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GENERALIZED JACOBI SPECTRAL GALERKIN METHOD FOR FRACTIONAL-ORDER VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS*

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Abstract

In this paper, we present a generalized Jacobi spectral Galerkin method for fractional Volterra integro-differential equations (FVIDEs). The basis functions of the proposed method are generalized Jacobi functions, which serve as natural basis functions for appropriately designed spectral methods for FVIDEs. We establish a convergence analysis of the generalized Jacobi spectral Galerkin method under reasonable assumptions. Numerical experiments are provided to demonstrate the effectiveness of the proposed method.

Mathematics subject classification: 65L05, 65L20, 65L50.
Key words: Fractional-order Volterra integro-differential equations, Spectral Galerkin method, Generalized Jacobi function.

1. Introduction

The fractional calculus has gained a tremendous development in both theory and application, and its appearance and development have made up for defects of the classical calculus of integer-order to a certain extent. Progress in the last two decades has demonstrated that fluid flow in porous materials, anomalous diffusion transport, acoustic wave propagation in viscoelastic materials, dynamics in self-similar structures, signal processing, financial theory, electric conductance of biological systems are more accurately described by involving fractional calculus (cf. [11, 17, 20, 25, 28] and the references therein).

Fractional integral equations and integro-differential equations are usually very difficult to solve analytically and the exact solutions to this type of problem are very scarce. To solve these problems numerically, a great deal of interest has been emphasized by several researchers. In the last thirty years, there have been a great deal of methods to solve fractional-order integro-differential equations numerically, for instance Adomian decomposition method (ADM) [22],

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fractional differential transform method (FDTM) [23], the collocation method [30], variational iteration method (VIM) [3], Laplace transform method [27] and homotopy analysis method [15].

This paper is concerned with the numerical solutions of the fractional-order Volterra integro-differential equations

$$\begin{cases} {}_0D_t^{\mu} y(t) = y(t) + \displaystyle\int_0^t K(t,\tau)y(\tau)d\tau + g(t), \quad t \in I := [0,T], \\ y(0) = y_0, \end{cases} \tag{1.1}$$

where $0 < \mu < 1$, $K \in C(D)$ with

$$D := \{(t,\tau) : 0 \leqslant \tau \leqslant t \leqslant T\}, \quad g(t) \in C(I).$$

${}_0D_t^{\mu}$ denotes the left-sided Reimann-Liouville fractional derivative of order μ.

Many mathematical modeling of various physical phenomena contains fractional-order Volterra integro-differential equations, for instance, heat conduction in materials with memory in [24]. In addition, such kinds of equations are also encountered in conduction, convection and radiation problems (see, e.g. [1, 5, 26] and the references therein).

In recent years, more and more researchers have made contributions to the fractional-order Volterra integro-differential equations. Yang [39] proposed general spectral and pseudo-spectral Jacobi-Galerkin methods for fractional-order integro-differential equations of Volterra type. Zhu and Fan [42] proposed a second Chebyshev wavelet method (SCWM) to solve nonlinear fractional-order Volterra integro-differential equations. In [13], the Legendre wavelet method (LWM) has been applied to solve the fractional population growth model which is the form of the fractional Volterra-Fredholm integro-differential equation (FVFIDE). The nonlinear FVFIDEs with mixed conditions have been solved by Nystrom and Newton-Kantorovitch method. In addition, orthogonal polynomial approximations have been applied to fractional integro-differential equations, viz., Bernoulli polynomials [35], least squares method and shifted Chebyshev Polynomial [21], Bernstein operational matrix [2], etc.

The spectral Galerkin method is an efficient method that has been applied by many researchers to solve different types of partial integro-differential equations. In [32], a generalized Jacobi-Galerkin method is applied to nonlinear Volterra integral equations with weakly singular kernels. Xie *et al.* [38] proposed Jacobi spectral Galerkin approaches for the second kind Volterra integral equation. Wan *et al.* [36] also proposed Legendre spectral Galerkin method for second kind VIEs. Yang [40] employed a Jacobi spectral Galerkin method for VIEs with a weakly singular kernel. In addition to this, Petrov-Galerkin method has been successively applied to solve many partial differential equations (see, e.g. [16, 31]).

When $\mu = 1$, the Eq. (1.1) is the classical Volterra integro-differential equations (VIDEs)

$$\begin{cases} y'(t) = y(t) + \displaystyle\int_0^t K(t,\tau)y(\tau)d\tau + g(t), \quad t \in [0,T], \\ y(0) = y_0. \end{cases} \tag{1.2}$$

Recently, many kinds of spectral collocation methods are proposed for solving VIEs with smooth kernels (cf. [9, 18, 34, 37] and the references therein). To solve VIEs with weakly singular kernels, many attempts have been made to overcome the difficulties caused by the singularities of the solutions. Chen and Tang [7, 8] proposed spectral collocation methods for weakly singular VIEs. In [19], linear Volterra integro-differential equations have been solved by Petrov–Galerkin

method. Huang *et al.* [14] studied the supergeometric convergence of spectral collocation methods for weakly singular Volterra/Fredholm integral equations, etc.

In this paper, we develop a non-polynomial spectral Galerkin method for FVIDEs. More precisely, we construct a spectral Galerkin method for FVIDEs (1.1), using the generalized Jacobi functions as basis functions. This kind of functions have been used by Chen *et al.* [6], and Zayernouri and Karniadakis [41] for approximating fractional-order differential equations. The main strategies and contributions are highlighted as follows:

- We propose a generalized Jacobi spectral Galerkin method for linear FVIDEs. The basis functions can be tuned to match the singularities of the underlying solutions and lead to an efficient implementation. The existing works can be referenced in [39].
- We carry out an error analysis of the proposed method for problem (1.1).

The organization of this paper is as follows. In the next section, we introduce some useful properties of fractional calculus and define some functional spaces. In Section 3, we present the generalized Jacobi spectral Galerkin method for FVIDEs (1.1). Some lemmas useful for the convergence analysis will be provided in Section 4. The convergence of the generalized Jacobi spectral Galerkin method is given in Section 5. Numerical results will be carried out in Section 6, which will be used to verify the theoretical results. In the final section, some concluding remarks are given.

2. Preliminaries

2.1. Fractional calculus

We start with some definitions of fractional calculus (see, e.g. [11, 28]). To fix the idea, we restrict our attentions to the interval $\Lambda := [-1, 1]$. For $\rho \in \mathbb{R}^+$ the left-sided and right-sided Reimann-Liouville integrals are respectively defined as

$$ {}_{-1}I_x^\rho y(x) = \frac{1}{\Gamma(\rho)} \int_{-1}^{x} \frac{y(s)}{(x-s)^{1-\rho}} ds, \quad x \in \Lambda, \tag{2.1a} $$

$$ {}_{x}I_1^\rho y(x) = \frac{1}{\Gamma(\rho)} \int_{x}^{1} \frac{y(s)}{(s-x)^{1-\rho}} ds, \quad x \in \Lambda, \tag{2.1b} $$

where $\Gamma(\cdot)$ is the usual Gamma function.

For $\nu \in [m-1, m)$ with $m \in \mathbb{N}$, the left-sided and right-sided Reimann-Liouville fractional derivative of order ν are defined by

$$ {}_{-1}D_x^\nu y(x) = \frac{1}{\Gamma(m-\nu)} \frac{d^m}{dx^m} \int_{-1}^{x} \frac{y(s)}{(x-s)^{\nu-m+1}} ds, \quad x \in \Lambda, \tag{2.2a} $$

$$ {}_{x}D_1^\nu y(x) = \frac{(-1)^m}{\Gamma(m-\nu)} \frac{d^m}{dx^m} \int_{x}^{1} \frac{y(s)}{(s-x)^{\nu-m+1}} ds, \quad x \in \Lambda. \tag{2.2b} $$

For $\nu \in [m-1, m)$ with $m \in \mathbb{N}$, the left-sided and right-sided Caputo fractional derivative of order ν are defined by

$$ {}^{C}_{-1}D_x^\nu y(x) = \frac{1}{\Gamma(m-\nu)} \int_{-1}^{x} \frac{y^{(m)}(s)}{(x-s)^{\nu-m+1}} ds, \quad x \in \Lambda, \tag{2.3a} $$

$$ {}_x^C D_1^\nu y(x) = \frac{(-1)^m}{\Gamma(m-\nu)} \int_x^1 \frac{y^{(m)}(s)}{(s-x)^{\nu-m+1}} ds, \quad x \in \Lambda. \tag{2.3b} $$

According to [11, Theorem 2.14], we have that for any absolutely integrable function f, and real $\nu \geqslant 0$,

$$ {}_{-1}D_x^\nu {}_{-1}I_x^\nu y(x) = y(x), \quad {}_xD_{1\,x}^\nu I_1^\nu y(x) = y(x), \quad x \in \Lambda. \tag{2.4} $$

The following lemma shows the relationship between the Riemann-Liouville and Caputo fractional derivatives.

