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ROTHE METHOD AND NUMERICAL ANALYSIS FOR A SUB-DIFFUSION EQUATION WITH CLARKE SUBDIFFERENTIAL*

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Abstract

This paper is devoted to the study of a sub-diffusion equation involving a Clarke subdifferential boundary condition. It describes transport of particles governed by the anomalous diffusion in media with boundary semipermeability. The weak formulation of the model problem results in a time fractional parabolic hemivariational inequality. We first construct an abstract hemivariational evolutionary inclusion and prove its unique solvability using a time-discretization approximation, known as the Rothe method. In addition, a numerical approach based on a finite difference scheme in time and finite dimensional approximation in space is proposed and analyzed for the abstract problem. These results are then applied to establish the convergence of the numerical solution of the model problem. Under appropriate regularity assumptions, an optimal order error estimate for the linear finite element method is derived. Some numerical examples are provided to support the theoretical results.

Mathematics subject classification: 35S10, 47J20, 49J27, 65M06, 65M12.
Key words: Numerical analysis, Fractional hemivariational inequality, Sub-diffusion equation, Rothe method, Clarke subdifferential.

1. Introduction

Anomalous diffusion has been an active area of research in the physics community since the introduction of continuous time random walks by Montroll and Weiss [45]. One of the main characteristics of anomalous diffusion is the nonlinear behaviour of the mean square displacement, i.e. $\langle x^2(t)\rangle \propto t^\alpha$ ($\alpha < 1$ and $\alpha > 1$ correspond to subdiffusion and superdiffusion, respectively), which is, in general, related to a stochastic process with non-Markovian characteristics. This kind of diffusion process is found in various complex systems, which usually no longer follow Gaussian statistics, and thus Fick's second law fails to describe the related transport behaviour. Examples can be found in various fields ranging from biophysics (e.g. transport of large molecules in living cells [15, 53]), to geophysics and ecology (e.g. tracer diffusion in subsurface hydrology [7]).

Fractional partial differential equations (FPDEs) have been widely regarded as a complementary tool in the description of anomalous transport processes [42]. For the classical theory of fractional derivatives and FPDEs, the reader is referred to [10,48]. In particular, time fractional

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diffusion equations (TFDEs), which is obtained from the standard diffusion equations by replacing the first-order time derivative with a fractional derivative of order α $(0 < \alpha < 1)$, appear to be useful for the mathematical description of anomalous subdiffusion, related to long waiting times between particle jumps [41]. A number of scholars are devoted to the investigation of the analytical solutions to the TFDEs or the multi-term TFDEs [1,11,24–26]. Luchko and Yamamoto [38] obtained some uniqueness and existence results for a general TFDE. Liu *et al.* [36] extended the time fractional diffusion wave equation to a generalized form in the sense of the regularized version of the k-Hilfer-Prabhakar fractional operator involving the k-Mittag function and found a novel and general solution.

From the numerical aspect, there exist a lot of work related to construction of time stepping methods for time-fractional differential equations; see, e.g. [2, 13, 14, 35, 39, 40, 52, 54, 57]. In the existing work, the finite difference approximation of the fractional derivative is the most studied one. This type of time stepping methods makes use of piecewise linear/quadratic interpolation approximation at each subinterval, results in the widely-used L1 scheme [35, 52], L2-1_σ scheme [2], and L2 scheme [40]. Some other work involves the storage reduction and singularity treatment at the starting point. Much efforts have been made in developing fast numerical methods to recover the desired convergence order for solutions of low regularity; see, e.g. [4, 5, 8, 17, 27, 28, 31, 32, 34, 37, 51].

In this paper, we focus on a time fractional diffusion equation involving a Clarke subdifferential boundary condition. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ with $d = 2, 3$ and the boundary $\partial\Omega$ is Lipschitz continuous. Assume $\partial\Omega$ is divided into two disjoint measurable parts Γ_1 and Γ_2 and the measure of Γ_1 is positive. The outward unit normal vector at $\partial\Omega$ is denoted by $\boldsymbol{\nu}$. We are concerned with the evolutionary process on the time interval $[0, T]$ with a given $T > 0$. The pointwise formulation of our model problem is

Problem 1.1. Find $u : \Omega \times (0, T) \to \mathbb{R}$ such that for all $t \in (0, T)$,

$$
\begin{aligned}
&{}_0D_t^\alpha u(x,t) - \Delta u(x,t) = f(x,t) && \text{in } \Omega \times (0,T),\\
&u(x,t) = 0 && \text{on } \Gamma_1 \times (0,T),\\
&-\frac{\partial u(x,t)}{\partial \nu} \in \partial j\big(u(x,t)\big) && \text{on } \Gamma_2 \times (0,T),\\
&u(x,0) = u_0(x) && \text{in } \Omega.
\end{aligned}
$$

Here the unknown function u represents the density of particles and ${}_0D_t^\alpha u(x,t)$ denotes the time Caputo fractional derivative of u introduced in the next section. In addition, ∂j denotes the Clarke subdifferential of a locally Lipschitz continuous function j. It is worth mentioning that the traditional diffusion equation with a Clarke subdifferential boundary condition has been considered in [47]. Han and Wang [23] studied the numerical solution of that model and derived an optimal error estimate of a finite element method.

Clarke subdifferential is a multivalued operator, widely used in fields such as contact mechanics and fluid dynamics to represent nonsmooth relations between variables that include jumps. Roughly speaking, the Clarke subdifferential introduced in Problem 1.1 as the boundary condition of the anomalous diffusion equation is targeted at simulating the process of pollutants in groundwater entering the aquifer through soil. As a porous medium, the complex structure and adsorption characteristics of soil will significantly affect the transport of pollutants. In addition, the difference in physical properties between aquifer and soil will also lead to a nonsmooth relationship between diffusion flux and solute concentration at the boundary. Therefore, compared with classical boundary conditions, the use of Clarke subdifferential can better describe the anomalous diffusion in porous media.

Subdiffusion also occurs in the transport process of substances (such as nutrients, drugs, etc.) in biological cells. Due to the complexity of the cell membrane structure, local chemical reactions and heterogeneity of the cell membrane, the flux of substances may change suddenly, and the relationship between the flux and the concentration of substances is usually nonsmooth. For example, subdiffusion inside a living cell with a semipermeable membrane was investigated in references [16, 30], which can also be modeled with Problem 1.1. Semipermeability problems were first studied by Duvaut and Lions [12] in the context of convexity, i.e. for monotone semipermeability relations. These relations lead to variational inequalities and arise in heat conduction, in electrostatics, in flow problems through porous media, etc. In 1980s, Panagiotopoulos [47] extended the problems by removing monotonicity assumptions. The arising boundary value problems (BVPs) lead to hemivariational inequalities (HVIs) instead of variational inequalities since the potentials are nonconvex. References [43, 47] introduced semipermeability conditions in more detail with some examples.

HVIs, arisen from nonmonotone inclusions involving Clarke subdifferential of a locally Lipschitz potential, require more sophisticated treatment than weak formulations of classical BVPs. The basic theory of HVIs can be found in [19,44,46]. Han *et al.* [20] explored the existence and uniqueness of the solution to a general dynamic history-dependent variational-hemivariational inequality. Zeng *et al.* [55] used the Rothe method to establish the existence of a solution to a class of fractional HVIs and applied the results to a fractional quasistatic contact problem. In addition, there are some researches on numerical schemes and algorithms of HVIs. Bartosz *et al.* [6] established a convergence result of a numerical semidiscrete scheme for a parabolic variational-hemivariational inequality and presented the convergence rate with numerical simulations. They used the primal-dual active set approach to find numerical solutions in every time step. Han *et al.* [22] studied the numerical solution of a general elliptic hemivariational inequality and provided optimal error estimates for various elliptic hemivariational inequalities arising in contact mechanics. Han and Sofonea [21] established a general convergence theory of penalty based numerical methods for elliptic constrained inequality problems, including variational inequalities, hemivariational inequalities, and variational-hemivariational inequalities.

The rest of the paper is structured as follows. In Section 2, we recall some basic notation and present several preliminary results. In Section 3, we introduce an abstract time fractional parabolic hemivariational inequality, which contains the model Problem 1.1 as a special case. The existence and uniqueness of the solution to the abstract problem is established with Rothe method. In Section 4, we propose a fully discrete numerical scheme based on a finite difference method in time and finite dimensional approximation in space. The main goal of this section is to derive a Céa's type inequality for the numerical scheme. In Section 5, the results obtained in Section 3 and Section 4 are applied to prove the unique solvability of the weak formulation of Problem 1.1. Under appropriate regularity assumptions, an optimal order error estimate of the numerical method is derived as well. Numerical examples are presented to verify the theoretically predicted convergence order in Section 6.

2. Notation and Preliminaries

In this section, we recall some basic notations and definitions that will be used in the rest of the paper. Let X be a real Banach space. Everywhere in the paper we will denote by $\|\cdot\|_X$, X^*

and $\langle \cdot\,,\cdot\rangle_{X^*\times X}$ the norm in X, its dual space and the duality pairing of X and X^*, respectively. We start by recalling the definitions and properties of the fractional integral and differential operators, for more details, we refer to [10, 48].

Let $0<\alpha<1, \Lambda=[a,b]$. The left Riemann-Liouville fractional integral operator of order α is defined by

$$ {}_aI_t^\alpha v(t)=\frac{1}{\Gamma(\alpha)}\int_a^t \frac{v(s)ds}{(t-s)^{1-\alpha}}, \quad \forall\, t\in\Lambda. $$

The right Riemann-Liouville fractional integral operator of order α is defined by

$$ {}_tI_b^\alpha v(t)=\frac{1}{\Gamma(\alpha)}\int_t^b \frac{v(s)ds}{(s-t)^{1-\alpha}}, \quad \forall\, t\in\Lambda. $$

The left Riemann-Liouville fractional differential operator of order α is defined by

$$ {}_a^RD_t^\alpha v(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t \frac{v(s)ds}{(t-s)^{\alpha}}, \quad \forall\, t\in\Lambda. $$

The right Riemann-Liouville fractional differential operator of order α is defined by

$$ {}_t^RD_b^\alpha v(t)=-\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^b \frac{v(s)ds}{(s-t)^{\alpha}}, \quad \forall\, t\in\Lambda. $$

The Caputo fractional differential operator of order α is defined by

$$ {}_aD_t^\alpha v(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t \frac{v'(s)ds}{(t-s)^{\alpha}}, \quad \forall\, t\in\Lambda. $$

Proposition 2.1. *Let $0<\alpha<1, \Lambda=[a,b]$. Then the following statements hold:*

(a) For $v\in AC(\Lambda;X)$,

$$ {}_aI_t^\alpha\, {}_aD_t^\alpha v(t)=v(t)-v(a) \quad \textit{for a.e.} \quad t\in\Lambda. $$

(b) For $v\in L^1(\Lambda;X)$,

$$ {}_a^RD_t^\alpha\, {}_aI_t^\alpha v(t)=v(t) \quad \textit{for a.e.} \quad t\in\Lambda. $$

(c) For $v\in L^1(\Lambda;X)$,

$$ {}_aD_t^\alpha\, {}_aI_t^\alpha v(t)=v(t) \quad \textit{for a.e.} \quad t\in\Lambda. $$

We prove the following integrability result for fractional integrals.

