3), we get

$$\begin{aligned}&\frac{\kappa_1\Delta t}{\mu_f}\big(Kd_t\nabla\xi_h^{n+1}, \nabla p_h^{n+1}\big)\\ &\le \frac{K_2^2\kappa_1^2}{2K_1\mu_f}\big\|\nabla\xi_h^{n+1} - \nabla\xi_h^n\big\|^2_{L^2(\Omega)} + \frac{K_1}{2\mu_f}\big\|\nabla p_h^{n+1}\big\|^2_{L^2(\Omega)}\\ &\le \frac{K_2^2\kappa_1^2c_1^2}{2K_1\mu_f h^2}\big\|\xi_h^{n+1} - \xi_h^n\big\|^2_{L^2(\Omega)} + \frac{K_1}{2\mu_f}\big\|\nabla p_h^{n+1}\big\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.22}$$

To bound the first term on the right-hand side of (3.22), we appeal to the inf-sup condition (3.2) and (3.4), (2.31) get

$$\begin{aligned}\big\|\xi_h^{n+1} - \xi_h^n\big\|_{L^2(\Omega)} &\le \frac{1}{\beta_1}\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{\big(\operatorname{div}\mathbf{v}_h, \xi_h^{n+1} - \xi_h^n\big)}{\|\nabla\mathbf{v}_h\|_{L^2(\Omega)}}\\ &= \frac{1}{\beta_1}\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{\big(\mathcal{N}\big(\nabla\mathbf{u}_h^{n+1}\big) - \mathcal{N}\big(\nabla\mathbf{u}_h^n\big), \varepsilon(\mathbf{v}_h)\big)}{\|\nabla\mathbf{v}_h\|_{L^2(\Omega)}}\\ &\le \frac{C_3\Delta t}{\beta_1}\big\|d_t\varepsilon(\mathbf{u}_h^{n+1})\big\|_{L^2(\Omega)}.\end{aligned} \tag{3.23}$$

Substituting (3.23) into (3.22) and combining it with (3.12) imply that (3.21) holds if

$$\Delta t \le \frac{\tilde{C}K_1\mu_f\beta_1^2h^2}{2C_3^2\kappa_1^2K_2^2c_1^2}.$$

The proof is complete. □

With the help of Lemmas 3.1-3.3, one can check that the following Theorem 3.1 holds. Since the proof of Theorem 3.1 is similar to Theorem 2.1, here we omit the more details.

Theorem 3.1. *The numerical solution* $\{\mathbf{u}_h^{n+1}, \xi_h^{n+1}, \eta_h^{n+1}\}_{n\geq 0}$ *of the problem* (3.4)-(3.6) *exists uniquely.*

4. Error Estimates

To derive the optimal order error estimates of the fully discrete multiphysics finite element method, for any $\varphi \in L^2(\Omega)$, we firstly define $L^2(\Omega)$-projection operator $\mathcal{Q}_h : L^2(\Omega) \to X_h^k$ by

$$(\mathcal{Q}_h\varphi, \psi_h) = (\varphi, \psi_h), \quad \psi_h \in X_h^k, \tag{4.1}$$

where $X_h^k := \{\psi_h \in C^0;\ \psi_h|_E \in P_k(E),\ \forall\, E \in \mathcal{T}_h\}$.

Next, for any $\varphi \in H^1(\Omega)$, we define its elliptic projection $\mathcal{S}_h : H^1(\Omega) \to X_h^k$ by

$$(K\nabla \mathcal{S}_h\varphi, \nabla\varphi_h) = (K\nabla\varphi, \nabla\varphi_h), \quad \forall\, \varphi_h \in X_h^k, \tag{4.2}$$

$$(\mathcal{S}_h\varphi, 1) = (\varphi, 1). \tag{4.3}$$

Finally, for any $\mathbf{v} \in \mathbf{H}^1(\Omega)$, we define its elliptic projection $\mathcal{R}_h : \mathbf{H}^1(\Omega) \to \mathbf{V}_h^k$ by

$$\big(\varepsilon(\mathcal{R}_h\mathbf{v}), \varepsilon(\mathbf{w}_h)\big) = \big(\varepsilon(\mathbf{v}), \varepsilon(\mathbf{w}_h)\big), \quad \forall\, \mathbf{w}_h \in \mathbf{V}_h^k, \tag{4.4}$$

where

$$\mathbf{V}_h^k := \big\{\mathbf{v}_h \in \mathbf{C}^0;\ \mathbf{v}_h|_{\mathcal{K}} \in \mathbf{P}_k(\mathcal{K}), (\mathbf{v}_h, \mathbf{r}) = 0,\ \forall\, \mathbf{r} \in \mathbf{RM}\big\},$$

k is the degree of the piecewise polynomial on $\mathcal{K}$. From [7], we know that $\mathcal{Q}_h, \mathcal{S}_h$ and $\mathcal{R}_h$ satisfy

$$\|\mathcal{Q}_h\varphi - \varphi\|_{L^2(\Omega)} + h\|\nabla(\mathcal{Q}_h\varphi - \varphi)\|_{L^2(\Omega)} \leq Ch^{s+1}\|\varphi\|_{H^{s+1}(\Omega)}, \quad \forall\, \varphi \in H^{s+1}(\Omega), \quad 0 \leq s \leq k, \tag{4.5}$$

$$\|\mathcal{S}_h\varphi - \varphi\|_{L^2(\Omega)} + h\|\nabla(\mathcal{S}_h\varphi - \varphi)\|_{L^2(\Omega)} \leq Ch^{s+1}\|\varphi\|_{H^{s+1}(\Omega)}, \quad \forall\, \varphi \in H^{s+1}(\Omega), \quad 0 \leq s \leq k, \tag{4.6}$$

$$\|\mathcal{R}_h\mathbf{v} - \mathbf{v}\|_{L^2(\Omega)} + h\|\nabla(\mathcal{R}_h\mathbf{v} - \mathbf{v})\|_{L^2(\Omega)} \leq Ch^{s+1}\|\mathbf{v}\|_{H^{s+1}(\Omega)}, \quad \forall\, \mathbf{v} \in \mathbf{H}^{s+1}(\Omega), \quad 0 \leq s \leq k. \tag{4.7}$$

To derive the error estimates, we introduce the following notations:

$$E_{\mathbf{u}}^n = \mathbf{u}(t_n) - \mathbf{u}_h^n, \quad E_\xi^n = \xi(t_n) - \xi_h^n, \quad E_\eta^n = \eta(t_n) - \eta_h^n,$$
$$E_p^n = p(t_n) - p_h^n, \quad E_q^n = q(t_n) - q_h^n.$$

It is easy to check that

$$E_p^n = \kappa_1 E_\xi^n + \kappa_2 E_\eta^n, \quad E_q^n = \kappa_3 E_\xi^n + \kappa_1 E_\eta^n. \tag{4.8}$$

Also, we denote

$$
\begin{aligned}
E_{\mathbf{u}}^n &= \mathbf{u}(t_n) - \mathcal{R}_h\big(\mathbf{u}(t_n)\big) + \mathcal{R}_h\big(\mathbf{u}(t_n)\big) - \mathbf{u}_h^n := Y_{\mathbf{u}}^n + Z_{\mathbf{u}}^n,\\
E_\xi^n &= \xi(t_n) - \mathcal{S}_h\big(\xi(t_n)\big) + \mathcal{S}_h\big(\xi(t_n)\big) - \xi_h^n := Y_\xi^n + Z_\xi^n,\\
E_\eta^n &= \eta(t_n) - \mathcal{S}_h\big(\eta(t_n)\big) + \mathcal{S}_h\big(\eta(t_n)\big) - \eta_h^n := Y_\eta^n + Z_\eta^n,\\
E_p^n &= p(t_n) - \mathcal{S}_h\big(p(t_n)\big) + \mathcal{S}_h\big(p(t_n)\big) - p_h^n := Y_p^n + Z_p^n,\\
E_\xi^n &= \xi(t_n) - \mathcal{Q}_h\big(\xi(t_n)\big) + \mathcal{Q}_h\big(\xi(t_n)\big) - \xi_h^n := F_\xi^n + G_\xi^n,\\
E_\eta^n &= \eta(t_n) - \mathcal{Q}_h\big(\eta(t_n)\big) + \mathcal{Q}_h\big(\eta(t_n)\big) - \eta_h^n := F_\eta^n + G_\eta^n,\\
E_p^n &= p(t_n) - \mathcal{Q}_h\big(p(t_n)\big) + \mathcal{Q}_h\big(p(t_n)\big) - p_h^n := F_p^n + G_p^n.
\end{aligned}
$$