Lemma 2.1 ([11, 28]). *For $s \in [k-1, k)$ with $k \in \mathbb{N}$, we have*

$$ {}_{-1}D_x^s v(x) = {}_{-1}^C D_x^s v(x) + \sum_{j=0}^{k-1} \frac{v^{(j)}(-1)}{\Gamma(1+j-s)} (1+x)^{j-s}, \tag{2.5a} $$

$$ {}_xD_1^s v(x) = {}_x^C D_1^s v(x) + \sum_{j=0}^{k-1} \frac{(-1)^j v^{(j)}(1)}{\Gamma(1+j-s)} (1-x)^{j-s}. \tag{2.5b} $$

Remark 2.1. We observe immediately from (2.5) that for $s \in [k-1, k)$ with $k \in \mathbb{N}$

$$ {}_{-1}D_x^s v(x) = {}_{-1}^C D_x^s v(x), \quad \text{if} \quad v^{(j)}(-1) = 0, \quad 0 \leq j \leq k-1, \tag{2.6a} $$

$$ {}_xD_1^s v(x) = {}_x^C D_1^s v(x), \quad \text{if} \quad v^{(j)}(+1) = 0, \quad 0 \leq j \leq k-1. \tag{2.6b} $$

2.2. Standard Jacobi polynomials

For $\alpha, \beta > -1$, let $J_n^{(\alpha,\beta)}(x)$, $x \in \Lambda := (-1, 1)$ be the standard Jacobi polynomial of degree n, and denote the weight function $\omega^{(\alpha,\beta)}(x) = (1-x)^\alpha (1+x)^\beta$. The set of Jacobi polynomials is a complete $L^2_{\omega^{(\alpha,\beta)}}(\Lambda)$-orthogonal system, i.e.

$$ \int_{-1}^1 J_n^{(\alpha,\beta)}(x) J_m^{(\alpha,\beta)}(x) \omega^{(\alpha,\beta)}(x) dx = \gamma_m^{(\alpha,\beta)} \delta_{mn}, \tag{2.7} $$

in which δ_{mn} denotes the Kronecker function, and

$$ \gamma_m^{(\alpha,\beta)} = \begin{cases} \dfrac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, & m = 0, \\ \dfrac{2^{\alpha+\beta+1}}{(2m+\alpha+\beta+1)} \dfrac{\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}{m!\Gamma(m+\alpha+\beta+1)}, & m \geq 1. \end{cases} \tag{2.8} $$

In particular, $J_0^{(\alpha,\beta)}(x) = 1$.

2.3. Generalized Jacobi functions

The generalized Jacobi functions of degree n is defined by (cf. [6])

$$ {}^+J_n^{(-\alpha,\beta)}(x) := (1-x)^\alpha J_n^{(\alpha,\beta)}(x) \quad \alpha > -1, \quad \beta \in \mathbb{R}, \tag{2.9} $$

$$ {}^-J_n^{(\alpha,-\beta)}(x) := (1+x)^\beta J_n^{(\alpha,\beta)}(x) \quad \alpha \in \mathbb{R}, \quad \beta > -1 \tag{2.10} $$

for all $x \in \Lambda$ and $n \geq 0$.

The orthogonal properties of the generalized Jacobi functions are as follows:

$$\begin{aligned}&\int_{-1}^{1} {}^{+}J_n^{(-\alpha,\beta)}(x){}^{+}J_m^{(-\alpha,\beta)}(x)\omega^{(-\alpha,\beta)}(x)dx,\\ =&\int_{-1}^{1} {}^{-}J_n^{(\alpha,-\beta)}(x){}^{-}J_m^{(\alpha,-\beta)}(x)\omega^{(\alpha,-\beta)}(x)dx=\gamma_m^{(\alpha,\beta)}\delta_{nm},\end{aligned}\tag{2.11}$$

where $\gamma_m^{(\alpha,\beta)}$ is defined in (2.8).

Lemma 2.2 ([6]). *Let $s\in\mathbb{R}^+, n\in\mathbb{N}_0$ and $x\in\Lambda$.*

1. *For $\alpha>s-1$ and $\beta\in\mathbb{R}$,*

$${}_xD_1^s\big\{{}^{+}J_n^{(-\alpha,\beta)}(x)\big\}=\frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha-s+1)}{}^{+}J_n^{(-\alpha+s,\beta+s)}(x).\tag{2.12a}$$

2. *For $\alpha\in\mathbb{R}$ and $\beta>s-1$,*

$${}_{-1}D_x^s\big\{{}^{-}J_n^{(\alpha,-\beta)}(x)\big\}=\frac{\Gamma(n+\beta+1)}{\Gamma(n+\beta-s+1)}{}^{-}J_n^{(\alpha+s,-\beta+s)}(x).\tag{2.12b}$$

Remark 2.2. Two important special cases of Lemma 2.2 are as follows:

1. For $\alpha>0$ and $\beta\in\mathbb{R}$,

$${}_xD_1^\alpha\big\{{}^{+}J_n^{(-\alpha,\beta)}(x)\big\}=\frac{\Gamma(n+\alpha+1)}{n!}{}^{+}J_n^{(0,\alpha+\beta)}(x)=\frac{\Gamma(n+\alpha+1)}{n!}J_n^{(0,\alpha+\beta)}(x).\tag{2.13a}$$

2. For $\alpha\in\mathbb{R}$ and $\beta>0$,

$${}_{-1}D_x^\beta\big\{{}^{-}J_n^{(\alpha,-\beta)}(x)\big\}=\frac{\Gamma(n+\beta+1)}{n!}{}^{+}J_n^{(\alpha+\beta),0}(x)=\frac{\Gamma(n+\beta+1)}{n!}J_n^{(\alpha+\beta),0}(x).\tag{2.13b}$$

For the case of non-homogeneous initial conditions when $y(0)=y_0\neq 0$, we employ the method of lifting a known solution, where we decompose the solution $y(t)$ into two parts as

$$y(t)=y_{\mathcal{H}}(t)+y_{\mathcal{D}},\tag{2.14}$$

in which $y_{\mathcal{H}}(t)$ corresponds to the homogeneous solution and $y_{\mathcal{D}}=y_0$ is the nonzero initial condition, given in (1.1). We substitute (2.14) into (1.1) to get

$$\begin{cases}{}_0D_t^\mu y_{\mathcal{H}}(t)=y_{\mathcal{H}}(t)+\displaystyle\int_0^t K(t,\tau)y_{\mathcal{H}}(\tau)d\tau+f(t),\quad t\in(0,T],\\ y_{\mathcal{H}}(0)=0,\end{cases}\tag{2.15}$$

where

$$f(t)=g(t)+y_{\mathcal{D}}\bigg(\int_0^t K(t,s)ds+1-\frac{1}{\Gamma(1-\mu)t^\mu}\bigg).\tag{2.16}$$

For the fractional-order Volterra integro-differential equations with the non-homogeneous initial value conditions, we can transform it into Eq. (2.15) of homogeneous initial value conditions to solve. Therefore, we only need to consider the case of $y(0)=0$.

3. Generalized Jacobi Spectral-Galerkin Method

To be able to apply the properties of orthogonal polynomials, we use the following interval transformation:

$$t = \frac{T}{2}(1+x), \quad \tau = \frac{T}{2}(1+s),$$

and let

$$u(x) = y\left(\frac{T}{2}(1+x)\right), \qquad {}_{-1}D_x^{\mu}u(x) = \left(\frac{T}{2}\right)^{\mu} {}_0D_t^{\mu}y\left(\frac{T}{2}(1+x)\right),$$
$$G(x) = \left(\frac{T}{2}\right)^{\mu} g\left(\frac{T}{2}(1+x)\right), \quad k(x,s) = \left(\frac{T}{2}\right)^{\mu+1} K\left(\frac{T}{2}(1+x), \frac{T}{2}(1+s)\right).$$

The fractional-order Volterra integro-differential equation (1.1) can then be transformed into the following form:

$$\begin{cases} {}_{-1}D_x^{\mu}u(x) = \epsilon u(x) + \displaystyle\int_{-1}^{x} k(x,s)u(s)ds + G(x), \quad x \in \Lambda, \\ u(-1) = 0, \end{cases} \tag{3.1}$$

where $\epsilon = (T/2)^{\mu}$.