Proposition 2.2. *Let $0<\alpha<1, \Lambda=[a,b], v\in L^2(\Lambda;X)$. Then ${}_aI_t^\alpha v\in L^2(\Lambda;X)$, ${}_tI_b^\alpha v\in L^2(\Lambda;X)$.*

Proof. A direct calculation gives

$$ \begin{aligned} \|{}_aI_t^\alpha v\|_{L^2(\Lambda;X)} &= \left(\int_a^b \left\|\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}v(s)ds\right\|_X^2 dt\right)^{\frac12} \\ &\le \frac{1}{\Gamma(\alpha)}\left[\int_a^b \left((t-a)^\alpha\int_0^1 (1-\tau)^{\alpha-1}\big\|v\big((t-a)\tau+a\big)\big\|_X d\tau\right)^2 dt\right]^{\frac12} \\ &\le \frac{(b-a)^\alpha}{\Gamma(\alpha)}\left[\int_a^b \left(\int_0^1 (1-\tau)^{\alpha-1}\big\|v\big((t-a)\tau+a\big)\big\|_X d\tau\right)^2 dt\right]^{\frac12}. \end{aligned} \tag{2.1} $$

Using the generalized Minkowski inequality [56, Theorem B], we have

$$\begin{aligned}&\left[\int_a^b \left(\int_0^1 (1-\tau)^{\alpha-1}\big\|v\big((t-a)\tau+a\big)\big\|_X d\tau\right)^2 dt\right]^{\frac{1}{2}}\\ &\le \int_0^1 \left(\int_a^b (1-\tau)^{2\alpha-2}\big\|v\big((t-a)\tau+a\big)\big\|_X^2 dt\right)^{\frac{1}{2}} d\tau.\end{aligned}$$

This, together with (2.1), gives

$$\begin{aligned}&\left(\int_a^b \left\|\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}v(s)ds\right\|_X^2 dt\right)^{\frac{1}{2}}\\ &\le \frac{(b-a)^\alpha}{\Gamma(\alpha)}\int_0^1 \left(\int_a^b (1-\tau)^{2\alpha-2}\big\|v\big((t-a)\tau+a\big)\big\|_X^2 dt\right)^{\frac{1}{2}} d\tau\\ &= \frac{(b-a)^\alpha}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\left(\int_a^{(b-a)\tau+a} \tau^{-1}\|v(\beta)\|_X^2 d\beta\right)^{\frac{1}{2}} d\tau\\ &\le \frac{(b-a)^\alpha}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\tau^{-\frac{1}{2}}\left(\int_a^b \|v(\beta)\|_X^2 d\beta\right)^{\frac{1}{2}} d\tau\\ &= \frac{(b-a)^\alpha}{\Gamma(\alpha)}\|v\|_{L^2(\Lambda;X)}B\left(\alpha,\frac{1}{2}\right) < \infty.\end{aligned}$$

This proves ${}_aI_t^\alpha v \in L^2(\Lambda;X)$. The property ${}_tI_b^\alpha v \in L^2(\Lambda;X)$ can be proved similarly. □

Remark 2.1. A simpler version of Proposition 2.2 with $X = R$ was stated in [50, Theorem 2.6] without a proof.

Let $\psi : X \to \mathbb{R}$ be a locally Lipschitz function. The Clarke generalized directional derivative of ψ at the point $x \in X$ in the direction $v \in X$, is defined by

$$\psi^0(x;v) = \lim_{y\to x}\sup_{\lambda\downarrow 0}\frac{\psi(y+\lambda v)-\psi(y)}{\lambda}.$$

The Clarke subdifferential of ψ at x is a subset of X^* given by

$$\partial\psi(x) = \big\{\xi \in X^* \mid \psi^0(x;v) \ge \langle \xi, v\rangle_{X^*\times X},\ \forall\, v \in X\big\}.$$

We also need to introduce the space $BV(0,T;X)$ of functions of bounded total variation on $(0,T)$. Let $\mathcal{T}$ denotes any finite partition $\{I_i = [t_{i-1},t_i]\}_{i=1}^n$ of $(0,T)$ such that $0 = t_0 < t_1 < \cdots < t_n = T$. Let $\mathcal{F}$ denotes the family of all such partitions. Then for a function $v : [0,T] \to X$ we define its total variation by

$$\|v\|_{BV(0,T;X)} = \sup_{\mathcal{T}\in\mathcal{F}}\left\{\sum_{I_i\in\mathcal{T}}\|v(t_{i+1})-v(t_i)\|_X\right\}.$$

For $1 \le p \le \infty$ and Banach spaces X, Z such that $X \subset Z$, we define the space

$$M^{p,1}(0,T;X,Z) = L^p(0,T;X) \cap BV(0,T;Z).$$

It is easy to see that $M^{p,1}(0,T;X,Z)$ is also a Banach space, endowed with the norm $\|\cdot\|_{L^p(0,T;X)}$ $+\|\cdot\|_{BV(0,T;Z)}$.

The following compactness result can be found in [19, Proposition 5.11].

Proposition 2.3. *Let $1 \le p < \infty$. Let $X_1 \subset X_2 \subset X_3$ be real Banach spaces such that X_1 is reflexive, the embedding $X_1 \subset X_2$ is compact and the embedding $X_2 \subset X_3$ is continuous. Then the embedding $M^{p,1}(0,T;X_1,X_3) \subset L^p(0,T;X_2)$ is compact.*

3. An Abstract Time Fractional Parabolic Hemivariational Inequality

Problem 1.1 will be studied as a special case of an abstract time fractional parabolic hemivariational inequality. We first introduce some spaces and notations. Let V be a real, reflexive, separable Banach space and H be a real, separable Hilbert space equipped with the inner product $(\cdot\,,\cdot)_H$. We assume that the spaces V, H and V^* form an evolution triple, i.e. $V \subset H \subset V^*$ with all embeddings being dense and continuous. Moreover, we assume that the embedding $V \subset H$ is compact. Let $i : V \to H$ be an embedding operator, for $v \in V$ we still denote $iv \in H$ by v. For all $u \in H$ and $v \in V$, we have $\langle u, v\rangle_{V^*\times V} = (u, v)_H$. We also introduce a reflexive Banach space U and the operator $\gamma \in \mathcal{L}(V, U)$. Given $0 < T < +\infty$, we introduce spaces $\mathcal{V} = L^2(0, T; V), \mathcal{V}^* = L^2(0, T; V^*), \mathcal{H} = L^2(0, T; H), \mathcal{U} = L^2(0, T; U), \mathcal{U}^* = L^2(0, T; U^*)$. In addition, we denote

$$
\begin{aligned}
\langle u, v\rangle_{\mathcal{V}^*\times\mathcal{V}} &= \int_0^T \langle u(t), v(t)\rangle_{V^*\times V} dt, \quad \forall\, u \in \mathcal{V}^*, \quad v \in \mathcal{V},\\
(u, v)_{\mathcal{H}} &= \int_0^T \big(u(t), v(t)\big)_H dt, \qquad\quad \forall\, u, v \in \mathcal{H},\\
\langle u, v\rangle_{\mathcal{U}^*\times\mathcal{U}} &= \int_0^T \langle u(t), v(t)\rangle_{U^*\times U} dt, \quad \forall\, u \in \mathcal{U}^*, \quad v \in \mathcal{U}.
\end{aligned}
$$

Finally, we define the space

$$\mathcal{W} = \big\{v \in \mathcal{V} \mid {}_0D_t^\alpha v \in \mathcal{H}\big\}.$$

For simplicity, the subscripts of the duality pairing $\langle\cdot\,,\cdot\rangle_{V^*\times V}$ and the inner product $(\cdot\,,\cdot)_H$ are omitted in the rest of paper.

Now we present the abstract time fractional parabolic hemivariational problem.

Problem 3.1. Find $u \in \mathcal{W}$ such that $u(0) = u_0$ and for a.e. $t \in (0, T)$,

$$\big\langle {}_0D_t^\alpha u(t) + Au(t), v\big\rangle + J^0\big(\gamma u(t); \gamma v\big) \geq \langle f(t), v\rangle, \quad \forall\, v \in V. \tag{3.1}$$

The assumptions of Problem 3.1 are as follows:

$H(A)$. The operator $A : V \to V^*$ is linear, bounded, and coercive, i.e.

(i) $A \in \mathcal{L}(V, V^*)$.

(ii) $\langle Av, v\rangle \geq m_A\|v\|_V^2$ for all $v \in V$ with $m_A > 0$.

$H(J)$. The functional $J : U \to \mathbb{R}$ is such that

(i) J is locally Lipschitz.

(ii) ∂J satisfies the growth condition $\|\xi\|_{U^*} \leq c_J + \tilde{c}_J\|u\|_U$ for every $u \in U$ and $\xi \in \partial J(u)$ with $c_J, \tilde{c}_J > 0$.

(iii) ∂J is relaxed monotone, i.e. for a constant $m_J \geq 0$,

$$\langle \xi_1 - \xi_2, u_1 - u_2\rangle_{U^*\times U} \geq -m_J\|u_1 - u_2\|_U^2$$

for all $u_i \in U$, $\xi_i \in \partial J(u_i)$, $i = 1, 2$.

$H(\gamma)$. The mapping $\gamma : V \to U$ is linear, continuous, and compact. Moreover, the associated Nemytskii operator $\bar{\gamma}$: $M^{2,1}(0,T;V,V^*) \to \mathcal{U}$ defined by $(\bar{\gamma}v)(t) = \gamma v(t)$ for all $t \in [0,T]$ is also compact.

$H(f)$. $f \in C^1([0,T];H)$.

$H(0)$. $u_0 \in V$, and the following compatibility condition holds: There exists $\xi_0 \in \partial J(\gamma u_0)$ such that $Au_0 + \gamma^*\xi_0 - f(0) \in H$.

$H(1)$. $m_A > \max\{\tilde{c}_J\|\gamma\|^2, m_J\|\gamma\|^2\}$.

The following time fractional hemivariational evolution inclusion is equivalent to Problem 3.1.

Problem 3.2. Find $u \in \mathcal{W}$ such that $u(0) = u_0$ and for a.e. $t \in (0,T)$,

$$
{}_0D_t^\alpha u(t) + Au(t) + \gamma^*\partial J\big(\gamma u(t)\big) \ni f(t). \tag{3.2}
$$

This equivalence can be readily checked by the definition of the Clarke generalized directional derivative and the Clarke subdifferential.

Therefore, to deal with Problem 3.1, it suffices to establish the unique solvability of Problem 3.2. The main theorem in this section is as follows.

Theorem 3.1. *Assume that the conditions $H(A), H(J), H(\gamma), H(f), H(0)$ and $H(1)$ hold. Then Problem 3.2 admits a unique solution.*

The proof is based on a time-discretization approximation, known as the Rothe method. To begin with, let $N \in \mathbb{N}$ be fixed and $\tau = T/N$ be the time step. We denote $t_k = k\tau$ and $I_k = ((k-1)\tau, k\tau]$ for $k = 1, 2, \dots, N$. Hereafter, we denote by $c > 0$ a constant independent on τ and its value can differ from line to line.

The approximation of the Caputo fractional derivative of u at the time $t = t_k$ is constructed by

$$
\begin{aligned}
{}_0D_t^\alpha u(t_k) &= \frac{1}{\Gamma(1-\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_k - s)^{-\alpha} u'(s) ds \\
&\approx \frac{1}{\Gamma(1-\alpha)} \sum_{i=1}^{k} \frac{u(t_i) - u(t_{i-1})}{\tau} \int_{t_{i-1}}^{t_i} (t_k - s)^{-\alpha} ds \\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=1}^{k} \big(u(t_i) - u(t_{i-1})\big)[(k-i+1)^{1-\alpha} - (k-i)^{1-\alpha}] \\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=1}^{k} b_{k-i}\big(u(t_i) - u(t_{i-1})\big),
\end{aligned} \tag{3.3}
$$

where we use the notations $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}, j = 0, 1, \dots, k-1$, for the sake of simplification. Inspired by (3.3), we define the following temporally semi-discrete scheme (or called Rothe problem) of Problem 3.2.

Problem 3.3. For each $k = 1, 2, \dots, N$, find $u^k \in V$ such that

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=1}^{k} b_{k-i}(u^i - u^{i-1}) + Au^k + \gamma^* \xi^k = f(t_k) \tag{3.4}$$

with

$$\xi^k \in \partial J(\gamma u^k), \quad u^0 = u_0.$$

First of all, we need the existence result of Problem 3.3.