Lemma 4.1. *Let* $\{(\mathbf{u}_h^n, \xi_h^n, \eta_h^n)\}_{n\geq 0}$ *be generated by the MFEM and* $Y_{\mathbf{u}}^n, Z_{\mathbf{u}}^n, Y_\xi^n, Z_\xi^n, Y_\eta^n$ *and* Z_η^n *be defined as above, then there holds*

$$\begin{aligned}
&\mathcal{E}_h^{l+1} + \Delta t \sum_{n=0}^{l} \Big[\frac{\mathring{C}\Delta t}{2} \big\| d_t \varepsilon\big(Z_{\mathbf{u}}^{n+1}\big)\big\|^2_{L^2(\Omega)} + \frac{\kappa_2 \Delta t}{2} \big\| d_t G_\eta^{n+\theta}\big\|^2_{L^2(\Omega)} \\
&\qquad\qquad + \frac{\kappa_3\Delta t}{2}\big\|d_t G_\xi^{n+1}\big\|^2_{L^2(\Omega)} + \frac{1}{K\mu_f}\big(\nabla \hat{Z}_p^{n+1}, \nabla \hat{Z}_p^{n+1}\big)\Big] \\
&= \mathcal{E}_h^0 + \Delta t \sum_{n=0}^{l} \Big[\big(F_\xi^{n+1}, \operatorname{div} d_t Z_{\mathbf{u}}^{n+1}\big) - \big(\operatorname{div} d_t Y_{\mathbf{u}}^{n+1}, G_\xi^{n+1}\big)\Big] \\
&\quad + \kappa_1(1-\theta)(\Delta t)^2 \sum_{n=0}^{l} \big(d_t^2 \eta(t_{n+1}), G_\xi^{n+1}\big) + \Delta t \sum_{n=0}^{l} \big(R_h^{n+\theta}, \hat{Z}_p^{n+1}\big) \\
&\quad + \Delta t \sum_{n=0}^{l} \big(d_t G_\eta^{n+\theta}, Y_p^{n+1} - F_p^{n+1}\big) + (1-\theta)(\Delta t)^2 \sum_{n=0}^{l} \frac{\kappa_1}{\mu_f}\big(K d_t \nabla Z_\xi^{n+1}, \nabla \hat{Z}_p^{n+1}\big),
\end{aligned} \tag{4.9}$$

where the positive constant $\mathring{C}$ *satisfies* $C_4 \leq \mathring{C} \leq C_3$, *and*

$$\begin{aligned}
\hat{Z}_p^{n+1} &:= F_p^{n+1} - Y_p^{n+1} + \kappa_1 G_\xi^{n+1} + \kappa_2 G_\eta^{n+\theta}, \\
\mathcal{E}_h^{l+1} &:= \frac{1}{2}\Big[\mathring{C}\big\|\varepsilon\big(Z_{\mathbf{u}}^{l+1}\big)\big\|^2_{L^2(\Omega)} + \kappa_2 \big\|G_\eta^{l+\theta}\big\|^2_{L^2(\Omega)} + \kappa_3\big\|G_\xi^{l+1}\big\|^2_{L^2(\Omega)}\Big], \\
R_h^{n+1} &:= -\frac{1}{\Delta t}\int_{t_{n-1+\theta}}^{t_{n+\theta}} (s - t_n)\eta_{tt}(s)ds.
\end{aligned}$$

Proof. Subtracting (3.4) from (2.16), (3.5) from (2.17), (3.6) from (2.18), respectively, we get

$$\begin{aligned}
&\big(\mathcal{N}\big(\nabla \mathbf{u}(t_{n+1})\big) - \mathcal{N}\big(\nabla \mathbf{u}_h^{n+1}\big), \varepsilon(\mathbf{v}_h)\big) - \big(E_\xi^{n+1}, \operatorname{div} \mathbf{v}_h\big) = 0, \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h, && (4.10)\\
&\kappa_3\big(E_\xi^{n+1}, \varphi_h\big) + \big(\operatorname{div} E_{\mathbf{u}}^{n+1}, \varphi_h\big) \\
&= \kappa_1\big(E_\eta^{n+\theta}, \varphi_h\big) + \kappa_1(1-\theta)\Delta t\big(d_t\eta(t_{n+1}), \varphi_h\big), \qquad \forall\, \varphi_h \in M_h, && (4.11)\\
&\big(d_t E_\eta^{n+\theta}, \psi_h\big) + \frac{1}{\mu_f}\big(K\nabla E_p^{n+1}, \nabla\psi_h\big) \\
&\quad - (1-\theta)\frac{\kappa_1\Delta t}{\mu_f}\big(K d_t \nabla E_\xi^{n+1}, \nabla \psi_h\big) = \big(R_h^{n+\theta}, \psi_h\big), \qquad \forall\, \psi_h \in W_h, && (4.12)\\
&E_{\mathbf{u}}^0 = 0, \quad E_\xi^0 = 0, \quad E_\eta^{-1} = 0. && (4.13)
\end{aligned}$$

Using the definitions of the projection operators $\mathcal{Q}_h, \mathcal{S}_h$ and $\mathcal{R}_h$, we have

$$\begin{aligned}
&\big(\mathcal{N}\big(\nabla \mathbf{u}(t_{n+1})\big) - \mathcal{N}\big(\nabla \mathbf{u}_h^{n+1}\big), \varepsilon(\mathbf{v}_h)\big) - \big(G_\xi^{n+1}, \operatorname{div} \mathbf{v}_h\big) \\
&= (F_\xi^{n+1}, \operatorname{div} \mathbf{v}_h), \qquad \forall\, \mathbf{v}_h \in \mathbf{V}_h, && (4.14)\\
&\kappa_3(G_\xi^{n+1}, \varphi_h) + \big(\operatorname{div} Z_{\mathbf{u}}^{n+1}, \varphi_h\big) \\
&= \kappa_1\big(G_\eta^{n+\theta}, \varphi_h\big)\big(\operatorname{div} Y_{\mathbf{u}}^{n+1}, \varphi_h\big) + \kappa_1(1-\theta)\Delta t\big(d_t\eta(t_{n+1}), \varphi_h\big), \quad \forall\, \varphi_h \in M_h, && (4.15)\\
&\big(d_t G_\eta^{n+\theta}, \psi_h\big) + \frac{1}{\mu_f}\big(K\nabla \hat{Z}_p^{n+1}, \nabla \psi_h\big) - (1-\theta)\frac{\kappa_1\Delta t}{\mu_f}\big(K d_t \nabla E_\xi^{n+1}, \nabla\psi_h\big) \\
&= \big(R_h^{n+\theta}, \psi_h\big), \qquad \forall\, \psi_h \in W_h. && (4.16)
\end{aligned}$$

