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SUPERLINEARLY CONVERGENT ALGORITHMS FOR STOCHASTIC TIME-FRACTIONAL EQUATIONS DRIVEN BY WHITE NOISE*

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Abstract

The numerical analysis of stochastic time-fractional equations exhibits a significantly low-order convergence rate since the limited regularity of model caused by the nonlocal operator and the presence of noise. In this work, we consider stochastic time-fractional equations driven by integrated white noise, where ${}^C D_t^{\alpha}\psi(x,t), 0<\alpha<2$ and $I_t^{\gamma}\dot{W}(x,t), 0<\gamma<1$. We first establish the regularity of the mild solution. Then superlinear convergence rate

$$\left(\mathbb{E}\left\|\psi(\cdot,t_n)-\psi^n\right\|^2\right)^{\frac{1}{2}}=O\left(\tau^{\alpha+\gamma-\frac{\alpha d}{4}-\frac{1}{2}-\varepsilon}\right)$$

with sufficiently small ε term in the exponent is established based on the modified two-step backward difference formula methods. Here d represents the spatial dimension, ψ^n denotes the approximate solution at the n-th time step, and $\mathbb{E}$ is the expectation operator. Numerical experiments are performed to verify the theoretical results. To the best of our knowledge, this is the first topic on the superlinear convergence analysis for the stochastic time-fractional equations with integrated white noise.

Mathematics subject classification: 60H35, 34A08.
Key words: Stochastic fractional evolution equation, Integrated white noise, Superlinear convergence analysis.

1. Introduction

We are interested in the error estimates of modified two-step backward difference formula (BDF2) methods for solving the stochastic time-fractional evolution equation driven by integrated white noise [10, 15, 16], with $\alpha\in(1,2)$ and $\gamma\in(0,1)$,

$$\begin{cases} {}^C D_t^{\alpha}\psi(x,t)-A\psi(x,t)=f(x,t)+I_t^{\gamma}\dot{W}(x,t), & (x,t)\in\mathcal{O}\times\mathbb{R}_+,\\ \psi(x,0)=\upsilon(x), \quad \partial_t\psi(x,0)=b(x), & x\in\mathcal{O}, \end{cases} \tag{1.1}$$

where $\mathcal{O}\subset\mathbb{R}^d$, $d=1,2,3$ is a bounded domain with Lipschitz boundary $\partial\mathcal{O}$ and d denotes the spatial dimension. The operator A denotes the Laplacian Δ on a convex polyhedral domain $\mathcal{O}$ with $\mathcal{D}(A)=H_0^1(\mathcal{O})\cap H^2(\mathcal{O})$.

Here $W(x,t)$ is a cylindrical Wiener process with a covariance operator $Q=I$ on $L^2(\mathcal{O})$ with respect to a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ on a probability space $(\Omega,\mathscr{F},\mathbf{P})$ in [27]. And white noise

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$\dot{W}(x,t)$ is the time derivative of $W(x,t)$ with

$$W(x,t)=\sum_{j=1}^{\infty}\beta_j(t)\varphi_j(x), \tag{1.2}$$

where $\beta_j(t)$ are the independently and identically distributed Brownian motions and $\varphi_j(x)$, $j=1,2,\ldots$, are the L^2-norm normalized eigenfunctions of the operator $-\Delta$ corresponding to the eigenvalues $\lambda_j,\ j=1,2,\ldots$, arranged in nondecreasing order.

Note that the additive noise $\dot{W}(x,t)$ is expressed by [8]

$$\dot{W}(x,t)=\frac{dW(x,t)}{dt}=\sum_{j=1}^{\infty}\sigma_j(t)\dot{\beta}_j(t)\varphi_j(x), \tag{1.3}$$

where $\sigma_j(t)$ is the rapidly decay function as j increases with $\sum_{j=1}^{\infty}\sigma_j^2<\infty$.

The deterministic problems associated with model (1.1) arise in many areas of the applied sciences, such as the dynamics of viscoelastic materials, through water around rocks, and the transport of chemical contaminants [11, 17, 22]. The fractionally integrated noise $I_t^{\gamma}\dot{W}(x,t)$ characterizes random effects on particle motion in medium with memory or particles subject to sticking and trapping [1, 5, 12, 15, 16].

The solution of model (1.1) may be decomposed into the solution of the stochastic problem

$$\begin{cases} {}^C D_t^{\alpha}u(x,t)-Au(x,t)=I_t^{\gamma}\dot{W}(x,t), & (x,t)\in\mathcal{O}\times\mathbb{R}_+,\\ u(x,0)=\upsilon(x),\quad \partial_t u(x,0)=b(x), & x\in\mathcal{O}, \end{cases} \tag{1.4}$$

plus the solution of the deterministic problem

$$\begin{cases} {}^C D_t^{\alpha}v(x,t)-Av(x,t)=f(x,t), & (x,t)\in\mathcal{O}\times\mathbb{R}_+,\\ v(x,0)=0,\quad \partial_t v(x,0)=0, & x\in\mathcal{O}. \end{cases} \tag{1.5}$$

The operators ${}^C D_t^{\alpha}$ and I_t^{γ} denote the Caputo fractional derivative of order $\alpha\in(1,2)$ and Riemann-Liouville integral of order $\gamma\in(0,1)$, respectively, defined by

$$\begin{aligned} {}^C D_t^{\alpha}\psi(t)&=\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}\frac{d^2}{ds^2}\psi(s)ds,\\ I_t^{\gamma}\psi(t)&=\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\psi(s)ds=\frac{1}{\Gamma(\gamma)}t^{\gamma-1}*\psi(t), \end{aligned}$$

where $*$ denotes the convolution integral operators

$$(f*g)(t)=\int_0^t f(t-\tau)g(\tau)d\tau.$$

Numerical methods of (1.5) have been widely investigated by various authors. If $f(x,t)$ is smooth in time, one approach involves employing variable time-stepping schemes, such as geometric meshes or graded meshes [23,24]. These schemes are particularly effective in capturing the singularities of the solution at $t=0$. Another method is the utilization of convolution quadrature, which can be generated using BDFk or Lagrange interpolation of degree k, as discussed in [7, 13, 14, 21].

However, the model (1.6) can involve the hyper-singular source term. Since the Riemann-Liouville fractional derivative of order α, is defined by [26, p. 62]

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_0^t (t-\tau)^{1-\alpha} u(\tau) d\tau, \quad 1 < \alpha < 2. \tag{1.6}$$

It makes sense to allow $\partial_t^\alpha u(t)$ to be hyper-singular at $t = 0$ if u is absolutely continuous, e.g.

$$\partial_t^\alpha 1 = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha} \quad \to \quad \infty \quad \text{as} \quad t \quad \to \quad 0, \quad 1 < \alpha < 2.$$

This leads to the fractional evolution equations involving the hyper-singular source term, see [11, Eq. (21)], [17, Eq. (4.2.57)], [25, Eq. (4)] and [6, Eq. (10)]. For this case, many predominant time stepping methods, including the correction of high-order BDF schemes [14], may suffer from a severe order reduction. To recover the desired k-th-order convergence, the authors propose a smoothing method for time stepping schemes, where the singular term is regularized by using an m-fold integral-differential calculus and the equation is discretized by the k-step BDF convolution quadrature, called IDm-BDFk method [29,30]. These framework theories are originally derived from the modified BDF2 schemes (IDm-BDF2), see [4].

In this work, we focus on the numerical approximation of the stochastic time-fractional PDE (1.4) with fractionally integrated white noise based on the modified BDF2 schemes. Note that there are already some important progress for numerically solving the stochastic time-fractional PDEs with fractionally integrated additive noise [1, 3, 12, 15, 16] and integrated white noise [10, 32]. However, these time-stepping methods exhibit a significantly low-order convergence rate and offer less than first-order accuracy.

To the best of our knowledge, we are unaware of any other published work on the high-order numerical approximation for stochastic time-fractional PDE (1.4) with fractionally integrated white noise. Although the high-order numerical algorithm is provided by the authors with fractionally integrated additive noise [3], where the mild solution remains to be provided. Moreover, the resolvent estimate in [3] is not applicable for fractionally integrated white noise, since

$$\sum_{j=1}^{\infty} \sigma_j^2 \quad \to \quad \infty, \quad \sigma_j = 1.$$

To fill in this gap, we first derive the mild solution of (1.5) as

$$u(x,t) \in C^\nu\big([0,T]; L^2\big(\Omega; L^2(\mathcal{O})\big)\big), \quad 0 < \nu < \min\left(\alpha + \gamma - \frac{\alpha d}{4} - \frac{1}{2}, \frac{1}{2}\right).$$

Then superlinear convergence rate

$$\big(\mathbb{E}\big\|\psi(\cdot, t_n) - \psi^n\big\|^2\big)^{\frac{1}{2}} = O(\tau^{\alpha+\gamma-\frac{\alpha d}{4}-\frac{1}{2}-\varepsilon})$$

with arbitrarily small ε term in the exponent is established based on the modified BDF2 methods.

Throughout this paper, we denote c as a generic constant that is independent of the step size τ, which could be different at different occurrences. Additionally, we always assume $\varepsilon > 0$ is sufficiently a small positive constant.