Lemma 3.1. *If the assumptions $H(A), H(J), H(\gamma), H(f), H(0)$ and $H(1)$ hold, Problem 3.3 has a unique solution $u^k \in V$, for $k = 1, 2, \dots, N$.*

Proof. We consider the following equivalent problem: find $u^k \in V$ such that

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} b_{k-1} u_0 + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=1}^{k-1} (b_{k-i-1} - b_{k-i}) u^i + f(t_k) \in T(u^k), \tag{3.5}$$

where $T : V \to 2^{V^*}$ is defined by

$$T(v) = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} v + Av + \gamma^* \partial J(\gamma v), \quad \forall\, v \in V.$$

It is sufficient to prove that problem (3.5) has a solution.

On one hand, T is pseudomonotone. In fact, by assumption $H(A)$, the operator $v \mapsto \tau^{-\alpha} v/\Gamma(2-\alpha) + Av$ is bounded, continuous and monotone, thus is pseudomonotone, according to [44, Theorem 3.69]. The operator $v \mapsto \gamma^* \partial J(\gamma v)$ is pseudomonotone by assumption $H(J)$(ii) and [19, Proposition 5.6]. Hence, the operator is pseudomonotone as well by [9, Proposition 1.3.68].

On the other hand, we need to prove the coercivity. Let $v^* \in Tv$ for $v \in V$,

$$v^* = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} v + Av + \gamma^* \xi$$

with $\xi \in \partial J(\gamma v)$. Then, using $H(A)$(ii) and $H(J)$(ii), we have

$$\begin{aligned}
\langle v^*, v \rangle &= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \|v\|_H^2 + \langle Av, v \rangle + \langle \xi, \gamma v \rangle_{U^* \times U} \\
&\geq m_A \|v\|_V^2 - (c_J + \tilde{c}_J \|\gamma v\|_U) \|\gamma v\|_U \\
&\geq m_A \|v\|_V^2 - (\varepsilon + \tilde{c}_J) \|\gamma v\|_U^2 - \frac{1}{4\varepsilon} c_J^2 \\
&\geq \big(m_A - (\varepsilon + \tilde{c}_J) \|\gamma\|^2\big) \|v\|_V^2 - \frac{1}{4\varepsilon} c_J^2.
\end{aligned}$$

Take

$$\varepsilon = \frac{m_A - \tilde{c}_J \|\gamma\|^2}{2\|\gamma\|^2}$$

in the above inequality to get

$$\langle v^*, v \rangle \geq \frac{m_A - \tilde{c}_J \|\gamma\|^2}{2} \|v\|_V^2 - \frac{c_J^2 \|\gamma\|^2}{2\big(m_A - \tilde{c}_J \|\gamma\|^2\big)}.$$

The operator T is coercive from $H(1)$. Hence, by [19, Proposition 5.10], T is surjective and thus problem (3.5) has a solution.

The uniqueness of the solution is proved by contradiction. Assume $u_1^k, u_2^k \in V$ are two solutions to Problem 3.3. It is easy to get

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\big(u_1^k - u_2^k\big) + A\big(u_1^k - u_2^k\big) + \gamma^*\big(\xi_1^k - \xi_2^k\big) = 0.$$

We test the above equation with $(u_1^k - u_2^k)$ to obtain

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\big\|u_1^k - u_2^k\big\|_H^2 + \big\langle A\big(u_1^k - u_2^k\big), u_1^k - u_2^k\big\rangle + \big\langle \xi_1^k - \xi_2^k, \gamma\big(u_1^k - u_2^k\big)\big\rangle_{U^*\times U} = 0.$$

As a result, by $H(A)$(ii) and $H(J)$(ii), we have

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\big\|u_1^k - u_2^k\big\|_H^2 + \big(m_A - m_J\|\gamma\|^2\big)\big\|u_1^k - u_2^k\big\|_V^2 \le 0.$$

Thus $u_1^k = u_2^k$, the proof is complete. □

Then we present some boundedness results of the solution to Problem 3.3.

Lemma 3.2. *Assume that $\{u^k\}_{k=1}^N$ is the solution sequence of Problem 3.3. Under the assumptions $H(A), H(J), H(\gamma), H(f), H(0)$ and $H(1)$, for sufficiently small $\tau > 0$, we have*

$$\max_{k=1,2,\dots,N}\left\|\frac{u^k - u^{k-1}}{\tau^\alpha}\right\|_H \le c, \tag{3.6}$$

$$\sum_{k=1}^N \|u^k - u^{k-1}\|_H \le c, \tag{3.7}$$

$$\max_{k=1,2,\dots,N}\sum_{k=1}^n b_{n-k}\left\|\frac{u^k - u^{k-1}}{\tau^\alpha}\right\|_H \le c, \tag{3.8}$$

$$\max_{k=1,2,\dots,N}\|u^k\|_V \le c, \tag{3.9}$$

$$\max_{k=1,2,\dots,N}\|\xi^k\|_{U^*} \le c. \tag{3.10}$$

Proof. Replacing k with $k-1$ in Eq. (3.4) to get

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{k-1} b_{k-i-1}(u^i - u^{i-1}) + Au^{k-1} + \gamma^*\xi^{k-1} = f(t_{k-1}), \tag{3.11}$$

where $\xi^{k-1} \in \partial J(\gamma u^{k-1}), k = 2, 3, \dots, N$. We denote

$$\delta u^k = \frac{u^k - u^{k-1}}{\tau^\alpha}.$$

Test (3.4) with δu^k, test (3.11) with $-\delta u^k$ and add the resulting two equations to obtain

$$\begin{aligned}\frac{1}{\Gamma(2-\alpha)}\|\delta u^k\|_H^2 &+ \frac{1}{\Gamma(2-\alpha)}\left(\sum_{i=1}^{k-1}(b_{k-i} - b_{k-i-1})\delta u^i, \delta u^k\right) + \tau^\alpha\langle A\delta u^k, \delta u^k\rangle \\ &+ \tau^{-\alpha}\langle \xi^k - \xi^{k-1}, \gamma u^k - \gamma u^{k-1}\rangle_{U^*\times U} = \big(f(t_k) - f(t_{k-1}), \delta u^k\big).\end{aligned} \tag{3.12}$$

From $H(J)$(iii), we have

$$\langle \xi^k - \xi^{k-1}, \gamma u^k - \gamma u^{k-1} \rangle_{U^* \times U} \geq -m_J \|\gamma u^k - \gamma u^{k-1}\|_U^2 = -\tau^{2\alpha} m_J \|\gamma\|^2 \|\delta u^k\|_V^2. \tag{3.13}$$

Apply (3.13) to (3.12) to get

$$\begin{aligned}
&\frac{1}{\Gamma(2-\alpha)} \|\delta u^k\|_H^2 + \tau^\alpha \big(m_A - m_J \|\gamma\|^2\big) \|\delta u^k\|_V^2 \\
&\leq \frac{1}{\Gamma(2-\alpha)} \sum_{i=1}^{k-1} (b_{k-i-1} - b_{k-i}) \|\delta u^i\|_H \|\delta u^k\|_H + \|f(t_k) - f(t_{k-1})\|_H \|\delta u^k\|_H.
\end{aligned}$$

By $H(f)$, the following inequality holds:

$$\|f(t_k) - f(t_{k-1})\|_H \leq \|f\|_{C^1([0,T];H)} \tau.$$

Thus,

$$\|\delta u^k\|_H \leq \sum_{i=1}^{k-1} (b_{k-i-1} - b_{k-i}) \|\delta u^i\|_H + \Gamma(2-\alpha) \|f\|_{C^1([0,T];H)} \tau, \quad k = 2, 3, \ldots, N. \tag{3.14}$$

Hereafter, we omit the subscripts of the norm $\|f\|_{C^1([0,T];H)}$. Taking $k = 1$ in (3.4), we get

$$\frac{1}{\Gamma(2-\alpha)} \delta u^1 + A u^1 + \gamma^* \xi^1 = f(t_1). \tag{3.15}$$

From $H(0)$, there exists $v_0 \in H$, $\xi_0 \in \partial J(\gamma u_0)$ such that

$$A u_0 + \gamma^* \xi_0 - f(0) = v_0. \tag{3.16}$$

We test (3.15) with δu^1, test (3.16) with $-\delta u^1$ and add the resulting two equations to obtain

$$\begin{aligned}
&\frac{1}{\Gamma(2-\alpha)} \|\delta u^1\|_H^2 + \tau^\alpha \langle A \delta u^1, \delta u^1 \rangle + \tau^{-\alpha} \langle \xi^1 - \xi^0, \gamma u^1 - \gamma u^0 \rangle_{U^* \times U} \\
&= (f^1 - f^0, \delta u^1) + \big(v_0, \delta u^1\big).
\end{aligned} \tag{3.17}$$

Similarly, from $H(A)$(ii), $H(J)$(iii), $H(f)$ and (3.17), we find that

$$\|\delta u^1\|_H \leq \Gamma(2-\alpha) \|v_0\|_H + \Gamma(2-\alpha) \|f\| \tau. \tag{3.18}$$

We claim that for $k = 1, 2, \ldots, N$,

$$\|\delta u^k\|_H \leq \Gamma(2-\alpha) \|v_0\|_H + \Gamma(2-\alpha) \|f\| b_{k-1}^{-1} \tau. \tag{3.19}$$

The proof will be carried out by induction. The case for $k = 1$ is (3.18). Suppose now we have proven

$$\|\delta u^i\|_H \leq \Gamma(2-\alpha) \|v_0\|_H + \Gamma(2-\alpha) \|f\| b_{i-1}^{-1} \tau, \quad i = 1, 2, \ldots, k-1. \tag{3.20}$$

Apply (3.20) to (3.14) and obtain

$$\begin{aligned}
\|\delta u^k\|_H &\leq \sum_{i=1}^{k-1} (b_{k-i-1} - b_{k-i}) \big(\Gamma(2-\alpha) \|v_0\|_H + \Gamma(2-\alpha) \|f\| b_{i-1}^{-1} \tau\big) + \Gamma(2-\alpha) \|f\| \tau \\
&\leq \Gamma(2-\alpha) \|v_0\|_H + \left(\sum_{i=1}^{k-1} (b_{k-i-1} - b_{k-i}) + b_{k-1} \right) \Gamma(2-\alpha) \|f\| b_{k-1}^{-1} \tau \\
&= \Gamma(2-\alpha) \|v_0\|_H + \Gamma(2-\alpha) \|f\| b_{k-1}^{-1} \tau.
\end{aligned}$$

It is easy to verify that

$$k^{-\alpha}b_{k-1}^{-1} \le \frac{1}{1-\alpha}, \quad k = 1,2,\ldots,N. \tag{3.21}$$

By (3.19) and (3.21), we have

$$\begin{aligned}\|\delta u^k\|_H &\le \Gamma(2-\alpha)\|v_0\|_H + \Gamma(2-\alpha)\|f\|k^{-\alpha}b_{k-1}^{-1}\cdot k^\alpha\tau \\ &\le \Gamma(2-\alpha)\|v_0\|_H + \frac{\Gamma(2-\alpha)}{1-\alpha}\|f\|T^\alpha\tau^{1-\alpha}.\end{aligned}$$

Therefore, (3.6) holds for sufficiently small $\tau > 0$.