Using (2.23) and (2.31), we know that there exists $\mathring{C}$ satisfying $C_4 \leq \mathring{C} \leq C_3$ such that

$$\begin{aligned}&\big(\mathcal{N}\big(\nabla \mathbf{u}(t_{n+1})\big) - \mathcal{N}\big(\nabla \mathbf{u}_h^{n+1}\big), \varepsilon\big(\mathbf{u}(t_{n+1})\big) - \varepsilon\big(\mathbf{u}_h^{n+1}\big)\big)\\ &= \mathring{C}\big(\varepsilon\big(\mathbf{u}(t_{n+1})\big) - \varepsilon\big(\mathbf{u}_h^{n+1}\big), \varepsilon\big(\mathbf{u}(t_{n+1})\big) - \varepsilon\big(\mathbf{u}_h^{n+1}\big)\big).\end{aligned} \tag{4.17}$$

Setting $\mathbf{v}_h = d_t Z_{\mathbf{u}}^{n+1}$ in (4.14), we have

$$\big(\mathcal{N}\big(\nabla \mathbf{u}(t_{n+1})\big) - \mathcal{N}\big(\nabla \mathbf{u}_h^{n+1}\big), \varepsilon\big(d_t Z_{\mathbf{u}}^{n+1}\big)\big) - \big(G_\xi^{n+1}, \operatorname{div} d_t Z_{\mathbf{u}}^{n+1}\big) = \big(F_\xi^{n+1}, d_t \operatorname{div} Z_{\mathbf{u}}^{n+1}\big). \tag{4.18}$$

Setting $\varphi_h = G_\xi^{n+1}$ after applying the difference operator d_t to (4.15), we get

$$\begin{aligned}&\kappa_3\big(d_t G_\xi^{n+1}, G_\xi^{n+1}\big) + \big(\operatorname{div}\,(d_t Z_{\mathbf{u}}^{n+1}), G_\xi^{n+1}\big)\\ &= \kappa_1\big(d_t G_\eta^{n+\theta}, G_\xi^{n+1}\big) - \big(\operatorname{div}\,(d_t Y_{\mathbf{u}}^{n+1}), G_\xi^{n+1}\big)\\ &\quad + \kappa_1(1-\theta)\Delta t\big(d_t^2 \eta(t_{n+1}), G_\xi^{n+1}\big).\end{aligned} \tag{4.19}$$

Setting

$$\psi_h = \hat{Z}_p^{n+1} = F_p^{n+1} - Y_p^{n+1} + \kappa_1 G_\xi^{n+1} + \kappa_2 G_\eta^{n+\theta}$$

in (4.16), we obtain

$$\begin{aligned}&\big(d_t G_\eta^{n+\theta}, \hat{Z}_p^{n+1}\big) - (1-\theta)\frac{\kappa_1 \Delta t}{\mu_f}\big(K d_t \nabla Z_\xi^{n+1}, \nabla \hat{Z}_p^{n+1}\big) + \frac{1}{\mu_f}\big(K \nabla \hat{Z}_p^{n+1}, \nabla \hat{Z}_p^{n+1}\big)\\ &= \big(R_h^{n+\theta}, \hat{Z}_p^{n+1}\big).\end{aligned} \tag{4.20}$$

Adding (4.18)-(4.20), using (4.4) and (4.17), and applying the summation operator $\Delta t \sum_{n=0}^{l}$ to both sides, we see that (4.9) holds. The proof is complete. □

Theorem 4.1. *Let $\{(\mathbf{u}_h^n, \xi_h^n, \eta_h^n)\}_{n\geq 0}$ be defined by the MFEM, then there holds*

$$\begin{aligned}&\max_{0\leq n\leq l}\Big[\sqrt{\mathring{C}}\big\|\varepsilon\big(Z_{\mathbf{u}}^{n+1}\big)\big\|_{L^2(\Omega)} + \sqrt{\kappa_2}\big\|G_\eta^{n+\theta}\big\|_{L^2(\Omega)} + \sqrt{\kappa_3}\big\|G_\xi^{n+1}\big\|_{L^2(\Omega)}\Big]\\ &\quad + \left[\Delta t \sum_{n=0}^{l} \frac{K_1}{\mu_f}\big\|\nabla \hat{Z}_p^{n+1}\big\|_{L^2(\Omega)}^2\right]^{\frac{1}{2}} \leq \hat{C}_1(T)\Delta t + \hat{C}_2(T)h^2\end{aligned} \tag{4.21}$$

provided that $\Delta t = \mathcal{O}(h^2)$ when $\theta = 0$ and $\Delta t > 0$ when $\theta = 1$. Here

$$\hat{C}_1(T) = \hat{C}\|\eta_t\|_{L^2((0,T);L^2(\Omega))} + \hat{C}\|\eta_{tt}\|_{L^2((0,T);H^1(\Omega)')}, \tag{4.22}$$

$$\begin{aligned}\hat{C}_2(T) &= \hat{C}\|\xi_t\|_{L^2((0,T);H^2(\Omega))} + \hat{C}\|\xi\|_{L^\infty((0,T);H^2(\Omega))} + \hat{C}\|\eta_t\|_{L^2((0,T);H^2(\Omega))}\\ &\quad + \hat{C}\|\eta\|_{L^\infty((0,T);H^2(\Omega))} + \hat{C}\|\mathbf{u}\|_{L^\infty((0,T);H^2(\Omega))} + \hat{C}\|\operatorname{div}\mathbf{u}_t\|_{L^2((0,T);H^2(\Omega))}.\end{aligned} \tag{4.23}$$

Proof. Using (4.9) and the fact of $Z_{\mathbf{u}}^0 = \mathbf{0}, Z_\xi^0 = 0$ and $Z_\eta^{-1} = 0$, we have

$$\begin{aligned}&\mathcal{E}_h^{l+1} + \Delta t \sum_{n=0}^{l}\Big[\frac{1}{\mu_f}\big(K \nabla \hat{Z}_p^{n+1}, \nabla \hat{Z}_p^{n+1}\big) + \frac{\mathring{C}\Delta t}{2}\big\|d_t \varepsilon\big(Z_{\mathbf{u}}^{n+1}\big)\big\|_{L^2(\Omega)}^2\\ &\qquad + \frac{\kappa_2 \Delta t}{2}\big\|d_t G_\eta^{n+\theta}\big\|_{L^2(\Omega)}^2 + \frac{\kappa_3 \Delta t}{2}\big\|d_t G_\xi^{n+1}\big\|_{L^2(\Omega)}^2\Big]\\ &\leq \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5,\end{aligned} \tag{4.24}$$

where

$$
\begin{aligned}
\Phi_1 &= \Delta t \sum_{n=0}^{l} \left[\left(F_\xi^{n+1}, \operatorname{div} d_t Z_{\mathbf{u}}^{n+1}\right) - \left(\operatorname{div} d_t Y_{\mathbf{u}}^{n+1}, G_\xi^{n+1}\right) \right], \\
\Phi_2 &= \kappa_1 (1-\theta)(\Delta t)^2 \sum_{n=0}^{l} \left(d_t^2 \eta(t_{n+1}), G_\xi^{n+1}\right), \\
\Phi_3 &= \Delta t \sum_{n=0}^{l} \left(R_h^{n+\theta}, \hat{Z}_p^{n+1}\right), \\
\Phi_4 &= (1-\theta)(\Delta t)^2 \sum_{n=0}^{l} \frac{\kappa_1}{\mu_f} \left(K d_t \nabla Z_\xi^{n+1}, \nabla \hat{Z}_p^{n+1}\right), \\
\Phi_5 &= \Delta t \sum_{n=0}^{l} \left(d_t G_\eta^{n+\theta}, Y_p^{n+1} - F_p^{n+1}\right).
\end{aligned}
$$