We first define a linear integral operator $\mathcal{V} : C(\Lambda) \to C(\Lambda)$ by

$$(\mathcal{V}\psi)(x) := \int_{-1}^{x} k(x,s)\psi(s)ds.$$

On the other hand, we define the finite-dimensional fractional-polynomial space

$${}^{-}\mathcal{F}^{(\alpha,-\beta)}(\Lambda) = \left\{\phi = (1+x)^{\beta}\psi : \psi \in \mathcal{P}_N\right\} = \operatorname{span}\left\{{}^{-}J_n^{(\alpha,-\beta)} : 0 \le n \le N\right\}.$$

Let $u = u(x), D^{\mu}u = {}_{-1}D_x^{\mu}u(x)$. Then, the numerical scheme for problem (3.1) is to find $u_N \in {}^{-}\mathcal{F}^{(\alpha,-\beta)}(\Lambda)$ such that

$$\left(D^{\mu}u_N, v_N\right)_{\omega^{(\alpha,-\beta)}} = \left(\epsilon u_N + \mathcal{V}u_N + G, v_N\right)_{\omega^{(\alpha,-\beta)}}, \quad \forall\, v_N \in {}^{-}\mathcal{F}^{(\alpha,-\beta)}(\Lambda). \tag{3.2}$$

In order to apply the property of Lemma 2.2, we choose $\alpha = \beta = \mu$. In FVIDE (3.1), we seek an approximate solution of the form

$$u(x) \approx u_N(x) = \sum_{n=1}^{N} a_n {}^{-}J_n^{(\mu,-\mu)}(x), \tag{3.3}$$

where a_n are the unknown expansion coefficients to be determined. By plugging (3.3) into (3.1), we get the residual $R_N(t)$ for problem (3.1) as

$$R_N(x) = {}_{-1}D_x^{\mu}u_N(x) - \epsilon u_N(x) - \mathcal{V}u_N(x) - G(x).$$

Next, we need to derive the numerical discrete format of the generalized Jacobi spectral Galerkin method for (3.1). Through the definition of residual $R_N(t)$ and (3.3), we can get

$$\sum_{n=1}^{N} a_n \left[\int_{-1}^{1} \left({}_{-1}D_x^{\mu} {}^{-}J_n^{(\mu,-\mu)}(x)\right) {}^{-}J_k^{(\mu,-\mu)}(x)\omega^{(\mu,-\mu)}(x)dx\right]$$

$$
\begin{aligned}
&- \sum_{n=1}^{N} a_n \left[\epsilon \int_{-1}^{1} {}^{-}J_n^{(\mu,-\mu)}(x)\, {}^{-}J_k^{(\mu,-\mu)}(x)\omega^{(\mu,-\mu)}(x)dx \right] \\
&- \sum_{n=1}^{N} a_n \left[\int_{-1}^{1} \big(\mathcal{V}\, {}^{-}J_n^{(\mu,-\mu)}(x)\big)\, {}^{-}J_k^{(\mu,-\mu)}(x)\omega^{(\mu,-\mu)}(x)dx \right] \\
&= \int_{-1}^{1} G(x)\, {}^{-}J_k^{(\mu,-\mu)}(x)\omega^{(\mu,-\mu)}(x)dx. \qquad (3.4)
\end{aligned}
$$

Set

$$
\begin{aligned}
&\mathbf{a} = (a_1, a_2, \cdots, a_N)^{\top}, \quad \mathbf{D} = (d_{kn})_{1\leqslant k,n\leqslant N}, \\
&\mathbf{P} = (p_{kn})_{1\leqslant k,n\leqslant N}, \qquad \mathbf{V} = (v_{kn})_{1\leqslant k,n\leqslant N}, \\
&d_{kn} = \int_{-1}^{1} \big({}_{-1}D_x^{\mu}\, {}^{-}J_n^{(\mu,-\mu)}(x)\big) {}^{-}J_k^{(\mu,-\mu)}(x)\omega^{(\mu,-\mu)}(x)dx, \\
&p_{kn} = \epsilon \int_{-1}^{1} {}^{-}J_n^{(\mu,-\mu)}(x)\, {}^{-}J_k^{(\mu,-\mu)}(x)\omega^{(\mu,-\mu)}(x)dx = \left(\frac{T}{2}\right)^{\mu} \gamma_{kn}^{(\mu,\mu)}\delta_{kn}, \qquad (3.5) \\
&v_{kn} = \int_{-1}^{1} \big(\mathcal{V}\, {}^{-}J_n^{(\mu,-\mu)}(x)\big)\, {}^{-}J_k^{(\mu,-\mu)}(x)\omega^{(\mu,-\mu)}(x)dx, \\
&\mathbf{F} = (f_1, f_2, \cdots, f_N)^{\top}, \quad f_k = \int_{-1}^{1} G(x) {}^{-}J_k^{(\mu,-\mu)}(x)\omega^{(\mu,-\mu)}(x)dx, \\
&\mathbf{A} = \mathbf{D} - \mathbf{P} - \mathbf{V}.
\end{aligned}
$$

Then, the system (3.4) becomes

$$\mathbf{A}\mathbf{a} = \mathbf{F}. \qquad (3.6)$$

In actual computation, we use the Jacobi-Gauss quadrature formula to approximate the terms f_k and d_{kn}, namely,

$$f_k \approx \sum_{j=0}^{N} G(x_j) J_k^{(\mu,\mu)}\big(x_j^{(\mu,0)}\big)\omega_j^{(\mu,0)}, \qquad (3.7)$$

$$d_{kn} \approx \frac{\Gamma(n+\mu+1)}{n!} \sum_{j=0}^{N} J_n^{(2\mu,0)}\big(x_j^{(\mu,0)}\big) J_k^{(\mu,\mu)}\big(x_j^{(\mu,0)}\big)\omega_j^{(\mu,0)}, \qquad (3.8)$$

where in this context, $\{\omega_j^{(\mu,0)}\}_{j=0}^{N}$ and $\{x_j^{(\mu,0)}\}_{j=0}^{N}$ are the Jacobi-Gauss-Lobatto quadrature weights and points corresponding to the Jacobi weight function $\omega^{(\mu,0)}(x)$.

To describe v_{kn}, we first transform the integral interval $[-1, x]$ to $[-1, 1]$ using the transformation

$$s(x,\theta) = \frac{1+x}{2}\theta + \frac{x-1}{2}, \quad \theta \in \Lambda, \qquad (3.9)$$

then, we have

$$
\begin{aligned}
v_{kn} &= \frac{1}{2^{\mu+1}} \int_{-1}^{1} \left(\int_{-1}^{1} k\big(x, s(x,\theta)\big)(1+\theta)^{\mu} J_n^{(\mu,\mu)}\big(s(x,\theta)\big)d\theta \right) (1-x)^{\mu}(1+x)^{\mu+1} J_k^{(\mu,\mu)}(x)dx \\
&\approx \frac{1}{2^{\mu+1}} \sum_{i,j=0}^{N} J_k^{(\mu,\mu)}\big(x_i^{(\mu,\mu+1)}\big)\omega_i^{(\mu,\mu+1)} k\Big(x_i^{(\mu,\mu+1)}, s\big(x_i^{(\mu,\mu+1)}, x_j^{(0,\mu)}\big)\Big) \\
&\qquad \times J_n^{(\mu,\mu)}\Big(s\big(x_i^{(\mu,\mu+1)}, x_j^{(0,\mu)}\big)\Big)\omega_j^{(0,\mu)},
\end{aligned}
$$

in which $\{\omega_i^{(\mu,\mu+1)}\}_{i=0}^N$ and $\{x_i^{(\mu,\mu+1)}\}_{i=0}^N$ denote the Jacobi-Gauss-Lobatto weights and quadrature points associated with the weight function $\omega^{(\mu,\mu+1)}(x)$. In addition, where $\{\omega_j^{(0,\mu)}\}_{j=0}^N$ and $\{x_j^{(0,\mu)}\}_{j=0}^N$ denote the Jacobi-Gauss-Lobatto weights and quadrature points associated with the weight function $\omega^{(0,\mu)}(x)$.

4. Some Useful Lemmas

In this section, we will provide some elementary lemmas, which are important for the derivation of the main results in the subsequent section.