Now, we prove a result that will be used later. Let $\{a_k\}_{k=1}^n, \{c_k\}_{k=1}^n, \{d_k\}_{k=1}^n$ be three non-negative sequences. Then we have

$$\begin{aligned}\sum_{k=2}^n a_k \sum_{i=1}^{k-1} c_{k-i}d_i &= \sum_{k=2}^n a_k \sum_{j=1}^{k-1} c_j d_{k-j} = \sum_{j=1}^{n-1} c_j \sum_{k=j+1}^{n} a_k d_{k-j} \\ &= \sum_{j=1}^{n-1} c_j \sum_{i=1}^{n-j} a_{i+j}d_i = \sum_{k=1}^{n-1} c_{n-k}\sum_{i=1}^{k} a_{n-k+i}d_i.\end{aligned} \tag{3.22}$$

From (3.14) and (3.22), we have

$$\begin{aligned}\sum_{k=2}^n \|u^k - u^{k-1}\|_H &\le \sum_{k=2}^n\sum_{i=1}^{k-1}(b_{k-i-1} - b_{k-i})\|u^i - u^{i-1}\|_H + \Gamma(2-\alpha)\|f\|T\tau^\alpha \\ &= \sum_{k=1}^{n-1}(b_{n-k-1} - b_{n-k})\sum_{i=1}^{k}\|u^i - u^{i-1}\|_H + \Gamma(2-\alpha)\|f\|T\tau^\alpha.\end{aligned}$$

Combining the above inequality and (3.18) gives

$$\begin{aligned}\sum_{k=1}^n \|u^k - u^{k-1}\|_H \le& \sum_{k=1}^{n-1}(b_{n-k-1} - b_{n-k})\sum_{i=1}^{k}\|u^i - u^{i-1}\|_H \\ &+ \big(\Gamma(2-\alpha)\|f\|T + \Gamma(2-\alpha)\|v_0\|_H\big)\tau^\alpha + \Gamma(2-\alpha)\|f\|\tau^{1+\alpha}.\end{aligned}$$

One can use the mathematical induction likewise to prove the following bound:

$$\sum_{k=1}^n \|u^k - u^{k-1}\|_H \le \big(\Gamma(2-\alpha)\|f\|T + \Gamma(2-\alpha)\|v_0\|_H\big)b_{n-1}^{-1}\tau^\alpha + \Gamma(2-\alpha)\|f\|b_{n-1}^{-1}\tau^{1+\alpha}.$$

Then we use (3.21) to obtain

$$\sum_{k=1}^n \|u^k - u^{k-1}\|_H \le \frac{\Gamma(2-\alpha)}{1-\alpha}(\|f\|T + \|v_0\|_H)T^\alpha + \frac{\Gamma(2-\alpha)}{1-\alpha}\|f\|T^\alpha\tau.$$

Therefore (3.7) is proved.

The proof of (3.8) is basically similar to the previous one. From (3.14) and (3.22), we have

$$\begin{aligned}\sum_{k=2}^n b_{n-k}\|\delta u^k\|_H &\le \sum_{k=2}^n b_{n-k}\sum_{i=1}^{k-1}(b_{k-i-1} - b_{k-i})\|\delta u^i\|_H + \Gamma(2-\alpha)\|f\|T^{1-\alpha}\tau^\alpha \\ &= \sum_{k=1}^{n-1}(b_{n-k-1} - b_{n-k})\sum_{i=1}^{k} b_{k-i}\|\delta u^i\|_H + \Gamma(2-\alpha)\|f\|T^{1-\alpha}\tau^\alpha.\end{aligned}$$

Combining the above inequality and (3.18) gives

$$\sum_{k=1}^{n} b_{n-k}\|\delta u^k\|_H \le \sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})\sum_{i=1}^{k} b_{k-i}\|\delta u^i\|_H + \Gamma(2-\alpha)\|v_0\|_H b_{n-1} + \Gamma(2-\alpha)\|f\|b_{n-1}\tau + \Gamma(2-\alpha)\|f\|T^{1-\alpha}\tau^{\alpha}.$$

By induction, it is not difficult to prove

$$\begin{aligned}\sum_{k=1}^{n} b_{n-k}\|\delta u^k\|_H &\le \Gamma(2-\alpha)\|v_0\|_H + \Gamma(2-\alpha)\|f\|\tau + \Gamma(2-\alpha)\|f\|T^{1-\alpha}b_{n-1}^{-1}\tau^{\alpha}\\ &\le \Gamma(2-\alpha)\|v_0\|_H + \Gamma(2-\alpha)\|f\|\tau + \frac{\Gamma(2-\alpha)}{1-\alpha}\|f\|T.\end{aligned}$$

We have verified (3.8) for sufficiently small $\tau > 0$.

Finally, we test (3.4) with u^k to obtain

$$\left(\frac{1}{\Gamma(2-\alpha)}\sum_{i=1}^{k} b_{k-i}\delta u^i, u^k\right) + \langle Au^k + \gamma^*\xi^k, u^k\rangle = \big(f(t_k), u^k\big). \tag{3.23}$$

Using the assumptions $H(A)$(ii) and $H(J)$(ii) in (3.23) gives

$$\begin{aligned}m_A\|u^k\|_V^2 \le{}& \frac{1}{\Gamma(2-\alpha)}\sum_{i=1}^{k} b_{k-i}\|\delta u^i\|_H\|u^k\|_H\\ &+ c_J\|\gamma\|\|u^k\|_V + \tilde{c}_J\|\gamma\|^2\|u^k\|_V^2 + \|f(t_k)\|_H\|u^k\|_H.\end{aligned}$$

Dividing by $\|u^k\|_V$ at both sides yields

$$\big(m_A - \tilde{c}_J\|\gamma\|^2\big)\|u^k\|_V \le \frac{1}{\Gamma(2-\alpha)}\sum_{i=1}^{k} b_{k-1}\|\delta u^i\|_H + c_J\|\gamma\| + \|f\|_{C([0,T];H)}.$$

Noticing the assumption $H(1)$ and (3.8), (3.9) is proved. The estimate (3.10) is a direct corollary of $H(J)$(ii) and (3.9). □

Now, we define following functions $\bar{u}_\tau, u_\tau, \bar{w}_\tau, w_\tau : (0,T] \to V$ and $\bar{\xi}_\tau : (0,T] \to U^*$ with the solution to Problem 3.3:

$$\begin{aligned}
&\bar{u}_\tau(t) = u^k, && t \in I_k,\\
&u_\tau(t) = u^k + \frac{t-\tau k}{\tau}(u^k - u^{k-1}), && t \in I_k,\\
&\bar{w}_\tau(t) = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{k} b_{k-i}(u^i - u^{i-1}), && t \in I_k,\\
&w_\tau(t) = \frac{1}{\Gamma(2-\alpha)}\sum_{i=1}^{k-1}\big[(t-t_{i-1})^{1-\alpha} - (t-t_i)^{1-\alpha}\big]\frac{u^i - u^{i-1}}{\tau}\\
&\qquad\quad + \frac{1}{\Gamma(2-\alpha)}(t-t_{k-1})^{1-\alpha}\frac{u^k - u^{k-1}}{\tau}, && t \in I_k,\\
&\bar{\xi}_\tau(t) = \xi^k, && t \in I_k.
\end{aligned}$$

We also define $\bar{f}_\tau : (0,T] \to H$ by $\bar{f}_\tau = f(t_k)$, $t \in I_k$.

For some of the above functions, we have estimates as follows.

Corollary 3.1. *Under the assumptions $H(A), H(J), H(\gamma), H(f), H(0)$ and $H(1)$, for sufficiently small $\tau > 0$, it holds*

$$\|\bar{u}_\tau\|_{M^{2,1}(0,T;V,V^*)} \le c, \tag{3.24}$$
$$\|\bar{w}_\tau\|_{\mathcal{H}} \le c, \tag{3.25}$$
$$\|\bar{\xi}_\tau\|_{\mathcal{U}^*} \le c. \tag{3.26}$$

Proof. It follows from (3.9)

$$\|\bar{u}_\tau\|_{\mathcal{V}}^2 = \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \|\bar{u}_\tau(t)\|_V^2 dt = \tau \sum_{k=1}^N \|u^k\|_V^2 \le c.$$

The estimate (3.7) gives

$$\|\bar{u}_\tau\|_{BV(0,T;V^*)} = \sum_{k=1}^N \|u^k - u^{k-1}\|_{V^*} \le c \sum_{k=1}^N \|u^k - u^{k-1}\|_H \le c.$$

This proves (3.24). The estimate (3.25) can be derived by using (3.8) as follows:

$$\begin{aligned}\|\bar{w}_\tau\|_{\mathcal{H}}^2 &= \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \|\bar{w}_\tau(t)\|_H^2 dt = \tau \sum_{k=1}^N \left\| \frac{1}{\Gamma(2-\alpha)} \sum_{i=1}^k b_{k-i} \delta u^i \right\|_H^2 \\ &\le \tau \sum_{k=1}^N \left(\frac{1}{\Gamma(2-\alpha)} \sum_{i=1}^k b_{k-i} \|\delta u^i\|_H \right)^2 \le c.\end{aligned}$$

Finally, (3.26) is obvious from (3.10). □

Corollary 3.2. *Under the assumptions $H(A), H(J), H(\gamma), H(f), H(0)$ and $H(1)$, the following convergences hold:*

$$\bar{u}_\tau - u_\tau \to 0 \quad \textit{in } \mathcal{H}, \tag{3.27}$$
$$\bar{w}_\tau - w_\tau \to 0 \quad \textit{in } \mathcal{H}. \tag{3.28}$$

Proof. By a direct calculation, we have

$$\|\bar{u}_\tau - u_\tau\|_{\mathcal{H}}^2 = \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \left\| \frac{t - \tau k}{\tau} (u^k - u^{k-1}) \right\|_H^2 dt = \frac{1}{3} \tau^{1+2\alpha} \sum_{k=1}^N \|\delta u^k\|_H^2.$$

Bringing (3.6) and (3.7) into this equality gives

$$\|\bar{u}_\tau - u_\tau\|_{\mathcal{H}}^2 \le c\tau^{1+2\alpha} \sum_{k=1}^N \|\delta u^k\|_H \le c\tau^{1+\alpha} \to 0 \quad \text{as} \quad \tau \to 0,$$

which proves (3.27). Again by a direct calculation, for $t \in I_n$, $n = 1, 2, \ldots, N$, we have

$$\begin{aligned}&\|\bar{w}_\tau(t) - w_\tau(t)\|_H \\ =& \left\| \frac{1}{\Gamma(2-\alpha)} \sum_{k=1}^{n-1} \left[\left(\frac{t - t_{k-1}}{\tau} \right)^{1-\alpha} - \left(\frac{t - t_k}{\tau} \right)^{1-\alpha} - b_{n-k} \right] \delta u^k \right. \\ &\left. + \frac{1}{\Gamma(2-\alpha)} \left[\left(\frac{t - t_{n-1}}{\tau} \right)^{1-\alpha} - b_0 \right] \delta u^n \right\|_H \\ \le& \frac{1}{\Gamma(2-\alpha)} \sum_{k=1}^{n-1} (b_{n-k-1} - b_{n-k}) \|\delta u^k\|_H + \frac{b_0}{\Gamma(2-\alpha)} \|\delta u^n\|_H.\end{aligned}$$

This gives

$$\begin{aligned}\|\bar{w}_\tau - w_\tau\|_{\mathcal{H}}^2 &= \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\|\bar{w}_\tau(t) - w_\tau(t)\|_H^2 dt\\ &\le \tau\sum_{n=1}^{N}\left(\frac{1}{\Gamma(2-\alpha)}\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})\|\delta u^k\|_H + \frac{b_0}{\Gamma(2-\alpha)}\|\delta u^n\|_H\right)^2.\end{aligned}$$