Next, we estimate each term on the right-hand of (4.24). As for Φ_1, using Korn's inequality, Cauchy-Schwarz inequality, Young inequality and the inequality of

$$
\|\operatorname{div} \mathbf{w}\|_{L^2(\Omega)} \le c_k \|\varepsilon(\mathbf{w})\|_{L^2(\Omega)}, \quad \forall\, \mathbf{w} \in \mathbf{H}_\perp^1(\Omega),
$$

we obtain

$$
\begin{aligned}
\Phi_1 &= \Delta t \sum_{n=0}^{l} \left[\left(F_\xi^{n+1}, \operatorname{div} d_t Z_{\mathbf{u}}^{n+1}\right) - \left(\operatorname{div} d_t Y_{\mathbf{u}}^{n+1}, G_\xi^{n+1}\right) \right] \\
&\le \Delta t \sum_{n=0}^{l} \left[\frac{c_k^2}{\mathring{C} \Delta t} \left\| F_\xi^{n+1} \right\|_{L^2(\Omega)}^2 + \frac{\Delta t \mathring{C}}{4 c_k^2} \left\| \operatorname{div} d_t Z_{\mathbf{u}}^{n+1}) \right\|_{L^2(\Omega)}^2 \right. \\
&\qquad \left. + \frac{1}{2} \left\| \operatorname{div} d_t Y_{\mathbf{u}}^{n+1} \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \left\| G_\xi^{n+1} \right\|_{L^2(\Omega)}^2 \right].
\end{aligned} \tag{4.25}
$$

When $\theta = 0$, using the integration by parts and $d_t \eta(t_0) = 0$, we get

$$
\begin{aligned}
\Phi_2 &= \kappa_1 (\Delta t)^2 \sum_{n=0}^{l} \left(d_t^2 \eta(t_{n+1}), G_\xi^{n+1}\right) \\
&= \kappa_1 (\Delta t)^2 \left[\frac{1}{\Delta t} \left(d_t \eta(t_{l+1}), G_\xi^{l+1}\right) - \sum_{n=1}^{l} \left(d_t \eta(t_{n+1}), d_t G_\xi^{n+1}\right) \right].
\end{aligned} \tag{4.26}
$$

Using the Cauchy-Schwarz inequality, Young inequality and (3.2), we have

$$
\begin{aligned}
&\frac{1}{\Delta t} \left(d_t \eta(t_{l+1}), G_\xi^{l+1}\right) \\
&\le \frac{1}{\Delta t} \|d_t \eta(t_{l+1})\|_{L^2(\Omega)} \left\| G_\xi^{l+1} \right\|_{L^2(\Omega)} \\
&\le \frac{1}{\Delta t} \|\eta_t\|_{L^2((t_l, t_{l+1});\Omega)} \cdot \frac{1}{\beta_1} \sup_{\mathbf{v}_h \in V_h} \frac{\left(\mathcal{N}\left(\nabla \mathbf{u}(t_{l+1})\right) - \mathcal{N}\left(\nabla \mathbf{u}_h^{l+1}\right), \varepsilon(\mathbf{v}_h)\right) - \left(F_\xi^{l+1}, \operatorname{div} \mathbf{v}_h\right)}{\|\nabla \mathbf{v}_h\|_{L^2(\Omega)}} \\
&\le \frac{1}{\beta_1 \Delta t} \|\eta_t\|_{L^2((t_l, t_{l+1});\Omega)} \left[C_3 \left\| \varepsilon\left(Z_{\mathbf{u}}^{l+1}\right) \right\|_{L^2(\Omega)} + C_3 \left\| \varepsilon\left(Y_{\mathbf{u}}^{l+1}\right) \right\|_{L^2(\Omega)} + c_k \left\| F_\xi^{l+1} \right\|_{L^2(\Omega)} \right] \\
&\le \frac{C}{\beta_1^2} \|\eta_t\|_{L^2((t_l, t_{l+1});\Omega)}^2 + \frac{\mathring{C}}{4 \kappa_1 \Delta t^2} \left\| \varepsilon\left(Z_{\mathbf{u}}^{l+1}\right) \right\|_{L^2(\Omega)}^2 \\
&\quad + \frac{\mathring{C}}{4 \kappa_1 \Delta t^2} \left\| \varepsilon\left(Y_{\mathbf{u}}^{l+1}\right) \right\|_{L^2(\Omega)}^2 + \frac{c_k^2}{4 \Delta t^2} \left\| F_\xi^{l+1} \right\|_{L^2(\Omega)}^2,
\end{aligned} \tag{4.27}
$$

$$
\begin{aligned}
&\sum_{n=1}^{l}\big(d_t\eta(t_{n+1}), d_t G_\xi^{n+1}\big)\\
&\le \sum_{n=0}^{l}\|d_t\eta(t_{n+1})\|_{L^2(\Omega)}\big\|d_t G_\xi^{n+1}\big\|_{L^2(\Omega)}\\
&\le \sum_{n=1}^{l}\|d_t\eta(t_{n+1})\|_{L^2(\Omega)}\cdot\frac{1}{\beta_1}\sup_{\mathbf{v}_h\in V_h}\frac{\big(d_t\mathcal{N}\big(\nabla\mathbf{u}(t_{n+1})\big)-d_t\mathcal{N}\big(\nabla\mathbf{u}_h^{n+1}\big),\varepsilon(\mathbf{v}_h)\big)-\big(d_tF_\xi^{n+1},\operatorname{div}\mathbf{v}_h\big)}{\|\nabla\mathbf{v}_h\|_{L^2(\Omega)}}\\
&\le \sum_{n=1}^{l}\frac{1}{\beta_1}\|d_t\eta(t_{n+1})\|_{L^2(\Omega)}\Big[C_3\big\|d_t\varepsilon\big(Z_{\mathbf{u}}^{n+1}\big)\big\|_{L^2(\Omega)}+C_3\big\|d_t\varepsilon\big(Y_{\mathbf{u}}^{n+1}\big)\big\|_{L^2(\Omega)}+c_k\big\|d_tF_\xi^{n+1}\big\|_{L^2(\Omega)}\Big]\\
&\le \frac{C}{\beta_1^2}\|\eta_t\|^2_{L^2((0,T);L^2(\Omega))}+\sum_{n=1}^{l}\bigg[\frac{\mathring{C}}{4\kappa_1}\big\|d_t\varepsilon\big(Z_{\mathbf{u}}^{n+1}\big)\big\|^2_{L^2(\Omega)}+\frac{\mathring{C}}{4\kappa_1}\big\|d_t\varepsilon\big(Y_{\mathbf{u}}^{n+1}\big)\big\|^2_{L^2(\Omega)}\\
&\qquad\qquad+\frac{c_k^2}{4}\big\|d_tF_\xi^{n+1}\big\|^2_{L^2(\Omega)}\bigg]. \qquad (4.28)
\end{aligned}
$$

The term of Φ_3 can be bounded by

$$
\begin{aligned}
&\left|\Delta t\sum_{n=0}^{l}\big(R_h^{n+\theta},\hat{Z}_p^{n+1}\big)\right|\\
&\le \Delta t\sum_{n=0}^{l}\big\|R_h^{n+\theta}\big\|_{H^1(\Omega)'}\big\|\nabla\hat{Z}_p^{n+1}\big\|_{L^2(\Omega)}\\
&\le \Delta t\sum_{n=0}^{l}\left[\frac{K}{4\mu_f}\big\|\nabla\hat{Z}_p^{n+1}\big\|^2_{L^2(\Omega)}+\frac{\mu_f}{K}\big\|R_h^{n+\theta}\big\|^2_{H^1(\Omega)'}\right]\\
&\le \Delta t\sum_{n=0}^{l}\left[\frac{K}{4\mu_f}\big\|\nabla\hat{Z}_p^{n+1}\big\|^2_{L^2(\Omega)}+\frac{\mu_f\Delta t}{3K}\|\eta_{tt}\|^2_{L^2((t_{n-1+\theta},t_{n+\theta});H^1(\Omega)')}\right], \qquad (4.29)
\end{aligned}
$$

where we used the fact that

$$\big\|R_h^{n+\theta}\big\|^2_{H^1(\Omega)'}\le\frac{\Delta t}{3}\int_{t_{n-1+\theta}}^{t_{n+\theta}}\|\eta_{tt}\|^2_{H^1(\Omega)'}\,dt.$$