Let $\mathcal{P}_N$ be the set of all algebraic (real-valued) polynomials of degree at most N. Define a weighted space $L^2_{\omega^{(\alpha,\beta)}}(\Lambda)$ as

$$L^2_{\omega^{(\alpha,\beta)}}(\Lambda) = \{v : v \text{ is measureable and } \|v\|_{\omega^{(\alpha,\beta)}} < \infty\}$$

with the inner product and norm

$$(u,v)_{\omega^{(\alpha,\beta)}} = \int_\Lambda u(x)v(x)\omega^{(\alpha,\beta)}(x)dx, \quad \|v\|_{\omega^{(\alpha,\beta)}} = \bigg(\int_\Lambda \omega^{\alpha,\beta}(x)v^2(x)dx\bigg)^{\frac{1}{2}}.$$

Furthermore, we define

$$H^m_{\omega^{(\alpha,\beta)}}(\Lambda) = \big\{u : \partial_x^k u \in L^2_{\omega^{(\alpha,\beta)}}(\Lambda),\ 0 \le k \le m\big\},$$

equipped with the norm

$$\|u\|_{H^m_{\omega^{(\alpha,\beta)}}(\Lambda)} = \left(\sum_{k=0}^m \left\|\frac{d^k u}{dx^k}\right\|^2_{\omega^{(\alpha,\beta)}}\right)^{\frac{1}{2}},$$

where $\partial_x^k v = d^k v/dx^k$.

Consider the $L^2_{\omega^{(\alpha,-\beta)}}$-orthogonal projection upon ${}^-\mathcal{F}_N^{(\alpha,-\beta)}(\Lambda)$, defined by

$$\big({}^-\pi_N^{(\alpha,-\beta)}u - u, v_N\big)_{\omega^{(\alpha,-\beta)}} = 0, \quad \forall\, v_N \in {}^-\mathcal{F}_N^{(\alpha,-\beta)}(\Lambda). \tag{4.1}$$

To characterize the regularity of u, we introduce two non-uniformly weighted spaces involving fractional derivatives

$${}^-\mathcal{B}^m_{\alpha,\beta}(\Lambda) := \big\{u \in L^2_{\omega^{(\alpha,-\beta)}}(\Lambda) : {}_{-1}D_x^{\beta+l}u \in L^2_{\omega^{(\alpha+\beta+l,l)}}(\Lambda)\big\},$$
$${}^-\bar{\mathcal{B}}^m_{\alpha,\beta}(\Lambda) := \big\{u \in L^2_{\omega^{(\alpha,-\beta)}}(\Lambda) : {}_{-1}D_x^{2\beta+l}u \in L^2_{\omega^{(\alpha+\beta+l,l)}}(\Lambda)\big\},$$

where $0 \le l \le m$ and $m \in \mathbb{N}_0$. We define space

$$\mathcal{C}^m_{\alpha,\beta}(\Lambda) := H^m_{\omega^{(\alpha,-\beta)}}(\Lambda) \cap {}^-\mathcal{B}^m_{\alpha,\beta}(\Lambda) \cap {}^-\bar{\mathcal{B}}^m_{\alpha,\beta}(\Lambda).$$

Lemma 4.1 ([6]). *Let $\alpha > -1$, $\beta > 0$ for any $u \in {}^-\mathcal{B}^m_{\alpha,\beta}(\Lambda)$ with integer $0 \le m \le N$, we have*

$$\big\|{}^-\pi_N^{(\alpha,-\beta)}u - u\big\|_{\omega^{(\alpha,-\beta)}} \le cN^{-(\beta+m)}\|D_-^{\beta+m}u\|_{\omega^{(\alpha+\beta+m,m)}}.$$

Lemma 4.2 ([4, 33]). *Suppose that $u \in H^m_{\omega^{(\alpha,\beta)}}(\Lambda)$ and $m \ge 1$,*

$$\|u - \pi_N u\|_\infty \le CN^{\frac{3}{4}-m}|u|_{H^{m;N}_{\omega^{(\alpha,\beta)}}(\Lambda)},$$
$$\|u - \Pi_N u\|_{\omega^{\alpha,\beta}} \le CN^{-m}|u|_{H^{m;N}_{\omega^{\alpha,\beta}}(\Lambda)},$$

where $|u|_{H^{m,N}_{\omega^{(\alpha,\beta)}}(\Lambda)}$ *denotes the seminorm defined by*

$$|u|_{H^{m;N}_{\omega^{(\alpha,\beta)}}(\Lambda)} = \left(\sum_{k=\min(m,N+1)}^{m} \left\| \frac{d^k u}{dx^k} \right\|^2_{\omega^{(\alpha,\beta)}} \right)^{\frac{1}{2}},$$

note that whenever $N \geq m-1$, *one has*

$$|u|_{H^{m;N}_{\omega^{(\alpha,\beta)}}(\Lambda)} = \|u^{(m)}\|_{L^2_{\omega^{(\alpha,\beta)}}(\Lambda)} = |u|_{H^m_{\omega^{(\alpha,\beta)}}(\Lambda)}.$$

Lemma 4.3 ([12]). *Suppose that* $u \in L^2_{\omega^{(\alpha,\beta)}}(\Lambda)$, *then*

$$\|\pi_N u\|_{\omega^{(\alpha,\beta)}} \leq C\|u\|_{\omega^{(\alpha,\beta)}}, \quad \|\pi_N u\|_\infty \leq C\|u\|_\infty.$$

Next, we will introduce the Hölder space. Set $m \geq 0$ and $\varrho \in (0,1), C^{m,\varrho}(\Lambda)$ consists of the m-times continuously differentiable function u and whose m-th derivatives are Hölder continuous with exponent ϱ. The norm is defined as follows:

$$\|u\|_{m,\varrho} = \sum_{k=0}^{m} \max_{z\in\Lambda} \left\|\partial_z^k u(z)\right\| + \sup_{\substack{z_1,z_2\in\Lambda \\ z_1\neq z_2}} \frac{\left\|\partial_z^m u(z_1) - \partial_z^m u(z_2)\right\|}{\|z_1 - z_2\|^\varrho},$$

and if $\varrho = 0$, then $C^{m,0}(\Lambda)$ represents the space of the m-times continuously derivative functions on Λ, it is also generally indicated by $C^m(\Lambda)$, and with norm $\|\cdot\|_m$.

Lemma 4.4 ([29]). *For a nonnegative integer* r *and* $\kappa \in (0,1)$, *there exists a constant* $C_{r,\kappa} > 0$ *such that for any function* $v \in C^{r,\kappa}([-1,1])$, *there exists a polynomial function* $\mathcal{T}_N v \in \mathcal{P}_N$ *such that*

$$\|v - \mathcal{T}_N v\|_\infty \leq C_{r,\kappa} N^{-(r+\kappa)} \|v\|_{r,\kappa},$$

where $\mathcal{T}_N$ *is a linear operator from* $C^{r,\kappa}([-1,1])$ *into* $\mathcal{P}_N$, *as stated in* [29].

Lemma 4.5 ([10]). *Let* $\kappa \in (0,1)$ *and let* $\mathcal{M}$ *be defined by*

$$(\mathcal{M}v)(x) = \int_{-1}^{x} (x-\tau)^{-\mu} K(x,\tau) v(\tau) d\tau.$$

Then, for any function $v \in C([-1,1])$, *there exists a positive constant* C *such that*

$$\frac{\|\mathcal{M}v(x') - {}^{(\mu)}\mathcal{M}v(x'')\|}{\|x' - x''\|} \leq C \max_{x\in[-1,1]} |v(x)|,$$

under the assumption that $0 < \kappa < 1-\mu$, *for any* $x', x'' \in [-1,1]$ *and* $x' \neq x''$. *This implies that*

$$\|\mathcal{M}v\|_{0,\kappa} \leq C \max_{x\in[-1,1]} |v(x)|, \quad 0 < \kappa < 1-\mu.$$

Lemma 4.6 (Grönwall Inequality). *Suppose* $L \geq 0, 0 < \mu < 1$, *and* u *and* v *are a nonnegative, locally integrable functions defined on* $[-1,1]$ *satisfying*

$$u(x) \leq v(x) + L \int_{-1}^{x} (x-\tau)^{-\mu} u(\tau) d\tau.$$

Then there exists a constant $C = C(\mu)$ such that

$$u(x) \le v(x) + CL \int_{-1}^{x} (x-\tau)^{-\mu} v(\tau) d\tau, \quad -1 \le x \le 1.$$

If a nonnegative integrable function $E(x)$ satisfies

$$E(x) \le L \int_{-1}^{x} E(s) ds + J(x), \quad -1 < x \le 1,$$

where $J(x)$ is an integrable function, then

$$\begin{aligned} &\|E\|_{L^\infty(\Lambda)} \le C \|J\|_{L^\infty(\Lambda)}, \\ &\|E\|_{L^p_{\omega^{(\alpha,\beta)}}(\Lambda)} \le C \|J\|_{L^p_{\omega^{(\alpha,\beta)}}(\Lambda)}, \quad p \ge 1. \end{aligned}$$

Here and below, C denotes a positive constant which is independent of N.