2. Modified BDF2 Methods

For the linear stochastic fractional PDEs with integrated white noise, the predominant time-stepping methods lead to low-order error estimates with $O(\tau^{\min\{\alpha+\gamma-\alpha d/4-1/2-\varepsilon,1\}})$ by ID1-BDF1 method, see [32]. To break the first-order barrier in such time-stepping methods, it motivates us to consider the modified BDF2 method proposed by the authors in [4]. However, if we use the ID1-BDF2 method, it still exists the first-order barrier, see Table 7.1. This motivated us to design the ID2-BDF2 method below.

Let $V(x,t) = u(x,t) - v(x) - tb(x)$, then (1.4) can be written as

$$\begin{cases} \partial_t^\alpha V(x,t) - AV(x,t) = Av(x) + tAb(x) + I_t^\gamma \dot{W}(x,t), & (x,t) \in \mathcal{O} \times \mathbb{R}_+, \\ V(x,0) = 0, \quad \partial_t V(x,0) = 0, & x \in \mathcal{O}. \end{cases} \tag{2.1}$$

Let an approximation of cylindrical Wiener process $W(x,t)$ in (1.2) be defined by

$$W_\ell(x,t) = \sum_{j=1}^{\ell} \beta_j(t)\varphi_j(x), \quad \ell \geq 1. \tag{2.2}$$

Correspondingly, the truncation noise $\dot{W}_\ell(x,t)$ is the time derivative of $W_\ell(x,t)$. Then perturbation equation of (2.1) can be written as

$$\begin{cases} \partial_t^\alpha V_\ell(x,t) - AV_\ell(x,t) = Av(x) + tAb(x) + I_t^\gamma \dot{W}_\ell(x,t), & (x,t) \in \mathcal{O} \times \mathbb{R}_+, \\ V_\ell(x,0) = 0, \quad \partial_t V_\ell(x,0) = 0, & x \in \mathcal{O}. \end{cases} \tag{2.3}$$

The noise term in (2.3) is subjected to a two-step process: first, integration, and then second, differentiation, using an m-fold integral-differential operator. According to indefinite Itô integral [9, p. 70] and Itô stochastic integral [9, p. 66], there exists

$$I_t^1 \dot{W}_\ell(\cdot,t) = \int_0^t dW_\ell = \sum_{j=1}^{\ell} \varphi_j(x) \int_0^t d\beta_j(t) = \sum_{j=1}^{\ell} \beta_j(t)\varphi_j(x) = W_\ell(\cdot,t),$$

where $W_\ell(\cdot,t)$ is uniformly Hölder continuous [9, p. 54] in time for each Hölder exponent $0 < \lambda < 1/2$. From [28, Chapter 1] with $I_t^m \dot{W}_\ell(\cdot,t)\big|_{t=0} = 0$, $m = 1, 2$, we have

$$I_t^\gamma \dot{W}_\ell(\cdot,t) = I_t^\gamma \partial_t^m \left(I_t^m \dot{W}_\ell(\cdot,t)\right) = \partial_t^{m-\gamma}\left(\frac{t^{m-1}}{\Gamma(m)} * \dot{W}_\ell(\cdot,t)\right). \tag{2.4}$$

Choosing $m = 2$ in (2.4) and denoting

$$g_\ell(\cdot,t) = t * \dot{W}_\ell(\cdot,t), \tag{2.5}$$

we have by (2.3),

$$\partial_t^\alpha V_\ell(t) - AV_\ell(t) = \partial_t\left(tAv + \frac{t^2}{2}Ab\right) + \partial_t^{2-\gamma} g_\ell(t). \tag{2.6}$$

Let V_ℓ^n be the approximate solution of $V_\ell(t_n)$ in (2.6). We obtain the following ID2-BDF2 time discretization scheme of (2.6):

$$\partial_\tau^\alpha V_\ell^n - AV_\ell^n = \partial_\tau\left(t_n Av + \frac{t_n^2}{2}Ab\right) + \partial_\tau^{2-\gamma} g_\ell(t_n), \quad n = 1, 2, \ldots, N. \tag{2.7}$$

Here, the discrete fractional-order derivative ∂_τ^β is defined by, with $\beta = 1, \alpha, 2-\gamma$,

$$\partial_\tau^\beta \varphi^n = \tau^{-\beta} \sum_{j=0}^{n} w_j^{(\beta)} \varphi^{n-j}, \tag{2.8}$$

and the weights $w_j^{(\beta)}$ are generated by $\delta_\tau^\beta(\xi) = \sum_{j=0}^\infty w_j^{(\beta)} \xi^j$ with

$$\delta_\tau(\xi) := \tau^{-1}\left(\frac{3}{2} - 2\xi + \frac{1}{2}\xi^2\right). \tag{2.9}$$

Remark 2.1. In this work, we mainly focus on the time semidiscrete schemes, since the spatial discretization is well understood. We shall also modify the initial values v and b in numerical schemes (2.7). Otherwise, it may drop down the convergence rate, see [4, 29, 30].

3. Solution Representation

It is well known that the operator A satisfies the resolvent estimate [21, 31]

$$\|(z-A)^{-1}\| \le c_\theta |z|^{-1}, \quad \forall\, z \in \Sigma_\theta$$

for all $\theta \in (\pi/2, \pi)$, where

$$\Sigma_\theta := \{z \in \mathbb{C}\backslash\{0\} : |\arg z| < \theta\}$$

is a sector of the complex plane. Choose the angle θ such that $\pi/2 < \theta < \min(\pi, \pi/\alpha)$, and it holds

$$\|(z^\alpha - A)^{-1}\| \le c|z|^{-\alpha}, \quad \forall\, z \in \Sigma_\theta. \tag{3.1}$$

Applying the Laplace transform in both sides of (1.5), it leads to

$$\widehat{v}(z) = (z^\alpha - A)^{-1}\widehat{f}(z).$$

By inverse Laplace transform, the deterministic problem (1.5) can be expressed by

$$v(t) = \frac{1}{2\pi i}\int_{\Gamma_{\theta,\kappa}} e^{zt}(z^\alpha - A)^{-1}\widehat{f}(z)dz = \int_0^t E(0, t-s)f(\cdot, s)ds.$$

Here $\Gamma_{\theta,\kappa}$ denotes the contour

$$\Gamma_{\theta,\kappa} = \{z \in \mathbb{C} : |z| = \kappa, |\arg z| \le \theta\} \cup \{z \in \mathbb{C} : z = re^{\pm i\theta}, r \ge \kappa\} \tag{3.2}$$

with $\pi/2 < \theta < \min(\pi, \pi/\alpha)$ and $\kappa > 0$, and the operator $E(q, t)$ is defined by

$$E(q,t)\varphi = \frac{1}{2\pi i}\int_{\Gamma_{\theta,\kappa}} e^{zt}(z^\alpha - A)^{-1} z^{-q}\varphi dz. \tag{3.3}$$

Correspondingly, the mild solution of (2.6) and (2.1) are, respectively, defined as [10]

$$\begin{aligned} V_\ell(t) &= \frac{1}{2\pi i}\int_{\Gamma_{\theta,\kappa}} e^{zt}(z^\alpha - A)^{-1}\big(z^{-1}Av + z^{-2}Ab + z^{2-\gamma}\widehat{g_l}(z)\big)dz \\ &= E(1,t)Av + E(2,t)Ab + \int_0^t E(\gamma, t-s)dW_\ell(\cdot, s) \\ &= E(1,t)Av + E(2,t)Ab + \sum_{j=1}^{\ell}\int_0^t E(\gamma, t-s)\varphi_j d\beta_j(s), \end{aligned} \tag{3.4}$$

$$
\begin{aligned}
V(t) &= E(1,t)Av + E(2,t)Ab + \int_0^t E(\gamma, t-s)dW(\cdot, s) \\
&= E(1,t)Av + E(2,t)Ab + \sum_{j=1}^{\infty}\int_0^t E(\gamma, t-s)\varphi_j d\beta_j(s).
\end{aligned} \tag{3.5}
$$

Here the mild solution $V(t)$ and $V_\ell(t)$ are well-defined in $C^\nu([0,T]; L^2(\Omega; L^2(\mathcal{O})))$ for arbitrary $0 < \nu < \min(\alpha + \gamma - \alpha d/4 - 1/2, 1/2)$; see Section 4.

Given a sequence $(\kappa_n)_0^\infty$ and take $\widetilde{\kappa}(\zeta) = \sum_{n=0}^\infty \kappa_n \zeta^n$ to be its generating power series. The representation of the discrete solution in (2.7) is obtained by the following.

Lemma 3.1. *Let δ_τ be given in* (2.9) *and $\rho_j(\xi) = \sum_{n=1}^\infty n^j \xi^n$ with $j = 1, 2$. Let $g_\ell(t)$ be defined by* (2.5). *Then the discrete solution of* (2.7) *is represented by*

$$
\begin{aligned}
V_\ell^n = &\frac{1}{2\pi i}\int_{\Gamma_{\theta,\kappa}^\tau} e^{zt_n}\big(\delta_\tau^\alpha(e^{-z\tau}) - A\big)^{-1}\delta_\tau(e^{-z\tau})\tau\bigg(\rho_1(e^{-z\tau})\tau Av + \frac{\rho_2(e^{-z\tau})}{2}\tau^2 Ab\bigg)dz \\
&+ \frac{1}{2\pi i}\int_{\Gamma_{\theta,\kappa}^\tau} e^{zt_n}\big(\delta_\tau^\alpha(e^{-z\tau}) - A\big)^{-1}\delta_\tau^{2-\gamma}(e^{-z\tau})\tau\widetilde{g}_\ell(e^{-z\tau})dz
\end{aligned} \tag{3.6}
$$

with $\Gamma_{\theta,\kappa}^\tau = \{z \in \Gamma_{\theta,\kappa} : |\Im z| \le \pi/\tau\}$ and $\widetilde{g}_l(e^{-z\tau}) = \widetilde{(t * \dot{W}_\ell)(t_n)}(e^{-z\tau})$.