As for Φ_4, using (3.3), Cauchy-Schwarz inequality, Young inequality and (3.2), we have

$$
\begin{aligned}
\Phi_4 &= (\Delta t)^2\sum_{n=0}^{l}\frac{\kappa_1}{\mu_f}\big(K d_t\nabla Z_\xi^{n+1},\nabla\hat{Z}_p^{n+1}\big)\\
&\le (\Delta t)^2\sum_{n=0}^{l}\frac{c_1h^{-1}K_2\kappa_1}{\mu_f}\big\|d_tZ_\xi^{n+1}\big\|_{L^2(\Omega)}\big\|\nabla\hat{Z}_p^{n+1}\big\|_{L^2(\Omega)}\\
&\le (\Delta t)^2\sum_{n=0}^{l}\frac{c_1h^{-1}K_2\kappa_1}{\mu_f\beta_1}\\
&\quad\times\sup_{\mathbf{v}_h\in V_h}\frac{\big(d_t\mathcal{N}\big(\mathbf{u}(t_{n+1})\big)-d_t\mathcal{N}\big(\nabla\mathbf{u}_h^{n+1}\big),\varepsilon(\mathbf{v}_h)\big)-\big(d_tY_\xi^{n+1},\operatorname{div}\mathbf{v}_h\big)}{\|\nabla\mathbf{v}_h\|_{L^2(\Omega)}}\big\|\nabla\hat{Z}_p^{n+1}\big\|_{L^2(\Omega)}
\end{aligned}
$$

$$
\begin{aligned}
&\le (\Delta t)^2 \frac{c_1 K_2 \kappa_1}{h\mu_f \beta_1} \sum_{n=0}^{l} \Big[C_3 \big\| d_t \varepsilon\big(Z_{\mathbf{u}}^{n+1}\big)\big\|_{L^2(\Omega)} + C_3 \big\| d_t \varepsilon\big(Y_{\mathbf{u}}^{n+1}\big)\big\|_{L^2(\Omega)} \\
&\qquad\qquad + c_k \big\| d_t Y_\xi^{n+1}\big\|_{L^2(\Omega)} \Big] \big\|\nabla \hat{Z}_p^{n+1}\big\|_{L^2(\Omega)} \\
&\le (\Delta t)^2 \frac{K_2^2 \kappa_1^2 c_1^2 \Delta t}{\mu_f h^2 \beta_1^2} \sum_{n=0}^{l} \Big[\frac{C_3^2}{K_1} \big\| d_t \varepsilon\big(Z_{\mathbf{u}}^{n+1}\big)\big\|^2_{L^2(\Omega)} + \frac{C_3^2}{K_1} \big\| d_t \varepsilon\big(Y_{\mathbf{u}}^{n+1}\big)\big\|^2_{L^2(\Omega)} + \frac{c_k^2}{K_1} \big\| d_t Y_\xi^{n+1}\big\|^2_{L^2(\Omega)} \Big] \\
&\quad + \frac{3K_1 \Delta t}{4\mu_f} \sum_{n=0}^{l} \big\|\nabla \hat{Z}_p^{n+1}\big\|^2_{L^2(\Omega)}.
\end{aligned} \tag{4.30}
$$

Using the Cauchy-Schwarz inequality and Young inequality, we get

$$
\begin{aligned}
\Phi_5 &= \Delta t \sum_{n=0}^{l} \big(d_t G_\eta^{n+\theta}, Y_p^{n+1} - F_p^{n+1} \big) \\
&= \Delta t \Big[\frac{1}{\Delta t} (G_\eta^{l+\theta}, Y_p^{l+1} - F_p^{l+1}) - \sum_{n=1}^{l} (G_\eta^{n+\theta}, d_t Y_p^{n+1} - d_t F_p^{n+1}) \Big] \\
&\le \Delta t \Big[\frac{1}{\Delta t} \big\|G_\eta^{l+\theta}\big\|_{L^2(\Omega)} \big\|Y_p^{l+1} - F_p^{l+1}\big\|_{L^2(\Omega)} + \sum_{n=0}^{l} \big\|G_\eta^{n+\theta}\big\|_{L^2(\Omega)} \big\| d_t Y_p^{n+1} - d_t F_p^{n+1}\big\|_{L^2(\Omega)} \Big] \\
&\le \Delta t \Big[\frac{1}{\Delta t} \Big(\frac{\kappa_2}{4} \big\|G_\eta^{l+\theta}\big\|^2_{L^2(\Omega)} + \frac{1}{\kappa_2} \big\|Y_p^{l+1} - F_p^{l+1}\big\|^2_{L^2(\Omega)} \Big) + \kappa_2 \sum_{n=1}^{l} \big\|G_\eta^{n+\theta}\big\|_{L^2(\Omega)} \\
&\qquad + \frac{1}{2\kappa_2} \sum_{n=1}^{l} \Big(\big\| d_t F_p^{n+1}\big\|^2_{L^2(\Omega)} + \big\| d_t Y_p^{n+1}\big\|^2_{L^2(\Omega)} \Big) \Big].
\end{aligned} \tag{4.31}
$$

Substituting (4.25)-(4.31) into (4.24) and applying the discrete Gronwall inequality (cf. [34]), we obtain

$$
\begin{aligned}
&\mathring{C} \big\|\varepsilon(Z_{\mathbf{u}}^{l+1})\big\|^2_{L^2(\Omega)} + \kappa_2 \big\|G_\eta^{l+\theta}\big\|^2_{L^2(\Omega)} + \kappa_3 \big\|G_\xi^{l+1}\big\|^2_{L^2(\Omega)} + \Delta t \sum_{n=0}^{l} \frac{K}{\mu_f} \big\|\nabla \hat{Z}_p^{n+1}\big\|^2_{L^2(\Omega)} \\
&\le \hat{C} \Big[\|\eta_{tt}\|^2_{L^2((0,T);H^1(\Omega)')} + \|\eta_t\|^2_{L^2((0,T);L^2(\Omega))} + \big\|F_\xi^{l+1}\big\|^2_{L^2(\Omega)} + \big\|\varepsilon\big(Y_{\mathbf{u}}^{l+1}\big)\big\|^2_{L^2(\Omega)} \\
&\qquad + \big\|Y_p^{l+1}\big\|^2_{L^2(\Omega)} + \big\|F_p^{l+1}\big\|^2_{L^2(\Omega)} + \Delta t \sum_{n=0}^{l} \big\| d_t F_\xi^{n+1}\big\|^2_{L^2(\Omega)} + \Delta t \sum_{n=0}^{l} \big\| d_t Y_\xi^{n+1}\big\|^2_{L^2(\Omega)} \\
&\qquad + \Delta t \sum_{n=0}^{l} \big\|\operatorname{div} d_t Y_{\mathbf{u}}^{n+1}\big\|^2_{L^2(\Omega)} + \Delta t \sum_{n=0}^{l} \big\| d_t \varepsilon\big(Y_{\mathbf{u}}^{n+1}\big)\big\|^2_{L^2(\Omega)} \\
&\qquad + \Delta t \sum_{n=0}^{l} \big\|F_p^{n+1}\big\|^2_{L^2(\Omega)} + \Delta t \sum_{n=0}^{l} \big\|Y_p^{n+1}\big\|^2_{L^2(\Omega)} \Big] \\
&\le \hat{C} (\Delta t)^2 \Big(\|\eta_t\|^2_{L^2((0,T);L^2(\Omega))} + \|\eta_{tt}\|^2_{L^2((0,T);H^1(\Omega)')} \Big) \\
&\quad + \hat{C} h^4 \Big[\|\xi_t\|^2_{L^2((0,T);H^2(\Omega))} + \|\xi\|^2_{L^\infty((0,T);H^2(\Omega))} + \|\eta_t\|^2_{L^2((0,T);H^2(\Omega))} \\
&\qquad + \|\eta\|^2_{L^\infty((0,T);H^2(\Omega))} + \|\mathbf{u}\|^2_{L^\infty((0,T);H^2(\Omega))} + \|\operatorname{div} \mathbf{u}_t\|^2_{L^2((0,T);H^2(\Omega))} \Big]
\end{aligned}
$$

provided that

$$\Delta t \leq \frac{(h^2\beta_1^2\mu_f\mathring{C}K_1)}{(8K_2^2\kappa_1^2c_1^2C_3^2)},$$

when $\theta=0$ or $\Delta t>0$ when $\theta=1$. Hence, we deduce that (4.21) holds. □