From (3.6), (3.7) and (3.22), we obtain

$$\begin{aligned}\|\bar{w}_\tau - w_\tau\|_{\mathcal{H}}^2 &\le c\tau\sum_{n=1}^{N}\left(\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})\|\delta u^k\|_H + \|\delta u^n\|_H\right)\\ &= c\tau\sum_{n=2}^{N}\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})\|\delta u^k\|_H + c\tau\sum_{n=1}^{N}\|\delta u^n\|_H\\ &= c\tau\sum_{n=1}^{N-1}(b_{N-n-1}-b_{N-n})\sum_{k=1}^{n}\|\delta u^k\|_H + c\tau\sum_{n=1}^{N}\|\delta u^n\|_H\\ &\le c\tau^{1-\alpha} \quad\to\quad 0 \quad \text{as} \quad \tau \quad\to\quad 0.\end{aligned}$$

Hence (3.28) is proved. □

We introduce the Nemytskii operators $\mathcal{A}:\mathcal{V}\to\mathcal{V}^*$ and $\bar{\gamma}:\mathcal{V}\to\mathcal{U}$, defined by $(\mathcal{A}v)(t)=Av(t)$ and $(\bar{\gamma}v)(t) = \gamma v(t)$, respectively, for all $v \in \mathcal{V}$, a.e. $t \in [0,T]$. With these notations and functions, (3.4) can be rewritten as

$$\bar{w}_\tau + \mathcal{A}\bar{u}_\tau + \bar{\gamma}^*\bar{\xi}_\tau = \bar{f}_\tau \quad \text{in } \mathcal{V}^*, \tag{3.29}$$

where

$$\bar{\xi}_\tau(t) \in \partial J\big((\bar{\gamma}\bar{u}_\tau)(t)\big), \quad \forall\, t \in (0,T). \tag{3.30}$$

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. According to Corollary 3.1, there exists $u \in \mathcal{V}, w \in \mathcal{H}$ and $\xi \in \mathcal{U}^*$ such that, at least for subsequences,

$$\bar{u}_\tau \quad\to\quad u \quad \text{weakly in } \mathcal{V}, \tag{3.31}$$

$$\bar{w}_\tau \quad\to\quad w \quad \text{weakly in } \mathcal{H}, \tag{3.32}$$

$$\bar{\xi}_\tau \quad\to\quad \xi \quad \text{weakly in } \mathcal{U}^*. \tag{3.33}$$

From (3.27), (3.28), (3.31) and (3.32), we get

$$u_\tau \quad\to\quad u \quad \text{weakly in } \mathcal{H}, \tag{3.34}$$

$$w_\tau \quad\to\quad w \quad \text{weakly in } \mathcal{H}. \tag{3.35}$$

Using Proposition 2.2 gives ${}_0I_t^\alpha w_\tau \in \mathcal{H}$ and ${}_0I_t^\alpha w \in \mathcal{H}$. Next, we are going to show that

$${}_0I_t^\alpha w_\tau \quad\to\quad {}_0I_t^\alpha w \quad \text{weakly in } \mathcal{H}. \tag{3.36}$$

For any $v \in \mathcal{H}$,

$$\begin{aligned}
&\big({}_0I_t^\alpha(w_\tau - w), v\big)_{\mathcal{H}} \\
&= \int_0^T \left(\frac{1}{\Gamma(\alpha)}\int_0^t \big(w_\tau(s) - w(s)\big)(t-s)^{\alpha-1}ds, v(t)\right)dt \\
&= \frac{1}{\Gamma(\alpha)}\int_0^T\int_0^t (t-s)^{\alpha-1}\big(w_\tau(s) - w(s), v(t)\big)dsdt \\
&= \frac{1}{\Gamma(\alpha)}\int_0^T\int_s^T (t-s)^{\alpha-1}\big(w_\tau(s) - w(s), v(t)\big)dtds \\
&= \int_0^T \left(w_\tau(t) - w(t), \frac{1}{\Gamma(\alpha)}\int_t^T (s-t)^{\alpha-1}v(s)ds\right)dt \\
&= \big(w_\tau - w, {}_tI_T^\alpha v\big)_{\mathcal{H}}. \qquad (3.37)
\end{aligned}$$

Once again by Proposition 2.2, we have ${}_tI_T^\alpha v \in \mathcal{H}$. Together with (3.35) and (3.37), (3.36) is proved. Naturally, we extend u_τ by defining $u_\tau(0) = u^0$ so that $u_\tau \in AC(0,T;V)$. Besides, noting that ${}_0D_t^\alpha u_\tau = w_\tau$, it follows from Proposition 2.1(a):

$$u_\tau = {}_0I_t^\alpha w_\tau + u^0. \qquad (3.38)$$

Using the convergence results (3.34) and (3.36), passing to the limit on both sides of (3.38) gives

$$u = {}_0I_t^\alpha w + u^0. \qquad (3.39)$$

Since the Nemytskii operators $\mathcal{A}$ and $\bar{\gamma}^*$ are linear and bounded, they are weakly continuous. Thus from (3.31) and (3.33), we get

$$\mathcal{A}\bar{u}_\tau \;\to\; \mathcal{A}u \quad \text{weakly in } \mathcal{V}^*, \qquad (3.40)$$

$$\bar{\gamma}^*\bar{\xi}_\tau \;\to\; \bar{\gamma}^*\xi \quad \text{weakly in } \mathcal{V}^*. \qquad (3.41)$$

It is clear that

$$\bar{f}_\tau \;\to\; f \quad \text{in } \mathcal{H}. \qquad (3.42)$$

By (3.39) and Proposition 2.1(c), we have

$$w = {}_0D_t^\alpha u. \qquad (3.43)$$

Passing to the limit on both sides of Eq. (3.29) and invoking (3.32), (3.40)-(3.43), we obtain

$${}_0D_t^\alpha u + \mathcal{A}u + \bar{\gamma}^*\xi = f \quad \text{in } \mathcal{V}^*. \qquad (3.44)$$

From (3.39), it is obvious that

$$u(0) = u^0. \qquad (3.45)$$

The bound (3.24) and the assumption $H(\gamma)$ imply that $\bar{\gamma}\bar{u}_\tau \to \bar{\gamma}u$ in $\mathcal{U}$ as $\tau \to 0$. Moreover, noticing $\bar{\xi}_\tau(t) \in \partial J(\bar{\gamma}\bar{u}_\tau(t))$ for a.e. $t \in (0,T)$ and (3.33), according to the convergence theorem of Aubin and Cellina [3], we have

$$\xi(t) \in \partial J\big(\gamma u(t)\big) \quad \text{for a.e.} \quad t \in (0,T). \qquad (3.46)$$

Combining (3.44)-(3.46), we conclude that $u \in \mathcal{W}$ is a solution to Problem 3.2.

Now we turn to the proof of the uniqueness. Since $w \in \mathcal{H} \subset L^1(0,T;H)$, we observe from Proposition 2.1(b)(c) that

$$ {}_0D_t^\alpha {}_0I_t^\alpha w = {}_0^RD_t^\alpha {}_0I_t^\alpha w = w, $$

which means

$$ {}_0D_t^\alpha (u-u_0) = {}_0^RD_t^\alpha (u-u_0). $$

Thus,

$$ {}_0D_t^\alpha u = {}_0^RD_t^\alpha u - \frac{1}{\Gamma(1-\alpha)}\frac{u_0}{t^\alpha}. \tag{3.47} $$

Assume $u_1, u_2 \in \mathcal{W}$ are two solutions to Problem 3.2. Taking into account (3.47), we have for $i = 1, 2$,

$$ \begin{aligned} &\int_0^T \big({}_0^RD_t^\alpha u_i(t), v(t)\big) + \langle Au_i(t) + \gamma^*\xi_i(t), v(t)\rangle dt \\ &= \int_0^T \left(f(t) + \frac{1}{\Gamma(1-\alpha)}\frac{u_0}{t^\alpha}, v(t)\right)dt, \quad \forall\, v \in \mathcal{V} \end{aligned} \tag{3.48} $$

with

$$ \xi_i(t) \in \partial J\big(\gamma u_i(t)\big), \quad \text{a.e.} \quad t \in (0,T). $$

In the equality (3.48) we take $v = u_1 - u_2$ for $i = 1, v = u_2 - u_1$ for $i = 2$ and add the resulting two equalities to get

$$ \begin{aligned} &\int_0^T \big({}_0^RD_t^\alpha (u_1-u_2)(t), (u_1-u_2)(t)\big)dt \\ &\quad + \int_0^T \langle A(u_1-u_2)(t), (u_1-u_2)(t)\rangle + \langle \xi_1(t) - \xi_2(t), \gamma u_1(t) - \gamma u_2(t)\rangle_{U^*\times U} dt = 0. \end{aligned} \tag{3.49} $$

Using [33, Lemmas 2.6, 2.8] gives

$$ \begin{aligned} &\int_0^T \big({}_0^RD_t^\alpha (u_1-u_2)(t), (u_1-u_2)(t)\big)dt \\ &= \Big({}_0^RD_t^{\frac{\alpha}{2}}(u_1-u_2), {}_t^RD_T^{\frac{\alpha}{2}}(u_1-u_2)\Big)_{\mathcal{H}} \\ &\ge c\Big\|{}_0^RD_t^{\frac{\alpha}{2}}(u_1-u_2)\Big\|_{\mathcal{H}}^2 \ge 0. \end{aligned} \tag{3.50} $$

By $H(A)$(ii) and $H(J)$(iii), we have

$$ \begin{aligned} &\int_0^T \langle A(u_1-u_2)(t), (u_1-u_2)(t)\rangle + \langle \xi_1(t) - \xi_2(t), \gamma u_1(t) - \gamma u_2(t)\rangle_{U^*\times U} dt \\ &\ge \big(m_A - m_J\|\gamma\|^2\big)\|u_1-u_2\|_{\mathcal{V}}^2. \end{aligned} \tag{3.51} $$

Taking (3.50) and (3.51) into (3.49), we obtain $u_1 = u_2$. □

4. A Fully Discrete Scheme and Error Analysis

In this section, we consider a full discretization for Problem 3.1 and derive an error estimate for the discrete solution.

Let V_h be a finite dimensional subspace of V, where $h > 0$ is a spatial discretization parameter. Let $N \in \mathbb{N}$ be fixed and $\tau = T/N$ be the time step. We denote $t_k = k\tau$ for $k = 1, 2, \dots, N$. We denote $c > 0$ a generic constant whose value may differ at different occurrences, but it is always independent of h or τ. In the error analysis, we assume the regularity

$$u \in C([0, T]; V) \cap C^2([0, T]; H). \tag{4.1}$$

Let $u_h^0 \in V_h$ be an appropriate approximation of u_0, we consider the following full discretization for Problem 3.1.

Problem 4.1. For each $k = 1, 2, \dots, N$, find $u_h^k \in V_h$ such that

$$\begin{aligned}\left(\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=1}^{k} b_{k-i}\big(u_h^i - u_h^{i-1}\big), v_h\right)& \\ + \big\langle A u_h^k, v_h \big\rangle + J^0\big(\gamma u_h^k; \gamma v_h\big) &\geq \big(f(t_k), v_h\big), \quad \forall\, v_h \in V_h.\end{aligned} \tag{4.2}$$

With the definition of the Clarke subdifferential, (4.2) can be equivalently written as

$$\begin{aligned}\left(\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=1}^{k} b_{k-i}\big(u_h^i - u_h^{i-1}\big), v_h\right)& \\ + \big\langle A u_h^k, v_h \big\rangle + \big\langle \gamma^* \xi_h^k, v_h \big\rangle &= \big(f(t_k), v_h\big), \quad \forall\, v_h \in V_h\end{aligned}$$

with

$$\xi_h^k \in \partial J\big(\gamma u_h^k\big).$$

So we get the following equivalent problem of Problem 4.1: Find $u_h^k \in V_h$ such that

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} b_{k-1} u_h^0 + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=1}^{k-1} (b_{k-i-1} - b_{k-i}) u_h^i + f(t_k) \in T_h\big(u_h^k\big),$$

where $T_h : V_h \to 2^{V_h^*}$ is defined by

$$T_h(v_h) = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} v_h + A v_h + \gamma^* \partial J(\gamma v_h), \quad \forall\, v_h \in V_h.$$

Carrying out the same proof approach of Lemma 3.1 by simply replacing the space V with the approximation space V_h, we find that Problem 4.1 has a unique solution. It also follows from the proof of (3.9) that the scheme is unconditionally stable.