Proof. Multiplying the (2.7) by ξ^n and summing over n with $V_\ell^0 = 0$, we obtain

$$
\sum_{n=1}^\infty \partial_\tau^\alpha V_\ell^n \xi^n - \sum_{n=1}^\infty AV_\ell^n\xi^n = \sum_{n=1}^\infty \partial_\tau\bigg(t_nAv + \frac{t_n^2}{2}Ab\bigg) + \sum_{n=1}^\infty \partial_\tau^{2-\gamma} g_\ell(t_n)\xi^n.
$$

Note that

$$
\begin{aligned}
\sum_{n=1}^\infty \partial_\tau^\alpha V_\ell^n\xi^n &= \sum_{n=1}^\infty \frac{1}{\tau^\alpha}\sum_{j=0}^n \omega_j^{(\alpha)} V_\ell^{n-j}\xi^n = \sum_{j=0}^\infty \frac{1}{\tau^\alpha}\sum_{n=j}^\infty \omega_j^{(\alpha)}V_\ell^{n-j}\xi^n \\
&= \sum_{j=0}^\infty \frac{1}{\tau^\alpha}\sum_{n=0}^\infty \omega_j^{(\alpha)}V_\ell^n \xi^{n+j} = \frac{1}{\tau^\alpha}\sum_{j=0}^\infty \omega_j^{(\alpha)}\xi^j\sum_{n=0}^\infty V_\ell^n\xi^n = \delta_\tau^\alpha(\xi)\widetilde{V}_\ell(\xi).
\end{aligned}
$$

Similarly, one has

$$
\sum_{n=1}^\infty \partial_\tau t_nAv\xi^n = \delta_\tau(\xi)\rho_1(\xi)\tau Av, \quad \sum_{n=1}^\infty \partial_\tau t_n^2 Ab\xi^n = \delta_\tau(\xi)\rho_2(\xi)\tau^2 Ab
$$

with $\rho_j(\xi) = \sum_{n=1}^\infty n^j\xi^n$, $j = 1, 2$ and

$$
\sum_{n=1}^\infty \partial_\tau^{2-\gamma} g_\ell(t_n)\xi^n = \sum_{n=1}^\infty \partial_\tau^{2-\gamma} g_\ell^n\xi^n = \delta_\tau^{2-\gamma}(\xi)\widetilde{g}_\ell(\xi).
$$

It leads to

$$
\begin{aligned}
\widetilde{V}_\ell(\xi) = &\big(\delta_\tau^\alpha(\xi) - A\big)^{-1}\delta_\tau(\xi)\bigg(\rho_1(\xi)\tau Av + \frac{\rho_2(\xi)}{2}\tau^2 Ab\bigg) \\
&+ \big(\delta_\tau^\alpha(\xi) - A\big)^{-1}\delta_\tau^{2-\gamma}(\xi)\widetilde{g}_\ell(\xi).
\end{aligned} \tag{3.7}
$$

According to Cauchy's integral formula, and the change of variables $\xi = e^{-z\tau}$, and Cauchy's theorem, one has [14]

$$V_\ell^n = \frac{1}{2\pi i}\int_{\Gamma_{\theta,\kappa}^\tau} e^{zt_n}\big(\delta_\tau^\alpha(e^{-z\tau}) - A\big)^{-1}\delta_\tau(e^{-z\tau})\tau\left(\rho_1(e^{-z\tau})\tau Av + \frac{\rho_2(e^{-z\tau})}{2}\tau^2 Ab\right)dz + \frac{1}{2\pi i}\int_{\Gamma_{\theta,\kappa}^\tau} e^{zt_n}\big(\delta_\tau^\alpha(e^{-z\tau}) - A\big)^{-1}\delta_\tau^{2-\gamma}(e^{-z\tau})\tau\widetilde{g}_\ell(e^{-z\tau})dz$$

with $\Gamma_{\theta,\kappa}^\tau = \{z \in \Gamma_{\theta,\kappa} : |\Im z| \le \pi/\tau\}$. The proof is complete. □

4. Regularity of the Mild Solution

For the stochastic time-fractional equation (2.1) with $0 < \alpha < 1, \gamma = 1 - \alpha$ and $v = b = 0$, the sharp error estimate is established in [10], which convergence rate is consistent with the Hölder regularity of the mild solution

$$u(x,t) \in C^\nu\big([0,T]; L^2\big(\Omega; L^2(\mathcal{O})\big)\big), \quad 0 < \nu < \frac{1}{2} - \frac{\alpha d}{4}.$$

Based on the idea of [10], we next derive the mild solution in (2.1) as

$$u(x,t) \in C^\nu\big([0,T]; L^2\big(\Omega; L^2(\mathcal{O})\big)\big), \quad 0 < \nu < \min\left(\alpha + \gamma - \frac{\alpha d}{4} - \frac{1}{2}, \frac{1}{2}\right).$$

However, the superlinear convergence rate will be established in this work.

Lemma 4.1 ([10, 19]). *Let $\mathcal{O}$ denote a bounded domain in $\mathbb{R}^d$, $d \in 1, 2, 3$. Suppose λ_j denotes the j-th eigenvalue of the Dirichlet boundary problem for the Laplacian operator $-\Delta$ in $\mathcal{O}$. With $|\mathcal{O}|$ denoting the volume of $\mathcal{O}$, we have that*

$$\lambda_j \ge \frac{C_d d}{d+2} j^{\frac{2}{d}} |\mathcal{O}|^{-\frac{2}{d}}, \quad \forall j \ge 1, \tag{4.1}$$

where $C_d = (2\pi)^2 B_d^{-2/d}$ and B_d denotes the volume of the unit d-dimensional ball.

Lemma 4.2 ([10]). *Let $\alpha \ge 0$ and $d = 1, 2, 3$. Then there exist constants C and C_φ such that*

$$\sum_{j=1}^\infty \left(\frac{r^\alpha}{r^\alpha + \lambda_j}\right)^2 \le C r^{\frac{\alpha d}{2}}, \quad \forall r > 0, \tag{4.2}$$

$$\left|\frac{1}{z + \lambda_j}\right| \le \frac{C_\varphi}{|z| + \lambda_j}, \qquad \forall z \in \textstyle\sum_\varphi, \quad \varphi \in (0,\pi), \tag{4.3}$$

where the constant C depends on the dimension d and the volume of the domain $\mathcal{O}$, and C_φ depends on the angle $\varphi \in (0,\pi)$.

Proposition 4.1 ([16, Itô Isometry Property]). *Let $\{\psi(s) : s \in [0,T]\}$ be a real-valued predictable process such that $\int_0^T \mathbb{E}|\psi(s)|^2 ds < \infty$. Let $B(t)$ denote a real-valued standard Brownian motion. Then the following isometry equality holds for $t \in (0,T]$:*

$$\mathbb{E}\left\|\int_0^t \psi(s)dB(s)\right\|^2 = \int_0^t \mathbb{E}\|\psi(s)\|^2 ds,$$

where $\mathbb{E}$ denotes the expectation.

Theorem 4.1. *Let $\upsilon = b = 0$. Then, for arbitrary $0 < \nu < \min(\alpha + \gamma - \alpha d/4 - 1/2, 1/2)$, the mild solution which is defined by* (3.5) *is in* $C^\nu([0,T]; L^2(\Omega; L^2(\mathcal{O})))$.