Theorem 4.2. *The solution of the MFEM satisfies the following error estimates:*

$$\max_{0\leq n\leq N}\Big[\sqrt{\mathring{C}}\big\|\nabla\big(\mathbf{u}(t_{n+1})-\mathbf{u}_h^{n+1}\big)\big\|_{L^2(\Omega)}+\sqrt{\kappa_2}\big\|\eta(t_{n+1})-\eta_h^{n+1}\big\|_{L^2(\Omega)}$$
$$+\sqrt{\kappa_3}\big\|\xi(t_{n+1})-\xi_h^{n+1}\big\|_{L^2(\Omega)}\Big]\leq \check{C}_1(T)\Delta t+\check{C}_2(T)h^2, \tag{4.32}$$

$$\left(\Delta t\sum_{n=0}^{N}\frac{K_1}{\mu_f}\big\|\nabla p(t_{n+1})-\nabla p_h^{n+1}\big\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}\leq \check{C}_1(T)\Delta t+\check{C}_2(T)h \tag{4.33}$$

provided that $\Delta t=\mathcal{O}(h^2)$ *when* $\theta=0$ *and* $\Delta t>0$ *when* $\theta=1$*. Here*

$$\check{C}_1(T)=\hat{C}_1(T),\quad \check{C}_2(T)=\hat{C}_2(T)+\|\nabla\mathbf{u}\|_{L^\infty((0,T);H^2(\Omega))}.$$

Proof. The above estimates follow immediately from an application of the triangle inequality on

$$\begin{aligned}
&\mathbf{u}(t_n)-\mathbf{u}_h^n=Y_{\mathbf{u}}^n+Z_{\mathbf{u}}^n, && \xi(t_n)-\xi_h^n=Y_\xi^n+Z_\xi^n=F_\xi^n+G_\xi^n,\\
&\eta(t_n)-\eta_h^n=Y_\eta^n+Z_\eta^n=F_\eta^n+G_\eta^n, && p(t_n)-p_h^n=Y_p^n+Z_p^n=F_p^n+G_p^n.
\end{aligned}$$

and appealing to (4.5)-(4.7) and Theorem 4.1. The proof is complete. □

5. Numerical Tests

In this section, we use Matlab as the programming language, and choose $\mathbf{P}_2-P_1-P_1$ element pairs to solve the reformulated problem.

Test 1. Let $\Omega=(0,1)\times(0,1)$, $\Gamma_1=\{(1,x_2);0\leq x_2\leq 1\}$, $\Gamma_2=\{(x_1,0);0\leq x_1\leq 1\}$, $\Gamma_3=\{(0,x_2);0\leq x_2\leq 1\}$, $\Gamma_4=\{(x_1,1);0\leq x_1\leq 1\}$, and $T=1$. The source functions are as follows:

$$\begin{aligned}
&\mathbf{f}=-(\lambda+\mu)t(1,1)'-2(\mu+\lambda)t^2(x_1,x_2)'+\alpha te^{x_1+x_2}(1,1)',\\
&\phi=c_0e^{x_1+x_2}-\frac{2K}{\mu_f}te^{x_1+x_2}+\alpha(x_1+x_2),
\end{aligned}$$

and the boundary and initial conditions are

$$\begin{aligned}
&p=te^{x_1+x_2} && \text{on } \partial\Omega_T,\\
&u_1=\frac{1}{2}x_1^2t && \text{on } \Gamma_j\times(0,T),\quad j=1,3,\\
&u_2=\frac{1}{2}x_2^2t && \text{on } \Gamma_j\times(0,T),\quad j=2,4,\\
&\sigma\mathbf{n}-\alpha p\mathbf{n}=\mathbf{f_1} && \text{on } \partial\Omega_T,\\
&\mathbf{u}(x,0)=\mathbf{0},\quad p(x,0)=0 && \text{in } \Omega,
\end{aligned}$$

where

$$\begin{aligned}\mathbf{f}_1(x,t) = &\lambda(x_1+x_2)(n_1,n_2)'t + \mu t(x_1n_1,x_2n_2)' + \mu t^2\big(x_1^2n_1,x_2^2n_2\big)' \\ &+ \lambda t^2\big(x_1^2+x_2^2\big)(n_1,n_2)' - \alpha(n_1,n_2)'te^{x_1+x_2}.\end{aligned}$$

The exact solution of this problem is

$$\mathbf{u}(x,t) = \frac{t}{2}\big(x_1^2,x_2^2\big)', \quad p(x,t) = te^{x_1+x_2}.$$

Table 5.2 displays the spatial errors and convergence orders of displacement $\mathbf{u}$ and the pressure p with the parameters of Table 5.1, which are consistent with the theoretical result. As for the convergence order of time, we define

$$\rho_{h,\Delta t} = \frac{\|v^{h,\Delta t} - v^{h,\Delta t/2}\|_{L^2}}{\|v^{h,\Delta t/2} - v^{h,\Delta t/4}\|_{L^2}},$$

where $v = \mathbf{u}, p$. In particular, $\rho_{h,\Delta t} \approx 2$ when the corresponding convergence orders in time are of $\mathcal{O}(\Delta t)$, one can refer to [27]. From Table 5.3, one can see that the time convergence order is first-order. Figs. 5.1-5.6 display that the numerical solution closely approximates the

Table 5.1: Values of parameters.

Parameters	Description	Values
ν	Poisson ratio	0.0455
α	Biot-Willis constant	1
E	Young's modulus	2.09e4
λ	Lamé constant	1e3
K	Permeability tensor	(1e-7) $\mathbf{I}$
μ	Lamé constant	1e4
c_0	Constrained specific storage coefficient	2

Table 5.2: Spatial errors and convergence rates (CR) of $\mathbf{u}$ and p.

h	$\\|\mathbf{u}-\mathbf{u}_h\\|_{H^1}$	CR	$\\|p-p_h\\|_{L^2}$	CR
$h=1/3$	7.402e-6		2.430e-2	
$h=1/6$	1.876e-6	1.980	4.900e-3	2.310
$h=1/12$	4.707e-7	1.995	1.100e-3	2.155
$h=1/24$	1.178e-7	1.999	2.367e-4	2.217

Table 5.3: Errors and time convergence rates of $\mathbf{u}$ and p with $h=1/6$.