The rest of this section focuses on error analysis of the numerical solution. For $k = 0, 1, \dots, N$, we denote $e^k = u(t_k) - u_h^k$ and split the error e^k into

$$e^k = \rho^k + \theta^k, \quad \rho^k \triangleq u(t_k) - v_h^k, \quad \theta^k \triangleq v_h^k - u_h^k,$$

where $v_h^k \in V_h$ be arbitrary but fixed. Taking $v_h = \theta^k$ in (4.2) yields

$$\left(\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=1}^{k} b_{k-i}\big(u_h^i - u_h^{i-1}\big), \theta^k\right) + \big\langle A u_h^k, \theta^k \big\rangle + J^0\big(\gamma u_h^k; \gamma \theta^k\big) \geq \big(f(t_k), \theta^k\big).$$

Taking $t = t_k$ and $v = -\theta^k$ in (3.1) gives

$$-\big({}_0D_t^\alpha u(t_k), \theta^k\big) - \big\langle A u(t_k), \theta^k \big\rangle + J^0\big(\gamma u(t_k); -\gamma \theta^k\big) \geq -\big(f(t_k), \theta^k\big).$$

Summing the above two inequalities, we obtain

$$\begin{aligned}&\left(\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{k} b_{k-i}\big(u_h^i - u_h^{i-1}\big) - {}_0D_t^{\alpha}u(t_k), \theta^k\right)\\&\quad - \langle Ae^k, \theta^k\rangle + J^0\big(\gamma u_h^k; \gamma\theta^k\big) + J^0\big(\gamma u(t_k); -\gamma\theta^k\big) \geq 0.\end{aligned} \tag{4.3}$$

We can write

$$\begin{aligned}&\left(\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{k} b_{k-i}\big(u_h^i - u_h^{i-1}\big) - {}_0D_t^{\alpha}u(t_k), \theta^k\right)\\&= \left(\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{k} b_{k-i}\big(v_h^i - v_h^{i-1}\big) - {}_0D_t^{\alpha}u(t_k), \theta^k\right)\\&\quad - \left(\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{k} b_{k-i}(\theta^i - \theta^{i-1}), \theta^k\right)\\&= -(r^k, \theta^k) - \left(\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{k} b_{k-i}(\rho^i - \rho^{i-1}), \theta^k\right)\\&\quad - \left(\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{k} b_{k-i}(\theta^i - \theta^{i-1}), \theta^k\right),\end{aligned} \tag{4.4}$$

where r^k is the truncation error defined by

$$r^k = {}_0D_t^{\alpha}u(t_k) - \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{k} b_{k-i}\big(u(t_i) - u(t_{i-1})\big).$$

The following estimate of r^k was derived in [35]:

$$|r^k| \leq c\tau^{2-\alpha}\max_{t\in[0,T]}|u''(t)|. \tag{4.5}$$

Thus,

$$-(r^k, \theta^k) \leq c\tau^{2-\alpha}\max_{t\in[0,T]}\|u''(t)\|_H\|\theta^k\|_H. \tag{4.6}$$

Note that $\theta^0 = 0$, by Cauchy-Schwarz inequality, we have

$$\begin{aligned}&-\left(\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{k} b_{k-i}(\theta^i - \theta^{i-1}), \theta^k\right)\\&= -\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\|\theta^k\|_H^2 + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{k-1}(b_{k-i-1} - b_{k-i})(\theta^i, \theta^k) + \frac{\tau^{-\alpha}b_{k-1}}{\Gamma(2-\alpha)}(\theta^0, \theta^k)\\&\leq -\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\|\theta^k\|_H^2 + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{k-1}(b_{k-i-1} - b_{k-i})\frac{\|\theta^i\|_H^2 + \|\theta^k\|_H^2}{2}\\&\leq -\frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}\|\theta^k\|_H^2 + \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}\sum_{i=1}^{k-1}(b_{k-i-1} - b_{k-i})\|\theta^i\|_H^2,\end{aligned} \tag{4.7}$$

$$
\begin{aligned}
&-\left(\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{k}b_{k-i}(\rho^i-\rho^{i-1}),\theta^k\right)\\
&\le\left\|\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{k}b_{k-i}(\rho^i-\rho^{i-1})\right\|_H\|\theta^k\|_H. \qquad (4.8)
\end{aligned}
$$

Taking (4.6)-(4.8) into (4.4), we obtain

$$
\begin{aligned}
&\left(\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{k}b_{k-i}\big(u_h^i-u_h^{i-1}\big)-{}_0D_t^{\alpha}u(t_k),\theta^k\right)\\
&\le c\tau^{2-\alpha}\|\theta^k\|_H+\left\|\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{k}b_{k-i}(\rho^i-\rho^{i-1})\right\|_H\|\theta^k\|_H\\
&\quad-\frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}\|\theta^k\|_H^2+\frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}\sum_{i=1}^{k-1}(b_{k-i-1}-b_{k-i})\|\theta^i\|_H^2. \qquad (4.9)
\end{aligned}
$$

From the assumption $H(A)$, we have

$$
-\langle Ae^k,\theta^k\rangle=-\langle A\rho^k,\theta^k\rangle-\langle A\theta^k,\theta^k\rangle\le\delta\|\theta^k\|_V^2+c(\delta)\|\rho^k\|_V^2-m_A\|\theta^k\|_V^2. \qquad (4.10)
$$

Here and after, $c(\delta)$ represents a constant that depends on δ and its value may change between lines. Note that Clarke generalized directional derivative is subadditive. Therefore,

$$
J^0\big(\gamma u_h^k;\gamma\theta^k\big)\le J^0\big(\gamma u_h^k;\gamma e^k\big)+J^0\big(\gamma u_h^k;-\gamma\rho^k\big), \qquad (4.11)
$$

$$
J^0\big(\gamma u(t_k);-\gamma\theta^k\big)\le J^0\big(\gamma u(t_k);-\gamma e^k\big)+J^0\big(\gamma u(t_k);\gamma\rho^k\big). \qquad (4.12)
$$

We deduce from $H(J)$(iii) that

$$
J^0\big(\gamma u_h^k;\gamma e^k\big)+J^0\big(\gamma u(t_k);-\gamma e^k\big)\le m_J\|\gamma\|^2\|e^k\|_V^2. \qquad (4.13)
$$

Using $H(J)$(ii) and the finite element interpolation error estimate in [18], we have

$$
\begin{aligned}
&J^0\big(\gamma u_h^k;-\gamma\rho^k\big)+J^0\big(\gamma u(t_k);\gamma\rho^k\big)\\
&\le\big(c_J+\tilde c_J\|\gamma u_h^k\|_U\big)\|\gamma\rho^k\|_U+(c_J+\tilde c_J\|\gamma u(t_k)\|_U)\|\gamma\rho^k\|_U\\
&\le 2(c_J+\tilde c_J\|\gamma u(t_k)\|_U)\|\gamma\rho^k\|_U+\tilde c_J\|\gamma e^k\|_U\|\gamma\rho^k\|_U\\
&\le c\Big(1+\max_{t\in[0,T]}\|u(t)\|_V\Big)\|\gamma\rho^k\|_U+\delta\|e^k\|_V^2+c(\delta)\|\rho^k\|_V^2. \qquad (4.14)
\end{aligned}
$$

Combining (4.11)-(4.14) gives

$$
\begin{aligned}
&J^0\big(\gamma u_h^k;\gamma\theta^k\big)+J^0\big(\gamma u(t_k);-\gamma\theta^k\big)\\
&\le m_J\|\gamma\|^2\|e^k\|_V^2+c\|\gamma\rho^k\|_U+\delta\|e^k\|_V^2+c(\delta)\|\rho^k\|_V^2. \qquad (4.15)
\end{aligned}
$$

Note that

$$
\begin{aligned}
&m_A\|e^k\|_V^2\le(m_A+\delta)\|\theta^k\|_V^2+c(\delta)\|\rho^k\|_V^2,\\
&\delta\|\theta^k\|_V^2\le 2\delta\|e^k\|_V^2+2\delta\|\rho^k\|_V^2.
\end{aligned}
$$

We take (4.9), (4.10) and (4.15) into (4.3) and use the above two inequalities to obtain

$$\begin{aligned}
&\frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}\|\theta^k\|_H^2 + \big(m_A - m_J\|\gamma\|^2 - 5\delta\big)\|e^k\|_V^2 \\
\le{}& \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}\sum_{i=1}^{k-1}(b_{k-i-1}-b_{k-i})\|\theta^i\|_H^2 + c\|\gamma\rho^k\|_U + c(\delta)\|\rho^k\|_V^2 \\
&+\left(c\tau^{2-\alpha} + \left\|\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{k} b_{k-i}(\rho^i-\rho^{i-1})\right\|_H\right)\|\theta^k\|_H. \qquad (4.16)
\end{aligned}$$

By Cauchy-Schwarz inequality and using (3.21), we have

$$\begin{aligned}
&\left(c\tau^{2-\alpha} + \left\|\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{k} b_{k-i}(\rho^i-\rho^{i-1})\right\|_H\right)\|\theta^k\|_H \\
\le{}& c\left(\tau^{4-2\alpha} + \left\|\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{k} b_{k-i}(\rho^i-\rho^{i-1})\right\|_H^2\right)\tau^{\alpha}b_{N-k}^{-1} + \frac{\tau^{-\alpha}b_{N-k}}{2\Gamma(2-\alpha)}\|\theta^k\|_H^2 \\
\le{}& c\left(\tau^{4-2\alpha} + \left\|\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{k} b_{k-i}(\rho^i-\rho^{i-1})\right\|_H^2\right)\frac{T^{\alpha}}{1-\alpha} + \frac{\tau^{-\alpha}b_{N-k}}{2\Gamma(2-\alpha)}\|\theta^k\|_H^2.
\end{aligned}$$

Thus, taking δ small enough in (4.16), together with the assumption $H(1)$, gives that there exists a constant $c_0 > 0$ such that

$$c_0\tau^{-\alpha}\|\theta^k\|_H^2 + \|e^k\|_V^2 \le c_0\tau^{-\alpha}\sum_{i=1}^{k-1}(b_{k-i-1}-b_{k-i})\|\theta^i\|_H^2 + c_0\tau^{-\alpha}b_{N-k}\|\theta^k\|_H^2 + E^k, \qquad (4.17)$$

where

$$E^k = c\left(\tau^{4-2\alpha} + \left\|\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{k} b_{k-i}(\rho^i-\rho^{i-1})\right\|_H^2 + \|\gamma\rho^k\|_U + \|\rho^k\|_V^2\right).$$

Let

$$G^k = \sum_{i=0}^{k}\|e^i\|_V^2 + c_0\tau^{-\alpha}\sum_{i=1}^{k} b_{k-i}\|\theta^i\|_H^2, \quad k = 0, 1, \ldots, N.$$

From (4.17), we have

$$\begin{aligned}
G^N - G^{N-1} &= \|e^N\|_V^2 + c_0\tau^{-\alpha}\|\theta^N\|_H^2 \\
&\quad - c_0\tau^{-\alpha}\sum_{k=1}^{N-1}(b_{N-k-1}-b_{N-k})\|\theta^k\|_H^2 \\
&\le c_0\tau^{-\alpha}b_{N-N}\|\theta^N\|_H^2 + E^N.
\end{aligned}$$

Thus,

$$\begin{aligned}
G^N &\le G^{N-1} + c_0\tau^{-\alpha}b_{N-N}\|\theta^N\|_H^2 + E^N \\
&\le G^0 + c_0\tau^{-\alpha}\sum_{k=1}^{N} b_{N-k}\|\theta^k\|_H^2 + \sum_{k=1}^{N}E^k.
\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{k=1}^{N}\|e^k\|_V^2 &\le G^0+\sum_{k=1}^{N}E^k\\ &\le c\tau^{3-2\alpha}+c\sum_{k=1}^{N}\left\|\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{k}b_{k-i}(\rho^i-\rho^{i-1})\right\|_H^2\\ &\quad+c\sum_{k=1}^{N}\left(\|\gamma\rho^k\|_U+\|\rho^k\|_V^2\right)+\|e^0\|_V^2. \end{aligned}\tag{4.18}$$

Summarizing the above arguments, we get the following Céa's type inequality, which is valid for any finite dimensional subspace V_h of V.