5. Convergence Analysis

In order to proceed smoothly in this part of the convergence analysis, we first introduce the relational expression about $u(x)$. Since $u(-1) = 0$, we have

$$ {}_{-1}I_x^{\mu}\big({}_{-1}D_x^{\mu} u(x)\big) = u(x). \tag{5.1}$$

Then $u(x)$ can be expressed as

$$u(x) = \frac{1}{\Gamma(\mu)} \int_{-1}^{x} (x-s)^{\mu-1} {}_{-1}D_x^{\mu} u(s) ds. \tag{5.2}$$

Similarly, we define a linear integral operator $\overline{\mathcal{V}} : C(\Lambda) \to C(\Lambda)$ by

$$(\overline{\mathcal{V}}\psi)(x) := \frac{1}{\Gamma(\mu)} \int_{-1}^{x} (x-s)^{\mu-1} \psi(s) ds.$$

Therefore, the problem (3.1) reads: Find $u = u(x)$ and $D^{\mu} u = D^{\mu} u(x)$ such that

$$\begin{aligned} &D^{\mu} u(x) = \epsilon u(x) + (\mathcal{V}u)(x) + G(x), \\ &u(x) = (\overline{\mathcal{V}} D^{\mu} u)(x). \end{aligned} \tag{5.3}$$

At this point, the problem (3.2) can be rewritten as: Find $u_N \in {}^{-}\mathcal{F}^{(\alpha,-\beta)}(\Lambda)$ such that

$$\begin{aligned} &\big(D^{\mu} u_N, v_N\big)_{\omega^{(-\mu,\mu)}} = (\epsilon u_N + \mathcal{V} u_N + G, v_N)_{\omega^{(-\mu,\mu)}}, \\ &(u_N, v_N)_{\omega^{(-\mu,\mu)}} = \big(\overline{\mathcal{V}} D^{\mu} u_N, v_N\big)_{\omega^{(-\mu,\mu)}}, \quad \forall\, v_N \in {}^{-}\mathcal{F}^{(\alpha,-\beta)}(\Lambda). \end{aligned} \tag{5.4}$$

Let $u_N^{\mu} = D^{\mu} u_N$, according to (5.4) and the definition of the projection operator ${}^{-}\pi_N^{(\mu,-\mu)}$, we obtain

$$\begin{aligned} &u_N^{\mu} = \epsilon u_N + {}^{-}\pi_N^{(\mu,-\mu)} \mathcal{V} u_N + {}^{-}\pi_N^{(\mu,-\mu)} G, \\ &u_N = {}^{-}\pi_N^{(\mu,-\mu)} \overline{\mathcal{V}} u_N^{\mu}. \end{aligned} \tag{5.5}$$

Theorem 5.1. *Suppose that u_N is the generalized Jacobi spectral Galerkin solution determined by (5.4), if the solution u of (3.1) satisfies $u \in \mathcal{C}^m_{\mu,\mu}(\Lambda)$, then we have the following error estimates:*

$$\begin{aligned}
&\|u-u_N\|_{\omega^{(\mu,-\mu)}}\\
&\le cN^{-(\mu+m)}\Big(\big\|{}_{-1}D_x^{\mu+m}u\big\|_{\omega^{(2\mu+m,m)}}+\big\|{}_{-1}D_x^{2\mu+m}u\big\|_{\omega^{(2\mu+m,m)}}\Big)\\
&\quad+CN^{\frac{3}{4}-m-\kappa}\Big(|u|_{H^{m;N}_{\omega^{(\mu,-\mu)}}(\Lambda)}+\big\|{}_{-1}D_x^{\mu}u\big\|_{H^{m;N}_{\omega^{(\mu,-\mu)}}(\Lambda)}\Big),\\
&\big\|D^\mu u-u_N^\mu\big\|_{\omega^{(\mu,-\mu)}}\\
&\le cN^{-(\mu+m)}\Big(\big\|{}_{-1}D_x^{\mu+m}u\big\|_{\omega^{(2\mu+m,m)}}+\big\|{}_{-1}D_x^{2\mu+m}u\big\|_{\omega^{(2\mu+m,m)}}\Big)\\
&\quad+CN^{\frac{3}{4}-m-\kappa}\Big(|u|_{H^{m;N}_{\omega^{(\mu,-\mu)}}(\Lambda)}+\big\|{}_{-1}D_x^{\mu}u\big\|_{H^{m;N}_{\omega^{(\mu,-\mu)}}(\Lambda)}\Big),
\end{aligned}\tag{5.6}$$

where $\kappa \in (0,\mu)$.

Proof. Let $\pi_N = {}^{-}\pi_N^{(\mu,-\mu)}$, $e = u-u_N$, $e^\mu = D^\mu u - u_N^\mu$, the combination of (5.3) and (5.5) leads to

$$\begin{aligned}
&D^\mu u-u_N^\mu=\epsilon(u-u_N)+\mathcal{V}u-\pi_N\mathcal{V}u_N+G-\pi_N G,\\
&u-u_N=\overline{\mathcal{V}}D^\mu u-\overline{\mathcal{V}}u_N^\mu.
\end{aligned}\tag{5.7}$$

We can directly calculate

$$\begin{aligned}
&\mathcal{V}u-\pi_N\mathcal{V}u_N\\
&=\mathcal{V}u-\pi_N\mathcal{V}u+\pi_N\mathcal{V}(u-u_N)\\
&=\mathcal{V}u-\pi_N\mathcal{V}u+\mathcal{V}(u-u_N)+[\pi_N\mathcal{V}(u-u_N)-\mathcal{V}(u-u_N)]\\
&=(D^\mu u-\epsilon u-G)-\pi_N(D^\mu u-\epsilon u-G)+\mathcal{V}(u-u_N)\\
&\quad+[\pi_N\mathcal{V}(u-u_N)-\mathcal{V}(u-u_N)]\\
&=(D^\mu u-\epsilon u-G)-\pi_N(D^\mu u-\epsilon u-G)+\mathcal{V}e+(\pi_N\mathcal{V}e-\mathcal{V}e).
\end{aligned}\tag{5.8}$$

On the other hand, we can similarly get

$$\begin{aligned}
&\overline{\mathcal{V}}D^\mu u-\pi_N\overline{\mathcal{V}}u_N^\mu\\
&=\overline{\mathcal{V}}D^\mu u-\pi_N\overline{\mathcal{V}}D^\mu u+\pi_N\overline{\mathcal{V}}\big(D^\mu u-u_N^\mu\big)\\
&=\overline{\mathcal{V}}D^\mu u-\pi_N\overline{\mathcal{V}}D^\mu u+\overline{\mathcal{V}}\big(D^\mu u-u_N^\mu\big)\\
&\quad+\big[\pi_N\overline{\mathcal{V}}\big(D^\mu u-u_N^\mu\big)-\overline{\mathcal{V}}\big(D^\mu u-u_N^\mu\big)\big]\\
&=u-\pi_N u+\overline{\mathcal{V}}e^\mu+\big(\pi_N\overline{\mathcal{V}}e^\mu-\overline{\mathcal{V}}e^\mu\big).
\end{aligned}\tag{5.9}$$

Substituting (5.8) and (5.9) into (5.7), we have

$$\begin{aligned}
e^\mu(x)&=\epsilon e(x)+\int_{-1}^{x}k(x,s)e(s)ds+D^\mu u-\pi_N D^\mu u+\epsilon(\pi_N u-u)+(\pi_N\mathcal{V}e-\mathcal{V}e)\\
&=\epsilon e(x)+\int_{-1}^{x}k(x,s)e(s)ds+I_1-\epsilon I_2+I_3,\\
e(x)&=\frac{1}{\Gamma(\mu)}\int_{-1}^{x}(x-s)^{\mu-1}e^\mu(s)ds+I_2+I_4,
\end{aligned}\tag{5.10}$$

where

$$I_1=D^\mu u-\pi_N D^\mu u,\quad I_2=u-\pi_N u,\quad I_3=\pi_N\mathcal{V}e-\mathcal{V}e,\quad I_4=\pi_N\overline{\mathcal{V}}e^\mu-\overline{\mathcal{V}}e^\mu.$$

In order to be able to directly establish the relationship between $e^{\mu}(x)$ and $e(x)$, we use the Dirichlet's formula which states that

$$\int_{-1}^{x}\int_{-1}^{\tau}\Phi(\tau,s)dsd\tau=\int_{-1}^{x}\int_{s}^{x}\Phi(\tau,s)d\tau ds. \tag{5.11}$$