Proof. From (3.5) and (3.3), it yields

$$\int_0^t E(\gamma, t-s)\varphi_j d\beta_j(s) = \varphi_j \int_0^t h_j(\gamma, t-s) d\beta_j(s) \tag{4.4}$$

with

$$h_j(\gamma, t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta,\kappa}} e^{zt}(z^\alpha + \lambda_j)^{-1} z^{-\gamma} dz. \tag{4.5}$$

According to (4.4), indefinite Itô integral [9, p. 70], it implies that each term in (3.5) is well defined in $C([0,T]; L^2(\Omega; L^2(\mathcal{O})))$. Since

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}\left\| \sum_{j=l}^{l+m} \int_0^t E(\gamma, t-s)\varphi_j d\beta_j(s) \right\|^2 \\
&= \sup_{t\in[0,T]} \int_0^t \sum_{j=l}^{l+m} \|E(\gamma, t-s)\varphi_j\|^2 ds \le \int_0^T \sum_{j=l}^{\infty} |h_j(\gamma, s)|^2 ds \\
&= \int_0^T \sum_{j=l}^{\infty} \left| \frac{1}{2\pi i} \int_{\Gamma_{\theta,\kappa}} e^{zs}(z^\alpha + \lambda_j)^{-1} z^{-\gamma} dz \right|^2 ds \\
&\le \int_0^T \sum_{j=l}^{\infty} \int_{\Gamma_{\theta,\kappa}} |e^{zs}|\,|z|^{-\beta}|dz| \int_{\Gamma_{\theta,\kappa}} |e^{zs}| \left| \frac{z^\alpha}{z^\alpha + \lambda_j} \right|^2 |z|^{-2\alpha-2\gamma+\beta}|dz| ds \\
&\le c \int_0^T s^{\beta-1} \int_\kappa^{+\infty} e^{rs\cos\theta} \sum_{j=l}^{\infty} \left(\frac{r^\alpha}{r^\alpha + \lambda_j} \right)^2 r^{-2\alpha-2\gamma+\beta} dr ds \\
&\quad + c \int_0^T s^{\beta-1} \int_{-\theta}^{\theta} e^{\kappa s \cos\varphi} \sum_{j=l}^{\infty} \left(\frac{\kappa^\alpha}{\kappa^\alpha + \lambda_j} \right)^2 \kappa^{-2\alpha-2\gamma+\beta+1} d\varphi ds.
\end{aligned} \tag{4.6}$$

Here, for the second to last inequality, we use

$$\begin{aligned}
&\int_{\Gamma_{\theta,\kappa}} |e^{zs}|\,|z|^{-\beta}|dz| \\
&= \int_\kappa^{+\infty} e^{rs\cos\theta} r^{-\beta} dr + \int_{-\theta}^{\theta} e^{\kappa s\cos\varphi} \kappa^{-\beta+1} d\varphi \\
&= s^{\beta-1} \left(\int_{\kappa s}^{+\infty} e^{t\cos\theta} t^{-\beta} dt + \int_{-\theta}^{\theta} e^{t\cos\varphi} t^{-\beta+1} d\varphi \right) \le c s^{\beta-1}
\end{aligned} \tag{4.7}$$

with $\beta \in (0,1)$ and $\cos\theta < 0$.

From Lemma 4.2, we have

$$\sum_{j=1}^{\infty} \left(\frac{r^\alpha}{r^\alpha + \lambda_j} \right)^2 \le C r^{\frac{\alpha d}{2}},$$

which implies that the remainder terms

$$\sum_{j=l}^{\infty} \left(\frac{r^\alpha}{r^\alpha + \lambda_j} \right)^2 \;\to\; 0 \quad \text{as} \quad l \;\to\; \infty,$$

and

$$\begin{aligned}
&\int_0^T s^{\beta-1}\int_\kappa^{+\infty} e^{rs\cos\theta}\sum_{j=l}^{\infty}\left(\frac{r^\alpha}{r^\alpha+\lambda_j}\right)^2 r^{-2\alpha-2\gamma+\beta}drds\\
\le{}& c\int_\kappa^{+\infty} r^{-2\alpha-2\gamma+\frac{\alpha d}{2}+\beta}dr\int_0^T s^{\beta-1}e^{rs\cos\theta}ds\\
\le{}& c\int_\kappa^{+\infty} r^{-2\alpha-2\gamma+\frac{\alpha d}{2}}dr \le c\kappa^{-2\alpha-2\gamma+\frac{\alpha d}{2}+1},
\end{aligned} \tag{4.8}$$

where $-2\alpha-2\gamma+\alpha d/2<-1$, and we use

$$\int_0^T s^{\beta-1}e^{rs\cos\theta}ds = r^{-\beta}\int_0^{rT}\eta^{\beta-1}e^{\eta\cos\theta}d\eta < r^{-\beta}\int_0^{\infty}\eta^{\beta-1}e^{\eta\cos\theta}d\eta \le cr^{-\beta}. \tag{4.9}$$

The Lebesgue dominated convergence theorem yields

$$\lim_{l\to\infty}\int_0^T s^{\beta-1}\int_\kappa^{+\infty} e^{rs\cos\theta}\sum_{j=l}^{\infty}\left(\frac{r^\alpha}{r^\alpha+\lambda_j}\right)^2 r^{-2\alpha-2\gamma+\beta}drds=0.$$

On the other hand,

$$\sum_{j=1}^{\infty}\left(\frac{\kappa^\alpha}{\kappa^\alpha+\lambda_j}\right)^2\le C\kappa^{\frac{\alpha d}{2}}$$

implies

$$\sum_{j=l}^{\infty}\left(\frac{\kappa^\alpha}{\kappa^\alpha+\lambda_j}\right)^2 \;\to\; 0 \quad \text{as} \quad l\;\to\;\infty,$$

and

$$\begin{aligned}
&\int_0^T s^{\beta-1}\int_{-\theta}^{\theta} e^{\kappa s\cos\varphi}\sum_{j=l}^{\infty}\left(\frac{\kappa^\alpha}{\kappa^\alpha+\lambda_j}\right)^2 \kappa^{-2\alpha-2\gamma+\beta+1}d\varphi ds\\
\le{}& c\int_{-\theta}^{\theta}\kappa^{-2\alpha-2\gamma+\frac{\alpha d}{2}+\beta+1}d\varphi\int_0^T s^{\beta-1}e^{\kappa s\cos\varphi}ds\\
\le{}& c\int_{-\theta}^{\theta}\kappa^{-2\alpha-2\gamma+\frac{\alpha d}{2}+1}d\varphi\le c\kappa^{-2\alpha-2\gamma+\frac{\alpha d}{2}+1}.
\end{aligned} \tag{4.10}$$

Again, using the Lebesgue dominated convergence theorem, we obtain

$$\lim_{l\to\infty}\int_0^T s^{\beta-1}\int_{-\theta}^{\theta} e^{\kappa s\cos\varphi}\sum_{j=l}^{\infty}\left(\frac{\kappa^\alpha}{\kappa^\alpha+\lambda_j}\right)^2 \kappa^{-2\alpha-2\gamma+\beta+1}d\varphi ds=0.$$

Therefore, we have

$$\sup_{t\in[0,T]}\mathbb{E}\left\|\sum_{j=l}^{l+m}\int_0^t E(\gamma,t-s)\varphi_j d\beta_j(s)\right\|^2 \;\to\; 0 \quad\text{as}\quad l\;\to\;\infty,$$

which means that the sequence $\sum_{j=1}^{l}\int_0^t E(\gamma,t-s)\varphi_j d\beta_j, l=1,2,\ldots,$ is a Cauchy sequence in $C([0,T];L^2(\Omega;L^2(\mathcal{O})))$. Furthermore, the Cauchy sequence converges to the mild solution $V\in C([0,T];L^2(\Omega;L^2(\mathcal{O})))$ in (3.5).

Let L_2^0 denote the space of Hilbert–Schmidt operators on $L^2(\mathcal{O})$. Its norm is defined by [10, 27, 33]

$$\|E(\gamma,t-s)\|_{L_2^0}=\left(\sum_{j=1}^{\infty}\|E(\gamma,t-s)\varphi_j\|_{L^2(\mathcal{O})}^2\right)^{\frac{1}{2}}.$$

By the above analysis, it shows that

$$\int_0^t\|E(\gamma,t-s)\|_{L_2^0}^2ds<\infty,$$

which implies that the stochastic integral in (3.5) is well-defined.

Next, we prove $V\in C^{\nu}([0,T];L^2(\Omega;L^2(\mathcal{O})))$ for $0<\nu<\min(\alpha+\gamma-\alpha d/4-1/2,1/2)$. From (3.5), we have

$$\begin{aligned}&\frac{V(t)-V(t-h)}{h^{\nu}}\\&=\sum_{j=1}^{\infty}\int_0^{t-h}\frac{E(\gamma,t-s)-E(\gamma,t-h-s)}{h^{\nu}}\varphi_j d\beta_j(s)\\&\quad+\frac{1}{h^{\nu}}\sum_{j=1}^{\infty}\int_{t-h}^{t}E(\gamma,t-s)\varphi_j d\beta_j(s).\end{aligned}$$

Then we can check that

$$\begin{aligned}&\sup_{t\in[0,T]}\mathbb{E}\left\|\frac{V(t)-V(t-h)}{h^{\nu}}\right\|^2\\&=\sup_{t\in[0,T]}\left(\sum_{j=1}^{\infty}\int_0^{t-h}\left\|\frac{E(\gamma,t-s)-E(\gamma,t-h-s)}{h^{\nu}}\varphi_j\right\|^2ds+\frac{1}{h^{2\nu}}\sum_{j=1}^{\infty}\int_{t-h}^{t}\|E(\gamma,t-s)\varphi_j\|^2ds\right)\\&\le\sum_{j=1}^{\infty}\int_0^{T}\left\|\frac{E(\gamma,s+h)-E(\gamma,s)}{h^{\nu}}\varphi_j\right\|^2ds+\sum_{j=1}^{\infty}\frac{1}{h^{2\nu}}\int_0^{h}\|E(\gamma,s)\varphi_j\|^2ds.\end{aligned}$$