Δt	$\\|\mathbf{u}-\mathbf{u}_h\\|_{L^2}$	CR	$\\|p-p_h\\|_{L^2}$	CR
$\Delta t=1/10$	7.572e-7		9.271e-5	
$\Delta t=1/20$	3.786e-7	2.000	4.636e-5	2.000
$\Delta t=1/40$	1.893e-7	2.000	2.318e-5	2.000
$\Delta t=1/80$	9.466e-8	2.000	1.159e-5	2.000

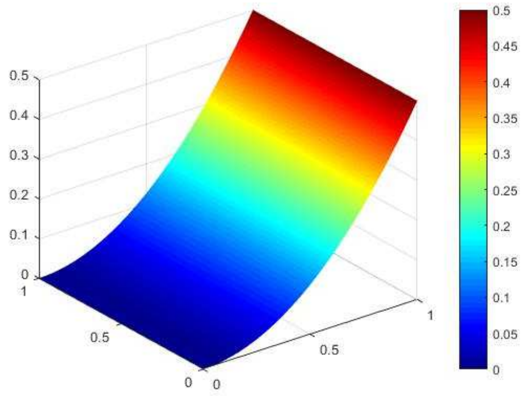

Fig. 5.1. The numerical displacement u_{1h}^{n+1} at $T = 1$ with the parameters of Table 5.1.

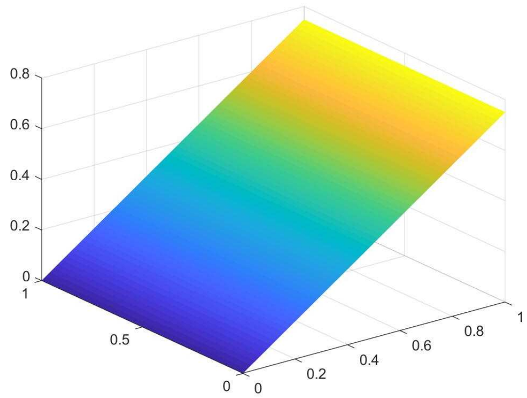

Fig. 5.2. Exact solution of displacement u_1 at $T = 1$.

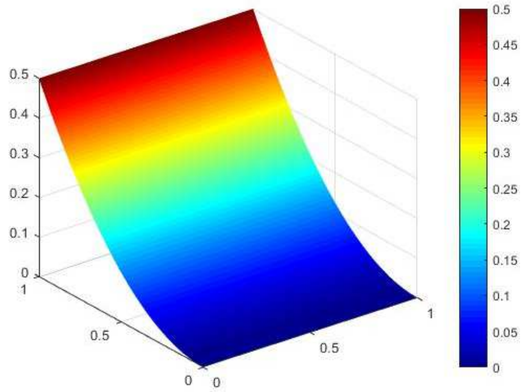

Fig. 5.3. The numerical displacement u_{2h}^{n+1} at $T = 1$ with the parameters of Table 5.1.

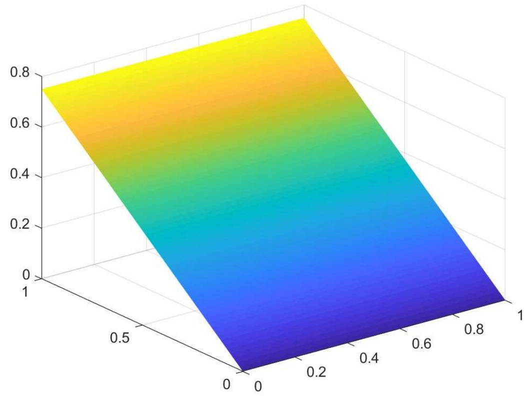

Fig. 5.4. Exact solution of displacement u_2 at $T = 1$.

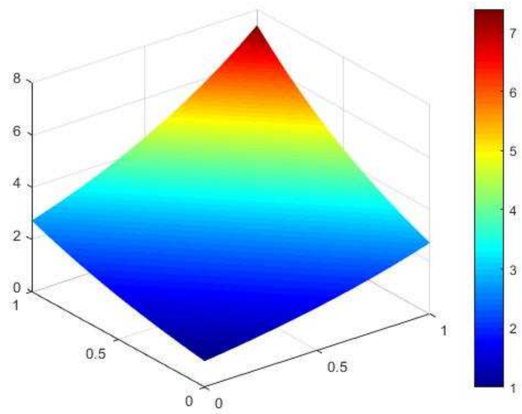

Fig. 5.5. The numerical pressure p_h^{n+1} at $T = 1$ with the parameters of Table 5.1.

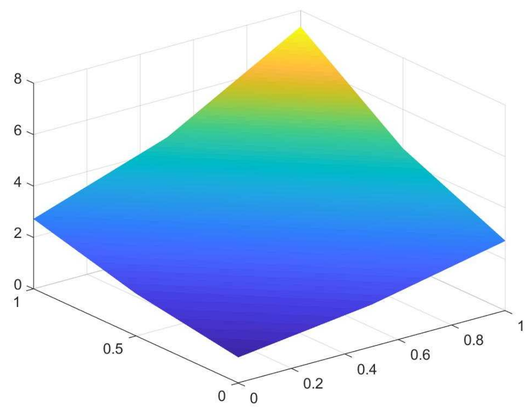

Fig. 5.6. Exact solution of pressure p at $T = 1$.

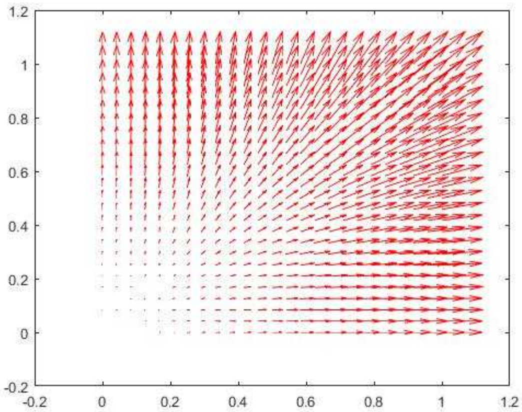

Fig. 5.7. Arrow plot of the computed displacement $\mathbf{u}$ at $T = 1$ with the parameters of Table 5.1.

exact solution. Fig. 5.7 presents an arrow plot of the computed displacement $\mathbf{u}$, illustrating the stability of the MFEM.

Test 2. The $\Omega, \Gamma_j, j = 1, 2, 3, 4$, the source functions, the boundary and initial conditions, the exact solution and the values of parameters are same as ones of Test 1, $T = 0.00001$.

Table 5.4 displays the spatial errors and convergence orders of displacement $\mathbf{u}$ and the pressure p at the terminal time $T = 0.00001$ with the parameters of Table 5.1, which are consistent with the theoretical result. Due to the time step Δt is so small, here we ignore the time error. From the exact solution

$$\mathbf{u}(x,t) = \frac{t}{2}\big(x_1^2, x_2^2\big)^\top, \quad p = t\sin(\pi x_1 + \pi x_2),$$

it is easy to check that $p_h^1 \geq 0$ and $\operatorname{div}\mathbf{u}_h^1 \geq 0$. In [18,33], there will appear $\operatorname{div}\mathbf{u}_h^1 \approx 0$ for the case of small Δt and low permeability K which leads to "locking phenomenon" in numerical pressure. Figs. 5.8-5.11 show that the numerical simulations of $\mathbf{u}$ and p at the terminal time $T = 0.00001$ are stable, indicating the absence of the "locking phenomenon".

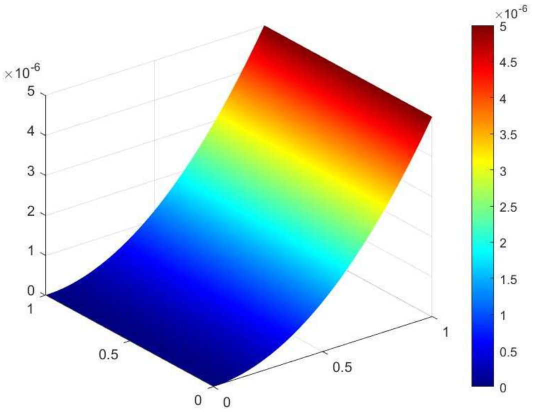

Fig. 5.8. The numerical displacement u_{1h}^{n+1} at $T = 0.00001$ with the parameters of Table 5.1.