Theorem 4.1. *Let u and u_h^k be solutions to Problems* 3.1 *and* 4.1*, respectively. Assume that the conditions $H(A), H(J), H(\gamma), H(f), H(0), H(1)$ and regularity* (4.1) *hold. Then we have the following inequality:*

$$\begin{aligned}\sum_{k=1}^{N}\left\|u(t_k)-u_h^k\right\|_V^2 \le c\tau^{3-2\alpha}&+c\sum_{k=1}^{N}\left\|\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{k}b_{k-i}\big(\big(u(t_i)-v_h^i\big)-\big(u(t_{i-1})-v_h^{i-1}\big)\big)\right\|_H^2\\ &+c\sum_{k=1}^{N}\left(\left\|\gamma\big(u(t_k)-v_h^k\big)\right\|_U+\left\|u(t_k)-v_h^k\right\|_V^2\right)\\ &+\left\|u_0-u_h^0\right\|_V^2,\quad \forall\, v_h^k\in V_h,\quad k=0,1,\ldots N. \end{aligned}\tag{4.19}$$

Remark 4.1. To obtain the optimal estimate, for each time step, v_h^k should be chosen as the best approximation of $u(t_k)$ in V_h. In specific situations, for example, V_h is a finite element space, then v_h^k can usually be selected as the projection or interpolation of $u(t_k)$ in V_h.

5. Weak Solvability and Numerical Analysis of Problem 1.1

In this section, we apply the result obtained in Sections 3 and 4 to the original model problem. The existence and uniqueness of the solution to the weak formulation of Problem 1.1 will be proved. We will also derive an error estimate for the numerical solution of Problem 1.1.

We make some assumptions on the data of Problem 1.1.

$H(j)$. The function $j:\mathbb{R}\to\mathbb{R}$ is such that

(i) is locally Lipschitz.

(ii) ∂j satisfies the growth condition $|\xi|\le c_j+\tilde{c}_j|r|$ for every $r\in\mathbb{R}$ and $\xi\in\partial j(r)$ with $c_j>0$.

(iii) There exists a constant $m_j\ge 0$ such that

$$j^0(r_1;r_2-r_1)+j^0(r_2;r_1-r_2)\le m_j|r_1-r_2|^2,\quad \forall\, r_1,r_2\in\mathbb{R}.$$

For the source function f and the initial value u_0, assume

$$f\in C^1\big([0,T];L^2(\Omega)\big),\quad u_0\in H^2(\Omega).\tag{5.1}$$

To derive the weak formulation, we denote

$$V = \{v \in H^1(\Omega) \,|\, v = 0 \text{ on } \Gamma_1\}, \quad H = L^2(\Omega), \quad U = L^2(\Gamma_2).$$

These spaces are real Hilbert spaces equipped with the canonical inner products. In addition, the space $\mathcal{W}$ is defined as in Section 3 with the spaces V and H. Let $\varepsilon \in (0, 1/2)$ and $Z = H^{1-\varepsilon}(\Omega)$. We also define the embedding $i_1 : V \to Z$, the trace operator $\gamma_1 : Z \to H^{1/2-\varepsilon}(\Gamma_2)$, and the embedding $i_2 : H^{1/2-\varepsilon}(\Gamma_2) \to U$. The trace operator $\gamma : V \to U$ is defined by $\gamma = i_2 \circ \gamma_1 \circ i_1$. By the Sobolev trace theorem,

$$\|\gamma v\|_U \le \|\gamma\| \|v\|_V, \quad \forall v \in V.$$

Since the measure of Γ_1 is positive, there exists a constant $m_1 > 0$ such that

$$\|\nabla v\|_H^2 \ge m_1 \|v\|_V^2, \quad \forall v \in V. \tag{5.2}$$

The weak formulation of Problem 1.1 is as follows.

Problem 5.1. Find $u \in \mathcal{W}$ such that $u(0) = u_0$ and for a.e. $t \in (0, T)$,

$$\big({}_0D_t^\alpha u(t), v\big) + \big(\nabla u(t), \nabla v\big) + \int_{\Gamma_2} j^0\big(\gamma u(t); \gamma v\big) d\sigma \ge \big(f(t), v\big), \quad \forall\, v \in V. \tag{5.3}$$

The unique solvability of Problem 5.1 is provided in the following theorem.

Theorem 5.1. *Under the assumptions stated before, if*

$$m_1 > \max\big\{\tilde{c}_j \|\gamma\|^2, m_j \|\gamma\|^2\big\}, \tag{5.4}$$

then Problem 5.1 *has a unique solution.*

Before proving this theorem, we first prove a result for an auxiliary problem. We define the operator $A : V \to V^*$ by

$$\langle Au, v\rangle_{V^* \times V} = (\nabla u, \nabla v), \quad u, v \in V.$$

Also, define the functional $J : U \to \mathbb{R}$ by

$$J(v) = \int_{\Gamma_2} j(v) d\sigma, \quad v \in U.$$

With these notations, we consider the following auxiliary problem.

Problem 5.2. Find $u \in \mathcal{W}$ such that $u(0) = u_0$ and for a.e. $t \in (0, T)$,

$$\big({}_0D_t^\alpha u(t), v\big) + \langle Au, v\rangle_{V^* \times V} + J^0\big(\gamma u(t); \gamma v\big) \ge \big(f(t), v\big), \quad \forall\, v \in V.$$

We apply Theorem 3.1 to studying Problem 5.2. For this purpose, the assumptions of Problem 3.1 need to be checked. By the definition of the operator A and (5.2), $H(A)$ holds with $m_A = m_1$. It is clear that $H(j)$(i) implies $H(J)$(i). From [44, Theorem 3.47], we get

$$J^0\big(\gamma u(t); \gamma v\big) \le \int_{\Gamma_2} j^0\big(\gamma u(t); \gamma v\big) d\sigma, \quad \forall\, v \in V, \quad \text{a.e.} \quad t \in (0, T). \tag{5.5}$$

Thus, $H(J)$(ii) is satisfied with $c_J = c_j\sqrt{m(\Gamma_2)}$ and $\tilde{c}_J = \tilde{c}_j$. Also, from $H(j)$(iii) and (5.5), we have

$$J^0(u_1; u_2 - u_1) + J^0(u_2; u_1 - u_2) \le m_j\|u_1 - u_2\|_U^2, \quad \forall\, u_1, u_2 \in U.$$

It is seen that the above property is equivalent to $H(J)$(iii) with $m_J = m_j$. Moreover, it is well known that the trace operator γ is linear, continuous and compact. Let $\{v_n\} \subset M^{2,1}(0,T;V,V^*)$ be a bounded sequence. In order to verify $H(\gamma)$, we will show that there exists $v \in \mathcal{U}$ such that, at least for a subsequence,

$$\gamma v_n \;\to\; v \quad \text{in } \mathcal{U} \quad \text{as} \quad n \;\to\; \infty. \tag{5.6}$$

Since the embedding $i_1 : V \to Z$ is compact and the embedding $Z \subset V^*$ is continuous, it follows from Proposition 2.3 that there exists $z \in L^2(0,T;Z)$ such that, at least for a subsequence,

$$\int_0^T \|i_1 v_n(t) - z(t)\|_Z^2 dt \;\to\; 0 \quad \text{as} \quad n \;\to\; \infty. \tag{5.7}$$

Let $v \in \mathcal{U}$ be given by $v = (i_2 \circ \gamma_1)z$. Then

$$\|\gamma v_n - v\|_{\mathcal{U}}^2 = \int_0^T \|\gamma v_n(t) - v(t)\|_U^2 dt \le \|i_2 \circ \gamma_1\|^2 \int_0^T \|i_1 v_n(t) - z(t)\|_Z^2 dt.$$

We combine this inequality with (5.7) to conclude that (5.6) holds. The assumption (5.1) implies $H(f)$ and we get $f(0) \in H$. By the definition of the operator A we have $Au_0 \in V$. In addition, we note that the set $\partial J(\gamma u_0)$ is nonempty. Then we obtain that $H(0)$ holds with some $\xi_0 \in \partial J(\gamma u_0)$. Finally, the assumption $H(1)$ comes from (5.4).

So far, all the assumptions of Problem 3.1 have been checked. Then we apply Theorem 3.1 to conclude that the auxiliary problem, i.e. Problem 5.2, has a unique solution $u \in \mathcal{W}$.

Proof of Theorem 5.1. It is readily seen from (5.5) that every solution of Problem 5.2 is a solution of Problem 5.1. Thus we have proved the existence of the solution to Problem 5.1. The uniqueness can be proved by contradiction as in Section 3. □

We will apply finite element method for spatial discretization. For simplicity, we set Ω be a polygonal domain. Let $\mathcal{T}_h$ be a regular triangulation of $\bar{\Omega}$ with mesh size h. We use the standard linear finite element space V_h over $\mathcal{T}_h$, defined by

$$V_h = \{v_h \in C(\bar{\Omega}) \mid \forall\, K \in \mathcal{T}_h,\ v_h|_K \in P_1(K) \text{ and } v_h = 0 \text{ on } \Gamma_1\},$$

where $P_1(K)$ denotes the space of polynomial functions of total degree lower or equal to one on K. The norm of $L^2(\Omega)$ is denoted by $\|\cdot\|_0$, and the norm of $H^m(\Omega)$ is denoted by $\|\cdot\|_m$. Let $I_h : C(\bar{\Omega}) \to V_h$ be the Lagrange interpolation operator. The full discretization for Problem 5.1 is as follows.

Problem 5.3. For each $k = 1, 2, \ldots, N$, find $u_h^k \in V_h$ such that

$$\left(\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{k} b_{k-i}\big(u_h^i - u_h^{i-1}\big), v_h\right) + \big(\nabla u_h^k, \nabla v_h\big) + \int_{\Gamma_2} j^0\big(\gamma u_h^k; \gamma v_h\big) d\sigma \ge \big(f(t_k), v_h\big), \quad \forall\, v_h \in V_h$$

with $u_h^0 = I_h u_0$.