Then, from (5.10) we can get

$$\begin{aligned}e^{\mu}(x)&=\epsilon e(x)+\int_{-1}^{x}\bigg(\int_{\tau}^{x}\frac{1}{\Gamma(\mu)}k(x,s)ds\bigg)(x-\tau)^{\mu-1}e^{\mu}(\tau)d\tau\\&\quad+\int_{-1}^{x}k(x,s)\big(I_2(s)+I_4(s)\big)ds+I_1(x)-\epsilon I_2(x)+I_3(x)\\&\le\epsilon|e(x)|+C\int_{-1}^{x}(x-\tau)^{\mu-1}\|e^{\mu}\|(\tau)d\tau+\|I_1(x)\|+C\|I_2(x)\|\\&\quad+\|I_3(x)\|+C\|I_4(x)\|.\end{aligned} \tag{5.12}$$

In terms of Grönwall inequality and (5.12), we obtain

$$\|e^{\mu}(x)\|_{\infty}\le C\bigg(\|e(x)\|_{\infty}+\sum_{i=1}^{4}\|I_i\|_{\infty}\bigg). \tag{5.13}$$

On the other hand, combining Lemmas 4.4, 4.5 and (5.10), we have

$$\begin{aligned}\|e(x)\|_{\infty}&\le C\|\mathcal{M}e^{\mu}\|_{\infty}+\|I_2\|_{\infty}+\|I_4\|_{\infty}\\&=C\big\|\big(\mathcal{M}e^{\mu}-\mathcal{T}_N{}^{(1-\mu)}\mathcal{M}e^{\mu}\big)\big\|_{\infty}+\|I_2\|_{\infty}+\|I_4\|_{\infty}\\&\le CN^{-\iota}\|\mathcal{M}e^{\mu}\|_{0,\iota}+\|I_2\|_{\infty}+\|I_4\|_{\infty}\\&\le CN^{-\iota}\|e^{\mu}\|_{\infty}+\|I_2\|_{\infty}+\|I_4\|_{\infty},\quad \iota\in(0,\mu).\end{aligned} \tag{5.14}$$

According to (5.13) and (5.14), we can get

$$\begin{aligned}\|e^{\mu}(x)\|_{\infty}&\le C\sum_{i=1}^{4}\|I_i\|_{\infty},\\\|e(x)\|_{\infty}&\le C\sum_{i=1}^{4}\|I_i\|_{\infty}.\end{aligned} \tag{5.15}$$

According to Lemma 4.2, we can get error estimates for I_1, I_2 and I_3 as

$$\begin{aligned}&\|I_1\|_{\infty}\le CN^{\frac{3}{4}-m}\|D^{\mu}u\|_{H^{m;N}_{\omega^{(\mu,-\mu)}}(\Lambda)},\\&\|I_2\|_{\infty}\le CN^{\frac{3}{4}-m}|u|_{H^{m;N}_{\omega^{(\mu,-\mu)}}(\Lambda)},\\&\|I_3\|_{\infty}\le CN^{-\frac{1}{4}}\bigg\|\int_{-1}^{x}k(x,s)e(s)ds\bigg\|_{H^{1;N}_{\omega^{(\mu,-\mu)}}(\Lambda)}\le CN^{-\frac{1}{4}}\|e\|_{\infty}.\end{aligned} \tag{5.16}$$

On the other hand, by Lemmas 4.3-4.5,

$$\begin{aligned}\|I_4\|_{\infty}&=\big\|(\pi_N-I)\mathcal{M}e^{\mu}\big\|_{\infty}\\&=\big\|(\pi_N-I)\big(\mathcal{M}e^{\mu}-\mathcal{T}_N\mathcal{M}e^{\mu}\big)\big\|_{\infty}\end{aligned}$$

$$\begin{aligned}
&\le \big\|\pi_N\big(\mathcal{M}e^\mu - \mathcal{T}_N\mathcal{M}e^\mu\big)\big\|_\infty + \big\|\mathcal{M}e^\mu - \mathcal{T}_N\mathcal{M}e^\mu\big\|_\infty\\
&\le C\big\|\mathcal{M}e^\mu - \mathcal{T}_N\mathcal{M}e^\mu\big\|_\infty\\
&\le CN^{-\kappa}\|\mathcal{M}e^\mu\|_{0,\kappa}\\
&\le CN^{-\kappa}\|e^\mu\|_\infty, \quad \kappa\in(0,\mu).
\end{aligned} \tag{5.17}$$

Together with (5.15)-(5.17), when N is large enough, we obtain

$$\begin{aligned}
&\|u-u_N\|_\infty \le CN^{\frac{3}{4}-m}\Big(|u|_{H^{m;N}_{\omega^{(\mu,-\mu)}}(\Lambda)} + \|D^\mu u\|_{H^{m;N}_{\omega^{(\mu,-\mu)}}(\Lambda)}\Big),\\
&\big\|D^\mu u - u_N^\mu\big\|_\infty \le CN^{\frac{3}{4}-m}\Big(|u|_{H^{m;N}_{\omega^{(\mu,-\mu)}}(\Lambda)} + \|D^\mu u\|_{H^{m;N}_{\omega^{(\mu,-\mu)}}(\Lambda)}\Big).
\end{aligned} \tag{5.18}$$

Now we investigate the $\|\cdot\|_{\omega^{(\mu,-\mu)}}$ -error estimates. It follows from (5.12) and Lemma 4.6 that

$$\begin{aligned}
\|e^\mu(x)\|_{\omega^{(\mu,-\mu)}} &\le C\sum_{i=1}^4 \|I_i\|_{\omega^{(\mu,-\mu)}},\\
\|e(x)\|_{\omega^{(\mu,-\mu)}} &\le C\sum_{i=1}^4 \|I_i\|_{\omega^{(\mu,-\mu)}}.
\end{aligned} \tag{5.19}$$

Due to Lemma 4.1, we have

$$\begin{aligned}
\|I_1\|_{\omega^{(\mu,-\mu)}} &\le cN^{-(\mu+m)}\big\|{}_{-1}D_x^{\mu+m}\big({}_{-1}D_x^\mu u\big)\big\|_{\omega^{(2\mu+m,m)}},\\
\|I_2\|_{\omega^{(\mu,-\mu)}} &\le cN^{-(\mu+m)}\big\|{}_{-1}D_x^{\mu+m}u\big\|_{\omega^{(2\mu+m,m)}}.
\end{aligned} \tag{5.20}$$

Because

$$u(-1)=0, \quad {}_{-1}D_x^{\mu+m}\big({}_{-1}D_x^\mu u\big) = {}_{-1}D_x^{2\mu+m}u$$

in the above formula holds. Therefore,

$$\|I_1\|_{\omega^{(\mu,-\mu)}} \le cN^{-(\mu+m)}\big\|{}_{-1}D_x^{2\mu+m}u\big\|_{\omega^{(2\mu+m,m)}}. \tag{5.21}$$

By Lemma 4.2, we can get the $\|\cdot\|_{\omega^{(\mu,-\mu)}}$-error estimate for I_3.

$$\|I_3\|_{\omega^{\alpha,\beta}} \le CN^{-1}\Big\|\int_{-1}^x k(x,s)e(x)\Big\|_{H^{1,N}_{\omega^{\alpha,\beta}}(\Lambda)} \le CN^{-1}\|e\|_\infty. \tag{5.22}$$

We find from Lemmas 4.3-4.5 that

$$\begin{aligned}
\|I_4\|_{\omega^{(\mu,-\mu)}} &= \big\|(\pi_N - I)\mathcal{M}e^\mu\big\|_{\omega^{(\mu,-\mu)}} = \big\|(\pi_N - I)\big(\mathcal{M}e^\mu - \mathcal{T}_N\mathcal{M}e^\mu\big)\big\|_{\omega^{(\mu,-\mu)}}\\
&\le \big\|\pi_N\big(\mathcal{M}e^\mu - \mathcal{T}_N\mathcal{M}e^\mu\big)\big\|_{\omega^{(\mu,-\mu)}} + \big\|\mathcal{M}e^\mu - \mathcal{T}_N\mathcal{M}e^\mu\big\|_{\omega^{(\mu,-\mu)}}\\
&\le C\big\|\mathcal{M}e^\mu - \mathcal{T}_N\mathcal{M}e^\mu\big\|_\infty\\
&\le CN^{-\kappa}\|\mathcal{M}e^\mu\|_{0,\kappa} \le CN^{-\kappa}\|e^\mu\|_\infty, \quad \kappa\in(0,\mu).
\end{aligned} \tag{5.23}$$