According to (4.2), (4.3) (4.7), (4.9) and $|(e^{zh}-1)/h^{\nu}|\le C|z|^{\nu}$ with $0<\nu\le 1$ on the contour $\Gamma_{\theta,\kappa}$, it leads to

$$\begin{aligned}&\sum_{j=1}^{\infty}\int_0^{T}\left\|\frac{E(\gamma,s+h)-E(\gamma,s)}{h^{\nu}}\varphi_j\right\|^2ds\\&=\int_0^T\sum_{j=1}^{\infty}\left|\frac{1}{2\pi i}\int_{\Gamma_{\theta,\kappa}}\frac{e^{zh}-1}{h^{\nu}}e^{zs}(z^{\alpha}+\lambda_j)^{-1}z^{-\gamma}dz\right|^2ds\\&\le c\int_0^T\sum_{j=1}^{\infty}\int_{\Gamma_{\theta,\kappa}}|e^{zs}|\,|z|^{-\beta}|dz|\int_{\Gamma_{\theta,\kappa}}|e^{zs}|\left|\frac{z^{\alpha}}{z^{\alpha}+\lambda_j}\right|^2|z|^{-2\alpha-2\gamma+\beta+2\nu}|dz|ds\\&\le c\int_0^T s^{\beta-1}\int_{\kappa}^{+\infty}e^{rs\cos\theta}\sum_{j=1}^{\infty}\left(\frac{r^{\alpha}}{r^{\alpha}+\lambda_j}\right)^2r^{-2\alpha-2\gamma+\beta+2\nu}drds\\&\quad+c\int_0^T s^{\beta-1}\int_{-\theta}^{\theta}e^{\kappa s\cos\varphi}\sum_{j=1}^{\infty}\left(\frac{\kappa^{\alpha}}{\kappa^{\alpha}+\lambda_j}\right)^2\kappa^{-2\alpha-2\gamma+\beta+2\nu+1}d\varphi ds\end{aligned}$$

$$
\begin{aligned}
&\leq c\int_{\kappa}^{+\infty} r^{-2\alpha-2\gamma+\frac{\alpha d}{2}+\beta+2\nu}dr\int_0^T s^{\beta-1}e^{rs\cos\theta}ds\\
&\quad + c\int_{-\theta}^{\theta}\kappa^{-2\alpha-2\gamma+\frac{\alpha d}{2}+\beta+2\nu+1}d\varphi\int_0^T s^{\beta-1}e^{\kappa s\cos\varphi}ds\\
&\leq c\int_{\kappa}^{+\infty} r^{-2\alpha-2\gamma+\frac{\alpha d}{2}+2\nu}dr + c\int_{-\theta}^{\theta}\kappa^{-2\alpha-2\gamma+\frac{\alpha d}{2}+2\nu+1}d\varphi\\
&\leq c\kappa^{-2\alpha-2\gamma+\frac{\alpha d}{2}+2\nu+1},
\end{aligned}
$$

which requires $-2\alpha-2\gamma+\alpha d/2+2\nu<-1$.

On the other hand, for arbitrary $\bar\beta>1$, we have

$$
\begin{aligned}
&\sum_{j=1}^{\infty}\frac{1}{h^{2\nu}}\int_0^h\|E(\gamma,s)\varphi_j\|^2ds\\
=&\frac{1}{h^{2\nu}}\int_0^h\sum_{j=1}^{\infty}\left|\frac{1}{2\pi i}\int_{\Gamma_{\theta,\kappa}}e^{zs}\left(z^\alpha+\lambda_j\right)^{-1}z^{-\gamma}dz\right|^2ds\\
\leq& c\frac{1}{h^{2\nu}}\int_0^h\sum_{j=1}^{\infty}\int_{\Gamma_{\theta,\kappa}}|z|^{-\bar\beta}|dz|\int_{\Gamma_{\theta,\kappa}}|e^{2zs}|\left|\frac{z^\alpha}{z^\alpha+\lambda_j}\right|^2|z|^{-2\alpha-2\gamma+\bar\beta}|dz|ds\\
\leq& c\kappa^{1-\bar\beta}\int_{\kappa}^{+\infty}r^{-2\alpha-2\gamma+\alpha d/2+\bar\beta}\int_0^h\frac{e^{-2rs|\cos\theta|}}{h^{2\nu}}dsdr\\
&+c\kappa^{1-\bar\beta}\int_{-\theta}^{\theta}\kappa^{-2\alpha-2\gamma+\alpha d/2+\bar\beta+1}\int_0^h\frac{e^{2\kappa s\cos\varphi}}{h^{2\nu}}dsd\varphi\\
\leq& c\kappa^{1-\bar\beta}\left(\int_{\kappa}^{+\infty}r^{-2\alpha-2\gamma+\frac{\alpha d}{2}+\bar\beta+2\nu-1}dr+\int_{-\theta}^{\theta}\kappa^{-2\alpha-2\gamma+\frac{\alpha d}{2}+\bar\beta+2\nu}d\varphi\right)\\
\leq& c\kappa^{-2\alpha-2\gamma+\frac{\alpha d}{2}+2\nu+1},
\end{aligned}
$$

where we require $-2\alpha-2\gamma+\alpha d/2+\bar\beta+2\nu-1<-1$ for last second inequality and use

$$
\int_0^h\frac{e^{-2rs|\cos\theta|}}{h^{2\nu}}ds\leq cr^{-1}\cdot\frac{1-e^{-2rh|\cos\theta|}}{h^{2\nu}}\leq cr^{-1}\cdot r^{2\nu},\quad 2\nu\leq 1.
$$

This requires $0<\nu<\min(\alpha+\gamma-\alpha d/4-1/2,1/2)$. The proof is complete. □

Theorem 4.2. *Let $V(t)$ and $V_\ell(t)$ are given in* (3.5) *and* (3.4)*, respectively. Let $\alpha+\gamma-\alpha d/4-1/2>0$ and $\sigma\in(0,2/(\alpha d))$. Then we have*

$$
\sup_{t\in[0,T]}\mathbb{E}\|V(t)-V_\ell(t)\|^2\leq\begin{cases}c\ell^{\max(-2\sigma(\alpha+\gamma-\frac{\alpha d}{4}-\frac{1}{2}),\,(1-\frac{4}{d})(1-\frac{\alpha\sigma d}{2}))}, & \gamma\leq\frac{1}{2},\\ c\ell^{1-\frac{4}{d}}, & \gamma>\frac{1}{2}.\end{cases}
$$

Proof. According to (4.6), there exists

$$
\begin{aligned}
&\sup_{t\in[0,T]}\mathbb{E}\|V(t)-V_\ell(t)\|^2\\
=&\sup_{t\in[0,T]}\mathbb{E}\left\|\sum_{j=\ell+1}^{\infty}\int_0^t E(\gamma,t-s)\varphi_j d\beta_j(s)\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\le c\int_0^T s^{\beta-1}\int_\kappa^{+\infty} e^{rs\cos\theta}\sum_{j=l+1}^{\infty}\left(\frac{r^\alpha}{r^\alpha+\lambda_j}\right)^2 r^{-2\alpha-2\gamma+\beta}drds\\
&\quad + c\int_0^T s^{\beta-1}\int_{-\theta}^{\theta} e^{\kappa s\cos\varphi}\sum_{j=l+1}^{\infty}\left(\frac{\kappa^\alpha}{\kappa^\alpha+\lambda_j}\right)^2 \kappa^{-2\alpha-2\gamma+1+\beta}d\varphi ds\\
&=\int_\kappa^{+\infty}\sum_{j=\ell+1}^{\infty}\left(\frac{r^\alpha}{r^\alpha+\lambda_j}\right)^2 r^{-2\alpha-2\gamma+\beta}dr\int_0^T s^{\beta-1}e^{rs\cos\theta}ds\\
&\quad+\int_{-\theta}^{\theta}\sum_{j=\ell+1}^{\infty}\left(\frac{\kappa^\alpha}{\kappa^\alpha+\lambda_j}\right)^2 \kappa^{-2\alpha-2\gamma+1+\beta}d\varphi\int_0^T s^{\beta-1}e^{\kappa s\cos\varphi}ds.
\end{aligned}
$$

Based on (4.8) and (4.10), we have

$$
\begin{aligned}
&\sup_{t\in[0,T]}\mathbb{E}\|V(t)-V_\ell(t)\|^2\\
&\le c\int_\kappa^{+\infty}\sum_{j=\ell+1}^{\infty}\left(\frac{r^\alpha}{r^\alpha+\lambda_j}\right)^2 r^{-2\alpha-2\gamma}dr\\
&\quad + c\int_{-\theta}^{\theta}\sum_{j=\ell+1}^{\infty}\left(\frac{\kappa^\alpha}{\kappa^\alpha+\lambda_j}\right)^2 \kappa^{-2\alpha-2\gamma+1}d\varphi\\
&= cJ_1+cJ_2.
\end{aligned}
$$

From (4.1) and (4.2), it yields

$$
\begin{aligned}
J_2 &\le \int_{-\theta}^{\theta}\sum_{j=\ell+1}^{\infty}\left(\frac{\kappa^\alpha}{cj^{2/d}}\right)^2 \kappa^{-2\alpha-2\gamma+1}d\varphi\\
&\le c\int_{-\theta}^{\theta}\kappa^{-2\gamma+1}d\varphi\int_\ell^\infty s^{-\frac{4}{d}}ds \le c\kappa^{1-2\gamma}\ell^{1-\frac{4}{d}},\quad \forall\,\gamma>0, \qquad (4.11)\\
J_1 &\le c\int_\kappa^{+\infty}\sum_{j=\ell+1}^{\infty} j^{-\frac{4}{d}}r^{-2\gamma}dr \le c\kappa^{1-2\gamma}\ell^{1-\frac{4}{d}},\qquad \forall\,\gamma>\frac{1}{2}. \qquad (4.12)
\end{aligned}
$$