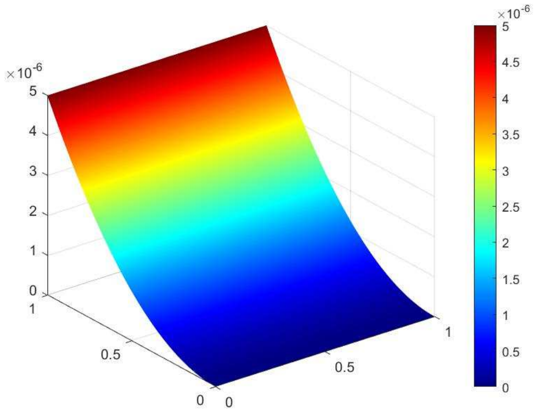

Fig. 5.9. The numerical displacement u_{2h}^{n+1} at $T = 0.00001$ with the parameters of Table 5.1.

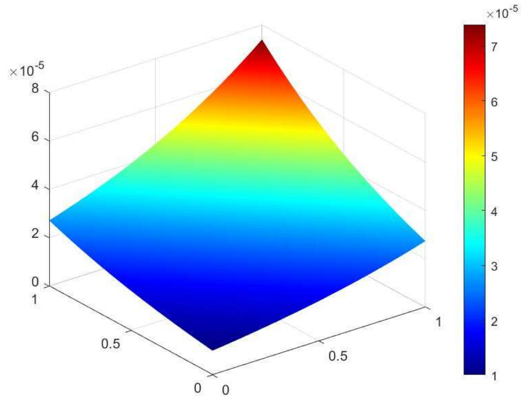

Fig. 5.10. The numerical pressure p_h^{n+1} at $T = 0.00001$ with the parameters of Table 5.1.

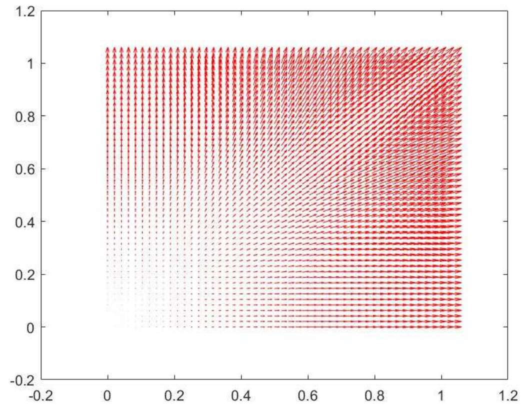

Fig. 5.11. Arrow plot of the computed displacement $\mathbf{u}$ at $T = 0.00001$ with the parameters of Table 5.1.

Table 5.4: Spatial errors and convergence rates of $\mathbf{u}$ and p.

h	$\|\mathbf{u}-\mathbf{u}_h\|_{H^1}$	CR	$\|p-p_h\|_{L^2}$	CR
$h=1/3$	1.356e-10		2.431e-7	
$h=1/6$	3.474e-11	1.965	4.933e-8	2.301
$h=1/12$	8.819e-12	1.978	1.052e-8	2.229
$h=1/24$	2.222e-12	1.989	2.367e-9	2.153

Test 3. The $\Omega, \Gamma_j, j=1,2,3,4$, and T are same as ones of Test 1. The source functions are $\mathbf{f}=0$ and $\phi=0$, and the boundary and initial conditions are

$$\begin{aligned}
&p = 0 && \text{on } \partial\Omega_T,\\
&u_1 = 0 && \text{on } \Gamma_j \times (0,T), \quad j=1,3,\\
&u_2 = 0 && \text{on } \Gamma_j \times (0,T), \quad j=2,4,\\
&\sigma\mathbf{n} - \alpha p\mathbf{n} = \mathbf{f_1} := (\mathbf{0}, \alpha\mathbf{p}) && \text{on } \partial\Omega_T,\\
&\mathbf{u}(x,0)=\mathbf{0}, \quad p(x,0)=0 && \text{in } \Omega,
\end{aligned}$$

where

$$p = \begin{cases} \sin t, & x_1 \in [0.2, 0.8) \times (0,T),\\ 0, & \text{otherwise}, \end{cases}$$

Figs. 5.12-5.13 present the numerical solution of pressure p_h^{n+1} and the arrow plot of the computed displacement $\mathbf{u}$, corresponding to the parameters in Table 5.5. Notably, the numerical pressure exhibits no oscillation even when $c_0 \to 0$ and for low permeability. And the arrows near the boundary math very well with the arrows on the boundary. Additionally, the displacement arrows near the boundary align well with those on the boundary. The MFEM includes a built-in mechanism to prevent the "locking phenomenon" often caused by specific parameter choices in the continuous Galerkin method.

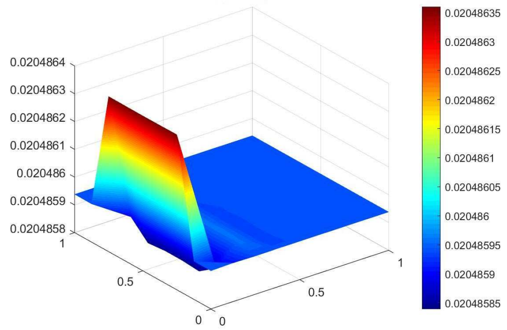

Fig. 5.12. The numerical pressure p_h^{n+1} at $T = 1$ with the parameters of Table 5.5.

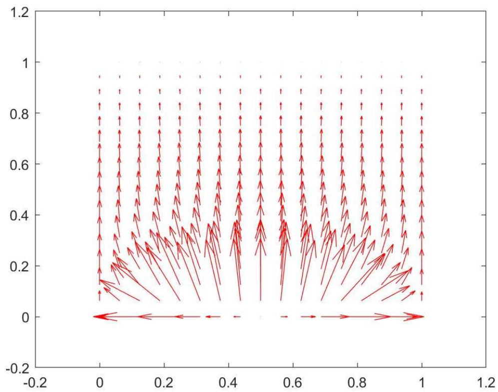

Fig. 5.13. Arrow plot of the computed displacement $\mathbf{u}$ at $T = 1$ with the parameters of Table 5.5.

Table 5.5: Values of parameters.

Parameters	Description	Values
ν	Poisson ratio	0.4
α	Biot-Willis constant	1
E	Young's modulus	2.8e3
λ	Lamé constant	4e3
K	Permeability tensor	(1e-7) $\mathbf{I}$
μ	Lamé constant	1e3
c_0	Constrained specific storage coefficient	1e-10

6. Conclusion

In this paper, we propose and analyze a multiphysics finite element method for the nonlinear poroelasticity model with the nonlinear constitutive relation

$$\tilde{\sigma}(\mathbf{u}) = \mu\tilde{\varepsilon}(\mathbf{u}) + \lambda tr(\tilde{\varepsilon}(\mathbf{u}))\mathbf{I}.$$

Using a multiphysics approach, we reformulate the nonlinear poroelasticity problem, transforming the original fluid-solid coupled problem into a fluid-fluid coupled system. We establish the growth, coercivity, and monotonicity of the nonlinear stress-strain relation, derive energy estimates, and apply Schaefer's fixed-point theorem to prove the existence and uniqueness of a weak solution. We also design a fully discrete time-stepping scheme using the multiphysics finite element method with $\mathbf{P}_2 - P_1 - P_1$ element pairs for spatial variables and the backward Euler method for time, and derive optimal convergence order error estimates. To the best of our knowledge, this is the first demonstration of the existence and uniqueness of a weak solution based on a multiphysics approach without assumptions on the nonlinear stress-strain relation, and the analysis of this fully discrete multiphysics finite element method for nonlinear poroelasticity is entirely novel.

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