The combination of (5.18)-(5.23) yields,

$$\begin{aligned}
&\|u-u_N\|_{\omega^{(\mu,-\mu)}}\\
&\le cN^{-(\mu+m)}\Big(\big\|{}_{-1}D_x^{\mu+m}u\big\|_{\omega^{(2\mu+m,m)}} + \big\|{}_{-1}D_x^{2\mu+m}u\big\|_{\omega^{(2\mu+m,m)}}\Big)\\
&\quad + CN^{\frac{3}{4}-m-\kappa}\Big(|u|_{H^{m;N}_{\omega^{(\mu,-\mu)}}(\Lambda)} + \big\|{}_{-1}D_x^\mu u\big\|_{H^{m;N}_{\omega^{(\mu,-\mu)}}(\Lambda)}\Big),
\end{aligned} \tag{5.24a}$$

$$
\begin{aligned}
&\left\|D^{\mu}u-u_N^{\mu}\right\|_{\omega^{(\mu,-\mu)}}\\
&\leq cN^{-(\mu+m)}\Big(\left\|{}_{-1}D_x^{\mu+m}u\right\|_{\omega^{(2\mu+m,m)}}+\left\|{}_{-1}D_x^{2\mu+m}u\right\|_{\omega^{(2\mu+m,m)}}\Big)\\
&\quad+CN^{\frac{3}{4}-m-\kappa}\Big(|u|_{H^{m;N}_{\omega^{(\mu,-\mu)}}(\Lambda)}+\left\|{}_{-1}D_x^{\mu}u\right\|_{H^{m;N}_{\omega^{(\mu,-\mu)}}(\Lambda)}\Big).
\end{aligned}
\tag{5.24b}
$$

provided N is large enough. Hence, the Theorem 5.1 is proved. □

6. Numerical Experiments

We provide some numerical examples to back up our analysis. To measure the efficiency of the results, $\|\cdot\|_\infty$ and $\|\cdot\|_\omega$ errors are used to assess the efficiency of the method. Matlab is used to perform all of the calculations.

Example 6.1. We consider the following the fractional-order Volterra integro-differential equation:

$$
\begin{cases}
{}_0D_t^{\mu}y(t)=y(t)+\displaystyle\int_0^t tsy(s)ds+g(t), \quad \mu\in(0,1),\\
y(0)=0,
\end{cases}
\tag{6.1}
$$

where $T=1$ and the exact solution is $y(t)=t^6+t^8+t^{10}$.

In Fig. 6.1 (left), we plot the discrete L^2_ω errors and the maximum errors of (6.1) when $\mu=0.5$. It is shown that the numerical errors decay exponentially as N increases. Fig. 6.1 (right) illustrates the numerical result of the generalized Jacobi spectral Galerkin method approximation solution for $N=24$ and exact solution of (6.1) when $\mu=0.5$, which are found in excellent agreement.

Table 6.1 lists the L^2_ω error of Example 6.1 when $\mu=0.1,0.5,0.9$. It is shown that the generalized Jacobi spectral Galerkin method has achieved higher accuracy when $N=6$, and as N increases, L^2_ω error of the numerical solution decreases rapidly. The numerical results show the high accuracy and effectiveness of the generalized Jacobi spectral Galerkin method.

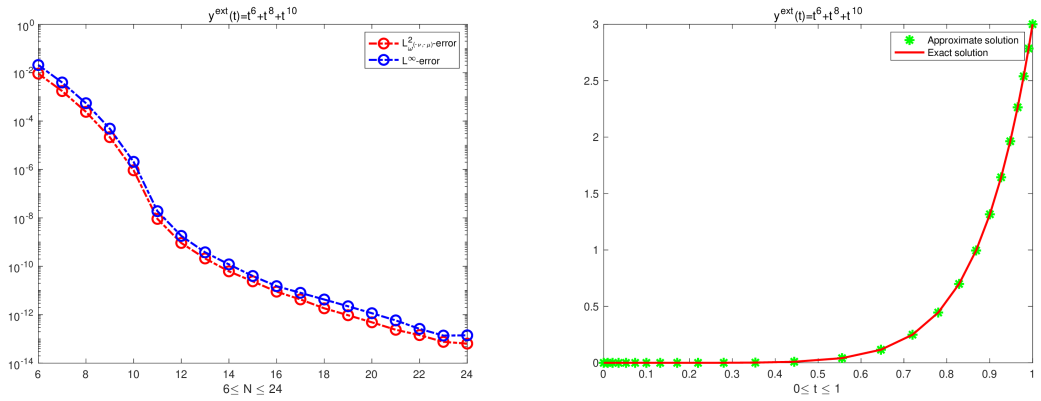

Fig. 6.1. Convergence of the generalized Jacobi spectral Galerkin method for Example 6.1 when $\mu=0.5$ (left). Comparison between approximate solution and exact solution of $y(t)$ with $\mu=0.5$ (right).

Table 6.1: When $0 < \mu < 1$, the $\|\cdot\|_\omega$ errors of Example 6.1.

N	$\mu = 0.1$	$\mu = 0.5$	$\mu = 0.9$
6	1.3144×10^{-2}	6.9936×10^{-3}	4.3049×10^{-3}
8	3.6636×10^{-4}	1.7385×10^{-4}	9.3401×10^{-5}
10	1.6350×10^{-6}	5.6658×10^{-7}	2.0503×10^{-7}
12	4.6222×10^{-10}	1.0834×10^{-9}	9.1720×10^{-10}
14	2.9136×10^{-11}	7.6845×10^{-11}	7.2640×10^{-11}
16	4.2183×10^{-12}	1.1113×10^{-11}	1.0711×10^{-11}
18	8.6186×10^{-13}	2.2826×10^{-12}	2.2409×10^{-12}
20	2.4637×10^{-13}	6.2234×10^{-13}	6.3388×10^{-13}
22	1.3758×10^{-13}	1.8789×10^{-13}	1.8534×10^{-13}
24	1.2403×10^{-13}	6.3452×10^{-14}	5.3382×10^{-14}

Example 6.2. We consider the following the fractional-order Volterra integro-differential equation:

$$\begin{cases} {}_0D_t^{0.3} y(t) = y(t) + \displaystyle\int_0^t t s^{\frac{2}{3}} y(s) ds + g(t), \\ y(0) = 0, \end{cases}$$

where $T = 1$, the exact solution $y(t) = t^{5+1/3} + t^{8+1/3} + t^{11+1/3}$. In addition, the functional formula of $g(t)$ is

$$\begin{aligned} g(t) = {} & \frac{\Gamma(6+1/3)}{\Gamma(6)} t^5 + \frac{\Gamma(9+1/3)}{\Gamma(9)} t^8 + \frac{\Gamma(12+1/3)}{\Gamma(12)} t^{11} \\ & - \frac{1}{7} t^8 - \frac{1}{10} t^{11} - \frac{1}{13} t^{14} - t^{5+\frac{1}{3}} - t^{8+\frac{1}{3}} - t^{11+\frac{1}{3}}. \end{aligned}$$

We implement the numerical scheme (3.4) based on the generalized Jacobi spectral Galerkin method to solve this example. The L^∞ and L^2_ω errors of the generalized Jacobi spectral Galerkin method are demonstrated Fig. 6.2 (left). We observe that the convergence rates are consistent with our error estimates.

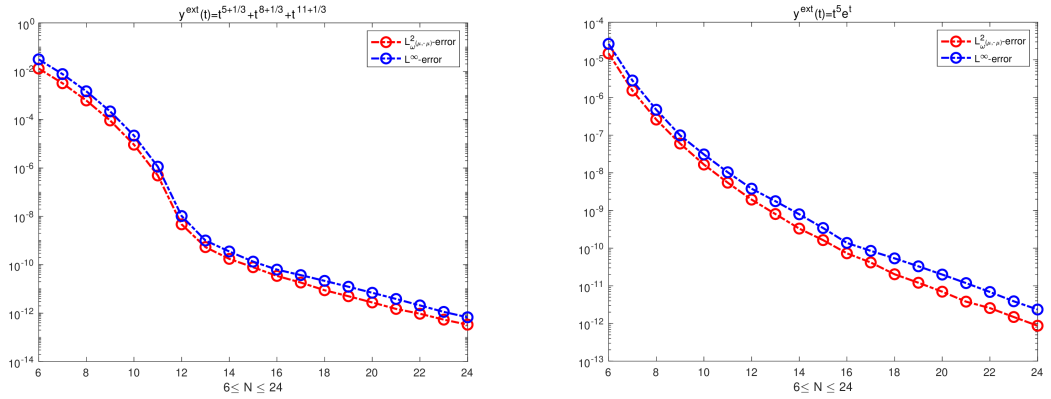

Fig. 6.2. Convergence of the generalized Jacobi spectral Galerkin method for the FVIDEs. Left: (Example 6.2) $y(t) = t^{5+1/3} + t^{8+1/3} + t^{11+1/3}$. Right: (Example 6.3) $y(t) = t^5 e^t$.