We next estimate J_1 with $0<\gamma\le 1/2$ for arbitrary $\sigma\in(0,2/(\alpha d))$. Since

$$
\begin{aligned}
J_1 &= \int_\kappa^{\ell^\sigma}\sum_{j=\ell+1}^{\infty}\left(\frac{r^\alpha}{r^\alpha+\lambda_j}\right)^2 r^{-2\alpha-2\gamma}dr\\
&\quad+\int_{\ell^\sigma}^{+\infty}\sum_{j=\ell+1}^{\infty}\left(\frac{r^\alpha}{r^\alpha+\lambda_j}\right)^2 r^{-2\alpha-2\gamma}dr\\
&=: I_1+I_2.
\end{aligned}
$$

From Lemmas 4.1 and 4.2, it follows that

$$
\begin{aligned}
I_2 &\le \int_{\ell^\sigma}^{+\infty} r^{-2\alpha-2\gamma+\frac{\alpha d}{2}}dr \le \ell^{-\sigma(2\alpha+2\gamma-\frac{\alpha d}{2}-1)}, \qquad (4.13)\\
I_1 &\le \int_\kappa^{\ell^\sigma}\sum_{j=\ell+1}^{\infty}\left(\frac{r^\alpha}{r^\alpha+cj^{2/d}}\right)^2 r^{-2\alpha-2\gamma}dr
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_{\kappa}^{\ell^{\sigma}} \int_{\ell}^{\infty} \left(\frac{r^{\alpha}}{r^{\alpha}+cs^{2/d}}\right)^{2} r^{-2\alpha-2\gamma} ds dr \\
&\leq \int_{\kappa}^{\ell^{\sigma}} r^{-2\alpha-2\gamma+\frac{\alpha d}{2}} dr \int_{\frac{\ell}{r^{\alpha d/2}}}^{\infty} \left(\frac{1}{1+c\xi^{2/d}}\right)^{2} d\xi \quad (s = r^{\frac{\alpha d}{2}}\xi) \\
&\leq c \int_{\kappa}^{\ell^{\sigma}} r^{-2\alpha-2\gamma+\frac{\alpha d}{2}} dr \int_{\ell^{1-\alpha\sigma d/2}}^{\infty} \xi^{-\frac{4}{d}} d\xi \qquad (r \leq \ell^{\sigma}) \\
&\leq c \int_{\kappa}^{\ell^{\sigma}} r^{-2\alpha-2\gamma+\frac{\alpha d}{2}} \ell^{(1-\frac{4}{d})(1-\frac{\alpha\sigma d}{2})} dr \\
&\leq c\kappa^{-2\alpha-2\gamma+\frac{\alpha d}{2}+1} \cdot \ell^{(1-\frac{4}{d})(1-\frac{\alpha\sigma d}{2})}.
\end{aligned} \tag{4.14}
$$

From (4.11)-(4.14), it yields

$$
\sup_{t\in[0,T]} \mathbb{E}\|V(t)-V_{\ell}(t)\|^{2} \leq \begin{cases} c\ell^{\max(-\sigma(2\alpha+2\gamma-\frac{\alpha d}{2}-1),(1-\frac{4}{d})(1-\frac{\alpha\sigma d}{2}))}, & \gamma \leq \frac{1}{2}, \\ c\ell^{1-\frac{4}{d}}, & \gamma > \frac{1}{2}. \end{cases}
$$

The proof is complete. □

5. Error Estimates

We first give some lemmas which will be used in the error estimates.

Lemma 5.1 ([14]). *Let $\delta_{\tau}(\xi)$ be given in (2.9). Then there exist the positive constants c_1, c_2, c and $\theta \in (\pi/2, \theta_{\varepsilon})$ with $\theta_{\varepsilon} \in (\pi/2, \pi)$ for any $\varepsilon > 0$ such that*

$$
\begin{aligned}
&c_1|z| \leq |\delta_{\tau}(e^{-z\tau})| \leq c_2|z|, \qquad |\delta_{\tau}(e^{-z\tau}) - z| \leq c\tau^{2}|z|^{3}, \\
&|\delta_{\tau}^{\alpha}(e^{-z\tau}) - z^{\alpha}| \leq c\tau^{2}|z|^{2+\alpha}, \quad \delta_{\tau}(e^{-z\tau}) \in \Sigma_{\pi/2+\varepsilon}, \quad \forall\, z \in \Gamma_{\theta,\kappa}^{\tau}.
\end{aligned}
$$

Lemma 5.2 ([29]). *Let $\delta_{\tau}(\xi)$ be given in (2.9) and $\rho_1(\xi) = \sum_{n=1}^{\infty} n\xi^{n}$. Then there exist a positive constant c such that*

$$
|\rho_1(e^{-z\tau})\tau^{2} - z^{-2}| \leq c\tau^{2},
$$

where $\theta \in (\pi/2, \pi)$ is sufficiently close to $\pi/2$.

Lemma 5.3 ([30]). *Let $\delta_{\tau}(\xi)$ be given by (2.9) and $\rho_j(\xi) = \sum_{n=1}^{\infty} n^{j}\xi^{n}$ with $j = 1, 2$. Then there exists a positive constants c such that*

$$
\left\| \left(\delta_{\tau}^{\alpha}(e^{-z\tau}) - A\right)^{-1} \delta_{\tau}(e^{-z\tau}) \frac{\rho_j(e^{-z\tau})}{\Gamma(j+1)} \tau^{j+1} A - (z^{\alpha} - A)^{-1} z^{-j} A \right\| \leq c\tau^{2}|z|^{2-j}.
$$

Lemma 5.4. *Let $\delta_{\tau}(\xi)$ be given in (2.9) and $\rho_1(\xi) = \sum_{n=1}^{\infty} n\xi^{n}$. Then there exist a positive constants c such that*

$$
\left|\delta_{\tau}^{2-\gamma}(e^{-z\tau})\rho_1(e^{-z\tau})\tau^{2} - z^{-\gamma}\right| \leq c\tau^{2}|z|^{2-\gamma}, \quad \forall\, z \in \Gamma_{\theta,\kappa}^{\tau},
$$

where $\theta \in (\pi/2, \pi)$ is sufficiently close to $\pi/2$.

Proof. Let

$$\delta_\tau^{2-\gamma}(e^{-z\tau})\rho_1(e^{-z\tau})\tau^2 - z^{-\gamma} = J_1 + J_2$$

with

$$\begin{aligned} J_1 &= \delta_\tau^{2-\gamma}(e^{-z\tau})\rho_1(e^{-z\tau})\tau^2 - \delta_\tau^{2-\gamma}(e^{-z\tau})z^{-2}, \\ J_2 &= \delta_\tau^{2-\gamma}(e^{-z\tau})z^{-2} - z^{-\gamma}. \end{aligned}$$

According to Lemmas 5.1 and 5.2, we have

$$\begin{aligned} |J_1| &\leq |\delta_\tau^{2-\gamma}(e^{-z\tau})|\,|\rho_1(e^{-z\tau})\tau^2 - z^{-2}| \leq c\tau^2|z|^{2-\gamma} \\ |J_2| &\leq |\delta_\tau^{2-\gamma}(e^{-z\tau}) - z^{2-\gamma}|\,|z^{-2}| \leq c\tau^2|z|^{2-\gamma}. \end{aligned}$$

By the triangle inequality, the desired result is obtained. □

Lemma 5.5. *Let $\delta_\tau(\xi)$ be given in (2.9) and $\rho_1(\xi) = \sum_{n=1}^{\infty} n\xi^n$. Then there exist a positive constant c such that*

$$\begin{aligned} &\left|\left(z^\alpha + \lambda_j\right)^{-1} z^{-\gamma} - \left(\delta_\tau^\alpha(e^{-z\tau}) + \lambda_j\right)^{-1}\delta_\tau^{2-\gamma}(e^{-z\tau})\rho_1(e^{-z\tau})\tau^2\right|^2 \\ &\leq c\tau^4\left(\frac{|z|^\alpha}{|z|^\alpha + \lambda_j}\right)^2 |z|^{-2\alpha-2\gamma+4}. \end{aligned}$$

Proof. Using the triangle inequality, we have

$$\left|\left(z^\alpha + \lambda_j\right)^{-1} z^{-\gamma} - \left(\delta_\tau^\alpha(e^{-z\tau}) + \lambda_j\right)^{-1}\delta_\tau^{2-\gamma}(e^{-z\tau})\rho_1(e^{-z\tau})\tau^2\right|^2 \leq cJ_1 + cJ_2$$

with

$$\begin{aligned} J_1 &= \left|\left(z^\alpha + \lambda_j\right)^{-1} - \left(\delta_\tau^\alpha(e^{-z\tau}) + \lambda_j\right)^{-1}\right|^2 |z|^{-2\gamma}, \\ J_2 &= \left|\delta_\tau^\alpha(e^{-z\tau}) + \lambda_j\right|^{-2}\left|z^{-\gamma} - \delta_\tau^{2-\gamma}(e^{-z\tau})\rho_1(e^{-z\tau})\tau^2\right|^2. \end{aligned}$$