For error estimation of the numerical solution, we assume the regularity

$$
\begin{aligned}
&u \in C\big([0,T]; H^2(\Omega)\big) \cap C^2\big([0,T]; L^2(\Omega)\big),\\
&{}_0D_t^{\alpha} u \in C\big([0,T]; H^1(\Omega)\big),\\
&u|_{\Gamma_2} \in C\big([0,T]; H^2(\Gamma_2)\big).
\end{aligned} \tag{5.8}
$$

Theorem 5.2. *Let u and u_h^k be solutions to Problems* 5.1 *and* 5.3*, respectively. Assume that the conditions $H(j)$,* (5.1) *and regularity* (5.8) *hold. Then we have the following optimal order error estimate:*

$$
\left(\tau \sum_{k=1}^{N} \big\|u(t_k) - u_h^k\big\|_1^2\right)^{\frac{1}{2}} \le c(\tau^{2-\alpha} + h).
$$

Proof. Based on applications of the inequality (5.5), the Céa's type inequality (4.19) holds for the numerical solution of Problem 5.3 with J^0 replaced by $\int_{\Gamma_2} j^0$. Now we choose $v_h^k = I_h u(t_k)$ in (4.19) to get

$$
\begin{aligned}
&\sum_{k=1}^{N} \big\|u(t_k) - u_h^k\big\|_1^2\\
&\le c\tau^{3-2\alpha} + c\sum_{k=1}^{N} \left\|(I - I_h)\left(\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=1}^{k} b_{k-i}\big(u(t_i) - u(t_{i-1})\big)\right)\right\|_0^2\\
&\quad + c\sum_{k=1}^{N} \left(\big\|\gamma\big(u(t_k) - I_h u(t_k)\big)\big\|_{L^2(\Gamma_2)} + \|u(t_k) - I_h u(t_k)\|_1^2\right) + \|u_0 - I_h u_0\|_1^2.
\end{aligned} \tag{5.9}
$$

From the finite element interpolation error estimates [18, 49] and recalling the regularity (5.8), we have

$$
\big\|\gamma\big(u(t_k) - I_h u(t_k)\big)\big\|_{L^2(\Gamma_2)} \le ch^2 \|u(t_k)\|_{H^2(\Gamma_2)} \le ch^2 \max_{t\in[0,T]} \|u(t)\|_{H^2(\Gamma_2)}, \tag{5.10}
$$

$$
\|u(t_k) - I_h u(t_k)\|_1^2 \le ch^2 \|u(t_k)\|_2^2 \le ch^2 \max_{t\in[0,T]} \|u(t)\|_2^2, \tag{5.11}
$$

$$
\|u_0 - I_h u_0\|_1^2 \le ch^2 \|u_0\|_2^2, \tag{5.12}
$$

and

$$
\begin{aligned}
&\left\|(I - I_h)\left(\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=1}^{k} b_{k-i}\big(u(t_i) - u(t_{i-1})\big)\right)\right\|_0^2\\
&\le ch^2 \left\|\left(\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=1}^{k} b_{k-i}\big(u(t_i) - u(t_{i-1})\big)\right)\right\|_1^2\\
&\le ch^2\left(\|r^k\|_1^2 + \big\|{}_0D_t^{\alpha} u(t_k)\big\|_1^2\right) \le ch^2\left(1 + \max_{t\in[0,T]} \big\|{}_0D_t^{\alpha} u(t)\big\|_1^2\right).
\end{aligned} \tag{5.13}
$$

Taking (5.10)-(5.13) to (5.9), we complete the proof. □

6. Numerical Examples

In this section, a numerical example will be provided to confirm our theoretical results. The main purpose is to verify the convergence rates with respect to the time step τ and mesh size h.

Let $\Omega = (0,1) \times (0,1)$, $\Gamma_2 = (0,1) \times \{0\}$ and $\Gamma_1 = \partial\Omega \backslash \Gamma_2$. We use uniform triangulations on the domain $\bar{\Omega}$ such that each side of Ω is split into equal sub-intervals. The time interval is $(0,T)$ with $T = 1$.

In Problem 1.1, we take

$$f(x,y,t) = 20t^2 \sin(\pi x) \sin(\pi y), \quad u_0(x,y) = 0,$$

and

$$j(r) = \begin{cases} 0, & \text{if} \quad r < 0, \\ -\frac{1}{2}r^2 + 2r, & \text{if} \quad 0 \le r \le 1, \\ r + \frac{1}{2}, & \text{if} \quad r > 1. \end{cases}$$

The Clarke subdifferential of j is given by (see Fig. 6.1)

$$\partial j(r) = \begin{cases} 0, & \text{if} \quad r < 0, \\ [0,2], & \text{if} \quad r = 0, \\ -r + 2, & \text{if} \quad 0 < r \le 1, \\ 1, & \text{if} \quad r > 1. \end{cases}$$

The active/inactive set method [6, 29] is used to find the numerical solution at each time step. Since the exact solution is hard to represent, we choose the numerical solution obtained by $h = 1/256$ and $\tau = 1/1024$ to be the reference solution u_{ref} in computing numerical solution errors. In the case of $\alpha = 0.5$, the reference solution at the moments $t = 0.25, 0.5, 0.75, 1$ are reported in Fig. 6.2. As the first part of tests, the spatial convergence rate of the numerical solutions is measured. We test for the mesh sizes $h = 1/4, 1/8, 1/16, 1/32, 1/64$ and fix the time step $\tau = 1/1024$. The errors computed by

$$\left(\tau \sum_{k=1}^{N} \|u_h^k - u_{ref}^k\|_1^2 \right)^{\frac{1}{2}} \tag{6.1}$$

are shown in Table 6.1, from which the first order convergence in h is observed.

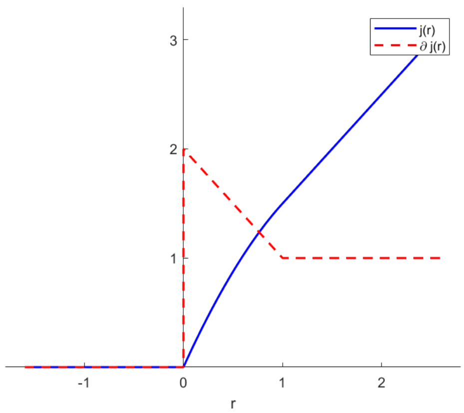

Fig. 6.1. j and its Clarke subdifferential ∂j.

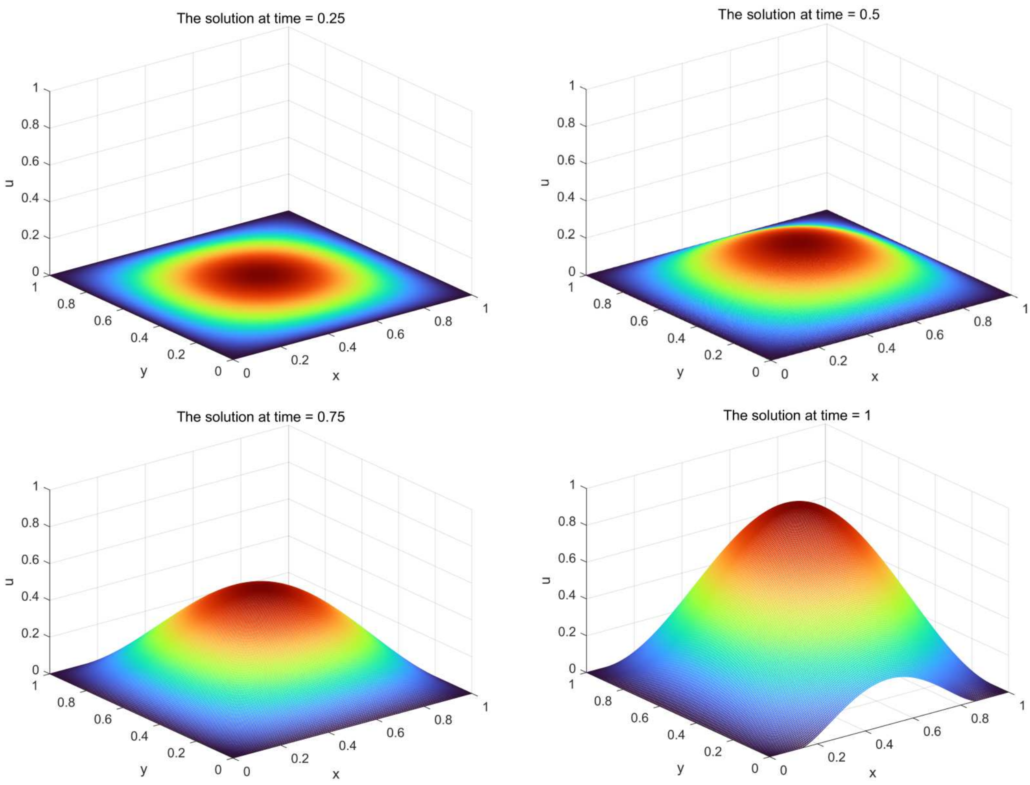

Fig. 6.2. Numerical solution at the moments $t = 0.25, 0.5, 0.75, 1$.

Table 6.1: Numerical errors and convergence orders in spatial direction with $\tau = 1/1024$.

h	$\alpha = 0.25$		$\alpha = 0.5$		$\alpha = 0.75$	
	Error	Order	Error	Order	Error	Order
1/4	3.5637e-01		3.4951e-01		3.4180e-01	
1/8	1.8253e-01	0.97	1.7926e-01	0.96	1.7553e-01	0.96
1/16	9.1859e-02	0.99	9.0191e-02	0.99	8.8924e-02	0.98
1/32	4.5769e-02	1.01	4.4932e-02	1.01	4.3984e-02	1.02
1/64	2.2351e-02	1.03	2.1941e-02	1.03	2.1478e-02	1.03

In order to test the convergence order with respect to the time step, we take $\tau = 1/8, 1/16, 1/32, 1/64, 1/128$ and fix the mesh size $h = 1/256$. The errors (6.1) and convergence rates are listed in Table 6.2, from which a convergence order in τ close to $(2 - \alpha)$ is observed.

Remark 6.1. For the errors in the temporal direction, the convergence orders shown by numerical examples may be slightly different from the theoretical value $(2 - \alpha)$, but it is acceptable. Compared to classical variational problems, the numerical solution of hemivariational inequalities is more difficult to compute. Some accuracy might be lost at the level of algorithm implementation. Although we use the numerical solution obtained by sufficiently small time step and mesh size as the reference solution, it still has some gaps from the exact solution. This further affects the error computation. Similar phenomena have also been observed in other references, for example, [6, 23].

Table 6.2: Numerical errors and convergence orders in temporal direction with $h = 1/256$.

τ	$\alpha = 0.25$		$\alpha = 0.5$		$\alpha = 0.75$	
	Error	Order	Error	Order	Error	Order
1/8	1.1006e-03		3.8024e-03		9.6409e-03	
1/16	3.5366e-04	1.64	1.3999e-03	1.44	4.0434e-03	1.25
1/32	1.0916e-04	1.70	4.8986e-04	1.51	1.6414e-03	1.30
1/64	3.3507e-05	1.70	1.7254e-04	1.51	6.7616e-04	1.28
1/128	1.0169e-05	1.72	6.0138e-05	1.52	2.7073e-04	1.32

7. Conclusions

In this paper, we have proposed a model problem describing anomalous diffusion in media with boundary semipermeability. Some pioneering research has been done. The weak formulation of the model is a time fractional parabolic hemivariational inequality, that has never been studied before. The existence and uniqueness of the solution to the hemivariational inequality has been established with Rothe method. In particular, we considered a full discretization to the hemivariational problem and analyzed its convergence properties. The employ of the well-known $L1$ time scheme for the time fractional derivative leads to globally (2-α)-order accuracy in time, which is consistent with the results of classical time fractional anomalous diffusion problems. For the spatial variable, P1-finite element method has been used and the first order convergence has been derived. Numerical experiments were performed to confirm our theoretical results.

It should be noted that results in this paper are coordinated with the known results of parabolic hemivariational inequalities. Taking $\alpha = 1$, the time fractional parabolic hemivariational inequality will become a parabolic hemivariational inequality. The unique solvability of the parabolic hemivariational inequality was proved using Rothe method in [6]. A fully discrete scheme based on first-order backward differentiation in time and P1-finite element in space was studied in [23], where the first order convergence in both temporal and spatial direction was derived.

Although this work provides error analysis of the numerical solution to Problem 1.1, some important issues remain to be investigated in the future. For instance, it has been known that the solutions to TFDEs may exhibit a weak singularity near $t = 0$ for smooth data. we plan to discuss the issue how the (2-α)-order convergence can be achieved for nonsmooth solutions. In addition, it is also interesting to address the storage reduction issue since the numerical scheme in this work unavoidably results in high storage costs on time variables.

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