Example 6.3. We consider the following the fractional-order Volterra integro-differential equation:

$$\begin{cases} {}_0D_t^{0.5}y(t) = y(t) + \displaystyle\int_0^t e^t y(s)ds + g(t), \quad t \in [0,3], \\ y(0) = 0. \end{cases}$$

Next, we take the exact solution $y(t) = t^5e^t$ from Example 6.3. The numerical errors for this example are plotted in Fig. 6.2 (right). We again see that the observed convergence rate agrees with the expected rate.

7. Conclusion

In this paper, we proposed a generalized Jacobi spectral Galerkin method for fractional-order Volterra integro-differential equations involving both Caputo and Riemann-Liouville fractional derivatives. This method can be implemented efficiently. We derived the error estimates for the spectral Galerkin method. Numerical experiments demonstrated that the proposed method is capable of proving very accurate results despite the solution singularity.

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References

[1] R.L. Bagley and P. Torvik, A theoretical basis for the application of fractional calculus to viscoelasticity, *J. Rheol.*, **27**:3 (1983), 201–210.

[2] B. Basirat and M.A. Shahdadi, Numerical solution of nonlinear integro-differential equations with initial conditions by Bernstein operational matrix of derivative, *Int'l J. of Modern Nonlinear Theory and Application*, **2** (2013), 141–149.

[3] M. Bayram, V.F. Hatipoğlu, S. Alkan, and S.E. Das, A solution method for integro-differential equations of conformable fractional derivative, *Therm. Sci.*, **22** (2017), 7–14.

[4] C. Canuto, M.Y. Hussaini, A. Quarteroni, and T.A. Zang, *Spectral methods: Fundamentals in Single Domains*, Springer, 2007.

[5] M. Caputo, Linear models of dissipation whose Q is almost frequency independent-II, *Geophys. J. Int.*, **13**:5 (1967), 529–539.

[6] S. Chen, J. Shen, and L.L. Wang, Generalized Jacobi functions and their applications to fractional differential equations, *Math. Comp.*, **85**:300 (2016), 1603–1638.

[7] Y. Chen and T. Tang, Spectral methods for weakly singular Volterra integral equations with smooth solutions, *J. Comput. Appl. Math.*, **233**:4 (2009), 938–950.

[8] Y. Chen and T. Tang, Convergence analysis of the Jacobi spectral-collocation methods for Volterra integral equations with a weakly singular kernel, *Math. Comp.*, **79**:269 (2010), 147–167.

[9] Z. Chen, Y. Chen, and Y. Huang, Piecewise spectral collocation method for second order Volterra integro-differential equations with nonvanishing delay, *Adv. Appl. Math. Mech.*, **14**:6 (2022), 1333–1356.

[10] D.L. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, in: *Applied Mathematical Sciences*, Vol. 93, Springer, 1998.

[11] K. Diethelm, *The Analysis of Fractional Differential Equations*, in: *Lecture Notes in Mathematics*, Vol. 2004, Springer, 2010.

[12] J. Douglas, T. Dupont, and L. Wahlbin, The stability in l^q of the l^2-projection into finite element function spaces, *Numer. Math.*, **23**:3 (1974), 193–197.

[13] M. Heydari, M. Hooshmandasl, C. Cattani, and M. Li, Legendre wavelets method for solving fractional population growth model in a closed system, *Math. Probl. Eng.*, **2013** (2013), 161030.

[14] C. Huang, T. Tang, and Z. Zhang, Supergeometric convergence of spectral collocation methods for weakly singular Volterra and Fredholm integral equations with smooth solutions, *J. Comput. Math.*, **29** (2011), 698–719.

[15] H. Jaradat, F. Awawdeh, and E. Rawashdeh, Analytic solution of fractional integro-differential equations, *Annals of the University of Craiova – Mathematics and Computer Science Series*, **38**:1 (2011), 1–10.

[16] E. Kharazmi, M. Zayernouri, and G.E. Karniadakis, Petrov-Galerkin and spectral collocation methods for distributed order differential equations, *SIAM J. Sci. Comput.*, **39**:3 (2017), A1003–A1037.

[17] A. Kilbas, *Theory and Applications of Fractional Differential Equations*, in: *North-Holland Mathematics Studies*, Vol. 204, Elsevier, 2006.

[18] X. Li, T. Tang, and C. Xu, Parallel in time algorithm with spectral-subdomain enhancement for Volterra integral equations, *SIAM J. Numer. Anal.*, **51**:3 (2013), 1735–1756.

[19] T. Lin, Y. Lin, M. Rao, and S. Zhang, Petrov-Galerkin methods for linear Volterra integro-differential equations, *SIAM J. Numer. Anal.*, **38**:3 (2000), 937–963.

[20] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*, World Scientific, 2010.

[21] D.S. Mohammed, Numerical solution of fractional integro-differential equations by least squares method and shifted Chebyshev polynomial, *Math. Probl. Eng.*, **2014** (2014), 31965.

[22] S. Momani and R. Qaralleh, An efficient method for solving systems of fractional integro-differential equations, *Comput. Math. Appl.*, **52**:3-4 (2006), 459–470.

[23] D. Nazari and S. Shahmorad, Application of the fractional differential transform method to fractional-order integro-differential equations with nonlocal boundary conditions, *J. Comput. Appl. Math.*, **234**:3 (2010), 883–891.

[24] J.W. Nunziato, On heat conduction in materials with memory, *Quart. Appl. Math.*, **29**:2 (1971), 187–204.

[25] K. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, 1974.

[26] W. Olmstead and R. Handelsman, Diffusion in a semi-infinite region with nonlinear surface dissipation, *SIAM Rev.*, **18**:2 (1976), 275–291.

[27] I. Podlubny, The Laplace transform method for linear differential equations of the fractional order, *arXiv:funct-an/9710005*, 1997.

[28] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, in: *Mathematics in Science and Engineering*, Vol. 198, Academic Press, 1999.

[29] D.L. Ragozin, Polynomial approximation on compact manifolds and homogeneous spaces, *Trans. Amer. Math. Soc.*, **150**:1 (1970), 41–53.

[30] E. Rawashdeh, Numerical solution of fractional integro-differential equations by collocation method, *Appl. Math. Comput.*, **176**:1 (2006), 1–6.

[31] F.H. Shekarabi, M. Khodabin, and K. Maleknejad, The Petrov-Galerkin method for numerical solution of stochastic Volterra integral equations, *IAENG Int. J. Appl. Math.*, **44**:4 (2014), 170–176.

[32] J. Shen, C. Sheng, and Z. Wang, Generalized Jacobi spectral-Galerkin method for nonlinear Volterra integral equations with weakly singular kernels, *J. Math. Study*, **48**:4 (2015), 315–329.

[33] J. Shen, T. Tang, and L.L. Wang, *Spectral Methods: Algorithms, Analysis and Applications*, in:

Springer Series in Computational Mathematics, Vol. 41, Springer, 2011.

[34] C. Sheng, Z. Wang, and B. Guo, A multistep Legendre-Gauss spectral collocation method for nonlinear Volterra integral equations, *SIAM J. Numer. Anal.*, **52**:4 (2014), 1953–1980.

[35] E. Tohidi, M. Ezadkhah, and S. Shateyi, Numerical solution of nonlinear fractional Volterra integro-differential equations via Bernoulli polynomials, *Abstr. Appl. Anal.*, **2014** (2014), 162896.

[36] Z. Wan, Y. Chen, and Y. Huang, Legendre spectral Galerkin method for second-kind Volterra integral equations, *Front. Math. China*, **4**:1 (2009), 181–193.

[37] Z. Wang and C. Sheng, An *hp*-spectral collocation method for nonlinear Volterra integral equations with vanishing variable delays, *Math. Comp.*, **85**:298 (2016), 635–666.

[38] Z. Xie, X. Li, and T. Tang, Convergence analysis of spectral Galerkin methods for Volterra type integral equations, *J. Sci. Comput.*, **53**:2 (2012), 414–434.

[39] Y. Yang, Jacobi spectral Galerkin methods for fractional integro-differential equations, *Calcolo*, **52**:4 (2015), 519–542.

[40] Y. Yang, Jacobi spectral Galerkin methods for Volterra integral equations with weakly singular kernel, *Bull. Korean Math. Soc.*, **53**:1 (2016), 247–262.

[41] M. Zayernouri and G.E. Karniadakis, Fractional Sturm-Liouville eigen-problems: Theory and numerical approximation, *J. Comput. Phys.*, **252** (2013), 495–517.

[42] L. Zhu and Q. Fan, Numerical solution of nonlinear fractional-order Volterra integro-differential equations by SCW, *Commun. Nonlinear Sci. Numer. Simul.*, **18**:5 (2013), 1203–1213.