From Lemmas 4.2, 5.1 and the equality

$$\left(z^\alpha + \lambda_j\right)^{-1} - \left(\delta_\tau^\alpha(e^{-z\tau}) + \lambda_j\right)^{-1} = \left(\delta_\tau^\alpha(e^{-z\tau}) - z^\alpha\right)\left(z^\alpha + \lambda_j\right)^{-1}\left(\delta_\tau^\alpha(e^{-z\tau}) + \lambda_j\right)^{-1},$$

it yields

$$\begin{aligned} J_1 &= c\left|\delta_\tau^\alpha(e^{-z\tau}) - z^\alpha\right|^2\left|z^\alpha + \lambda_j\right|^{-2}\left|\delta_\tau^\alpha(e^{-z\tau}) + \lambda_j\right|^{-2}|z|^{-2\gamma} \\ &= c\left|\delta_\tau^\alpha(e^{-z\tau}) - z^\alpha\right|^2\left(\frac{|z|^\alpha}{\left|z^\alpha + \lambda_j\right|}\right)^2\left|\delta_\tau^\alpha(e^{-z\tau}) + \lambda_j\right|^{-2}|z|^{-2\alpha-2\gamma} \\ &\leq c\tau^4\left(\frac{|z|^\alpha}{|z|^\alpha + \lambda_j}\right)^2|z|^{-2\alpha-2\gamma+4}. \end{aligned}$$

According to Lemmas 4.2 and 5.4, we obtain

$$J_2 \leq c\tau^4\left(\frac{|z|^\alpha}{|z|^\alpha + \lambda_j}\right)^2|z|^{-2\alpha-2\gamma+4}.$$

The proof is complete. □

We now turn to our main theorems.

Theorem 5.1. *Let $V_\ell(t_n)$ and V_ℓ^n be the solutions of (3.4) and (3.6), respectively. Let $\upsilon = b = 0$ and $\alpha+\gamma-\alpha d/4-1/2>0$. Then we have*

$$\left(\mathbb{E}\left\|V_\ell(t_n)-V_\ell^n\right\|^2\right)^{\frac{1}{2}} \le c\tau^{\alpha+\gamma-\frac{\alpha d}{4}-\frac{1}{2}-\varepsilon}t_n^{\varepsilon}.$$

Proof. From (3.4), we obtain

$$V_\ell(t_n) = \big(E(\gamma,t)*\dot{W}_\ell(\cdot,t)\big)(t_n) = \big(\big(\mathscr{E}(t)*t\big)*\dot{W}_\ell(\cdot,t)\big)(t_n) \tag{5.1}$$

with

$$\mathscr{E}(t) = \frac{1}{2\pi i}\int_{\Gamma_{\theta,\kappa}} e^{zt}(z^\alpha-A)^{-1}z^{2-\gamma}dz. \tag{5.2}$$

Using (3.7) with $\upsilon=b=0$ and $g_\ell^n = t_n*\dot{W}_\ell(\cdot,t)$, it yields

$$\begin{aligned}\widetilde{V}_\ell(\xi) &= \big(\delta_\tau^\alpha(\xi)-A\big)^{-1}\delta_\tau^{2-\gamma}(\xi)\widetilde{g}_\ell(\xi) = \widetilde{\mathscr{E}_\tau}(\xi)\widetilde{g}_\ell(\xi) = \sum_{n=0}^\infty \mathscr{E}_\tau^n\xi^n\sum_{j=0}^\infty g_\ell^j\xi^j\\ &= \sum_{n=0}^\infty\sum_{j=0}^\infty \mathscr{E}_\tau^n g_\ell^j\xi^{n+j} = \sum_{j=0}^\infty\sum_{n=j}^\infty \mathscr{E}_\tau^{n-j}g_\ell^j\xi^n = \sum_{n=0}^\infty\sum_{j=0}^n \mathscr{E}_\tau^{n-j}g_\ell^j\xi^n = \sum_{n=0}^\infty V_\ell^n\xi^n.\end{aligned}$$

Here

$$V_\ell^n = \sum_{j=0}^n \mathscr{E}_\tau^{n-j}g_\ell^j = \sum_{j=0}^n \mathscr{E}_\tau^{n-j}g_\ell(t_j),$$

and

$$\sum_{n=0}^\infty \mathscr{E}_\tau^n\xi^n = \widetilde{\mathscr{E}_\tau}(\xi) := \big(\delta_\tau^\alpha(\xi)-A\big)^{-1}\delta_\tau^{2-\gamma}(\xi). \tag{5.3}$$

From the Cauchy's integral formula and the change of variables $\xi = e^{-z\tau}$, we obtain the representation of the $\mathscr{E}_\tau^n$ as following:

$$\mathscr{E}_\tau^n = \frac{1}{2\pi i}\int_{|\xi|=\rho}\xi^{-n-1}\widetilde{\mathscr{E}_\tau}(\xi)d\xi = \frac{\tau}{2\pi i}\int_{\Gamma_{\theta,\kappa}^\tau} e^{zt_n}\big(\delta_\tau^\alpha(e^{-z\tau})-A\big)^{-1}\delta_\tau^{2-\gamma}(e^{-z\tau})dz, \tag{5.4}$$

where $\theta\in(\pi/2,\pi)$ is sufficiently close to $\pi/2$.

Let $\mathscr{E}_\tau(t) = \sum_{n=0}^\infty \mathscr{E}_\tau^n\delta_{t_n}(t)$ with δ_{t_n} being the Dirac delta function at t_n. Then

$$\big(\mathscr{E}_\tau(t)*g_\ell(t)\big)(t_n) = \left(\sum_{j=0}^\infty \mathscr{E}_\tau^j\delta_{t_j}(t)*g_\ell(t)\right)(t_n) = \sum_{j=0}^n \mathscr{E}_\tau^{n-j}g_\ell(t_j) = V_\ell^n. \tag{5.5}$$

Moreover, using the above equation, there exist

$$\begin{aligned}\widetilde{(\mathscr{E}_\tau*t)}(\xi) &= \sum_{n=0}^\infty(\mathscr{E}_\tau*t)(t_n)\,\xi^n = \sum_{n=0}^\infty\sum_{j=0}^n \mathscr{E}_\tau^{n-j}t_j\xi^n = \sum_{j=0}^\infty\sum_{n=0}^\infty \mathscr{E}_\tau^n t_j\xi^{n+j}\\ &= \sum_{n=0}^\infty \mathscr{E}_\tau^n\xi^n\sum_{j=0}^\infty t_j\xi^j = \widetilde{\mathscr{E}_\tau}(\xi)\tau\sum_{j=0}^\infty j\xi^j = \widetilde{\mathscr{E}_\tau}(\xi)\tau\rho_1(\xi)\end{aligned} \tag{5.6}$$

with $\rho_1(\xi) = \sum_{n=1}^\infty n\xi^n$.

From (5.2) and (5.4), we further denote

$$H_j(t) = \frac{1}{2\pi i}\int_{\Gamma_{\theta,\kappa}} e^{zt}\big(z^\alpha+\lambda_j\big)^{-1}z^{2-\gamma}dz, \quad j=1,2,\ldots, \tag{5.7}$$

$$H_{\tau j}^n = \frac{\tau}{2\pi i}\int_{\Gamma_{\theta,\kappa}^\tau} e^{zt_n}\big(\delta_\tau^\alpha(e^{-z\tau})+\lambda_j\big)^{-1}\delta_\tau^{2-\gamma}(e^{-z\tau})dz, \quad j=1,2,\ldots \tag{5.8}$$

with $H_{\tau j}(t)=\sum_{n=0}^\infty H_{\tau j}^n\delta_{t_n}(t)$. Then, using (2.2), (5.1), (5.2), (5.4), (5.5), Itô isometry property and the orthogonality of φ_j, we have

$$\begin{aligned}
&\mathbb{E}\big\|V_\ell(t_n)-V_\ell^n\big\|^2\\
&=\mathbb{E}\big\|\big([(\mathscr{E}-\mathscr{E}_\tau)*t]*\dot{W}_\ell(\cdot,t)\big)(t_n)\big\|^2\\
&=\mathbb{E}\left\|\sum_{j=1}^\ell\int_0^{t_n}\big[\big((\mathscr{E}-\mathscr{E}_\tau)*t\big)(t_n-s)\big]\varphi_j d\beta_j(s)\right\|^2\\
&=\mathbb{E}\left\|\sum_{j=1}^\ell\int_0^{t_n}\big[\big((H_j-H_{\tau j})*t\big)(t_n-s)\big]\varphi_j d\beta_j(s)\right\|^2\\
&=\sum_{j=1}^\ell\int_0^{t_n}\big\|\big[\big((H_j-H_{\tau j})*t\big)(t_n-s)\big]\varphi_j\big\|^2ds\\
&\le\sum_{j=1}^\ell\int_0^{t_n}\big|\big((H_j-H_{\tau j})*t\big)(t_n-s)\big|^2ds.
\end{aligned}\tag{5.9}$$

Next, we prove the following inequality for any $t\in(t_{n-1},t_n)$:

$$\sum_{j=1}^\ell\big|\big((H_j-H_{\tau j})*t\big)(t)\big|^2\le c\tau^{2\alpha+2\gamma-\frac{\alpha d}{2}-1-2\varepsilon}t^{-1+2\varepsilon}.$$

By Talor series expansion of $H_j(t)$ at $t=t_n$ with $j=1,2,\ldots$, we get

$$\big(H_j(t)*t\big)(t)=\big(H_j(t)*t\big)(t_n)+(t-t_n)\big(H_j(t)*1\big)(t_n)+\int_t^{t_n}(s-t)H_j(s)ds,$$

which also holds for the convolution $(H_{\tau j}*t)(t)$. On the other hand, using (5.2), (5.7) and Laplace transform pair, it leads to

$$\big(H_j(t)*t\big)(t_n)=\frac{1}{2\pi i}\int_{\Gamma_{\theta,k}}e^{zt_n}\big(z^\alpha+\lambda_j\big)^{-1}z^{-\gamma}dz. \tag{5.10}$$

From (5.3), (5.6), (5.8) and Cauchy's integral formula, it is easy to get

$$\big(H_{\tau j}(t)*t\big)(t_n)=\frac{\tau}{2\pi i}\int_{\Gamma_{\theta,k}^\tau}e^{zt_n}\big(\delta_\tau^\alpha(e^{-z\tau})+\lambda_j\big)^{-1}\delta_\tau^{2-\gamma}(e^{-z\tau})\rho_1(e^{-z\tau})\tau dz. \tag{5.11}$$

According to (5.10), (5.11) and the triangle inequality, we estimate

$$\sum_{j=1}^\ell\big|\big((H_j-H_{\tau j})*t\big)(t_n)\big|^2\le cJ_1+cJ_2$$

with

$$\begin{aligned}
J_1&=\sum_{j=1}^\ell\left|\frac{1}{2\pi i}\int_{\Gamma_{\theta,k}\backslash\Gamma_{\theta,k}^\tau}e^{zt_n}\big(z^\alpha+\lambda_j\big)^{-1}z^{-\gamma}dz\right|^2,\\
J_2&=\sum_{j=1}^\ell\left|\frac{1}{2\pi i}\int_{\Gamma_{\theta,k}^\tau}e^{zt_n}\big[\big(z^\alpha+\lambda_j\big)^{-1}z^{-\gamma}-\big(\delta_\tau^\alpha(e^{-z\tau})+\lambda_j\big)^{-1}\delta_\tau^{2-\gamma}(e^{-z\tau})\rho_1(e^{-z\tau})\tau^2\big]dz\right|^2.
\end{aligned}$$

From (4.7) and Lemma 4.2, choosing a number $\beta\in(0,1)$ it yields

$$\begin{aligned}
J_1 &\le c\sum_{j=1}^{\ell}\int_{\Gamma_{\theta,k}\backslash\Gamma_{\theta,k}^{\tau}}|e^{zt_n}|\,|z|^{-\beta}|dz|\int_{\Gamma_{\theta,k}\backslash\Gamma_{\theta,k}^{\tau}}|e^{zt_n}|\left|\frac{z^\alpha}{z^\alpha+\lambda_j}\right|^2|z|^{-2\alpha-2\gamma+\beta}|dz|\\
&\le ct_n^{\beta-1}\int_{\frac{\pi}{\tau\sin\theta}}^{+\infty}e^{rt_n\cos\theta}r^{-2\alpha-2\gamma+\frac{\alpha d}{2}+\beta}dr\\
&\le c\tau^4t_n^{\beta-1}\int_{\frac{\pi}{\tau\sin\theta}}^{+\infty}e^{rt_n\cos\theta}r^{-2\alpha-2\gamma+\frac{\alpha d}{2}+4+\beta}dr\\
&= c\tau^4t_n^{\beta-1}t_n^{2\alpha+2\gamma-\frac{\alpha d}{2}-5-\beta}\int_{\frac{\pi t_n}{\tau\sin\theta}}^{+\infty}e^{s\cos\theta}s^{-2\alpha-2\gamma+\frac{\alpha d}{2}+4+\beta}ds\\
&\le c\tau^4t_n^{2\alpha+2\gamma-\frac{\alpha d}{2}-6}\le c\tau^{2\alpha+2\gamma-\frac{\alpha d}{2}-1-2\varepsilon}t^{-1+2\varepsilon}.
\end{aligned}$$

On the other hand, using Lemmas 4.2, 5.5 and the Hölder inequality, it follows that

$$\begin{aligned}
J_2 &\le c\tau^4t_n^{\beta-1}\sum_{j=1}^{\ell}\int_{\Gamma_{\theta,\kappa}^{\tau}}|e^{zt_n}|\left(\frac{|z|^\alpha}{|z|^\alpha+\lambda_j}\right)^2|z|^{-2\alpha-2\gamma+4+\beta}|dz|\\
&\le c\tau^4t_n^{\beta-1}\bigg(\int_{\kappa}^{\frac{\pi}{\tau\sin\theta}}e^{rt_n\cos\theta}r^{-2\alpha-2\gamma+\frac{\alpha d}{2}+4+\beta}dr\\
&\qquad+\int_{-\theta}^{\theta}e^{\kappa t_n\cos\varphi}\kappa^{-2\alpha-2\gamma+\frac{\alpha d}{2}+5+\beta}d\varphi\bigg)\\
&\le c\tau^4t_n^{2\alpha+2\gamma-\frac{\alpha d}{2}-6}\le c\tau^{2\alpha+2\gamma-\frac{\alpha d}{2}-1-2\varepsilon}t^{-1+2\varepsilon}.
\end{aligned}$$

Then, we have

$$\sum_{j=1}^{\ell}\big|\big((H_j-H_{\tau j})*t\big)(t_n)\big|^2\le c\tau^{2\alpha+2\gamma-\frac{\alpha d}{2}-1-2\varepsilon}t^{-1+2\varepsilon}. \tag{5.12}$$

Similarly, it is easy to check

$$\sum_{j=1}^{\ell}\big|(t-t_n)\big((H_j-H_{\tau j})*1\big)(t_n)\big|^2\le c\tau^{2\alpha+2\gamma-\frac{\alpha d}{2}-1-2\varepsilon}t^{-1+2\varepsilon}. \tag{5.13}$$

Moreover, from (4.7), Lemma 4.2, for $\beta\in(0,1)$ and $n\ge 2$, one has

$$\begin{aligned}
&\sum_{j=1}^{\ell}\left|\int_t^{t_n}(s-t)H_j(s)ds\right|^2\\
&\le\sum_{j=1}^{\ell}\int_t^{t_n}(s-t)^2ds\int_t^{t_n}|H_j(s)|^2ds\\
&\le c\sum_{j=1}^{\ell}\tau^3\int_t^{t_n}\left|\frac{1}{2\pi i}\int_{\Gamma_{\theta,\kappa}}e^{zs}\big(z^\alpha+\lambda_j\big)^{-1}z^{2-\gamma}dz\right|^2ds\\
&\le c\sum_{j=1}^{\ell}\tau^3\int_t^{t_n}s^{\beta-1}\int_{\Gamma_{\theta,\kappa}}|e^{zs}|\left|\frac{z^\alpha}{z^\alpha+\lambda_j}\right|^2|z|^{-2\alpha-2\gamma+4+\beta}|dz|ds\\
&\le c\tau^3\int_t^{t_n}s^{\beta-1}\bigg[\int_\kappa^{+\infty}e^{rs\cos\theta}r^{-2\alpha-2\gamma+\frac{\alpha d}{2}+4+\beta}dr
\end{aligned}$$

$$
\begin{aligned}
&\qquad + \int_{-\theta}^{\theta} e^{\kappa s\cos\varphi} \kappa^{-2\alpha-2\gamma+\frac{\alpha d}{2}+5+\beta} d\varphi\Big] ds\\
&\leq c\tau^3 \int_t^{t_n} s^{2\alpha+2\gamma-\frac{\alpha d}{2}-6} ds \leq c\tau^4 t^{2\alpha+2\gamma-\frac{\alpha d}{2}-6}\\
&\leq c\tau^{2\alpha+2\gamma-\frac{\alpha d}{2}-1-2\varepsilon} t^{-1+2\varepsilon}. \qquad (5.14)
\end{aligned}
$$

In particular, for $n = 1$ and $t_1 = \tau$, it yields

$$
\begin{aligned}
&\sum_{j=1}^{\ell} \Big| \int_t^{t_1} (s-t) H_j(s) ds \Big|^2\\
&\leq \sum_{j=1}^{\ell} \int_t^{t_1} s^{-2+2\varepsilon} ds \int_t^{t_1} s^{2-2\varepsilon} (s-t)^2 |H_j(s)|^2 ds\\
&\leq c t^{-1+2\varepsilon} \int_t^{t_1} s^{2\alpha+2\gamma-\frac{\alpha d}{2}-2-2\varepsilon} ds\\
&\leq c\tau^{2\alpha+2\gamma-\frac{\alpha d}{2}-1-2\varepsilon} t^{-1+2\varepsilon}. \qquad (5.15)
\end{aligned}
$$

From (5.8) and $H_{\tau j} = \sum_{n=0}^{\infty} H_{\tau j}^n \delta_{t_n}(t), t \in (t_{n-1}, t_n)$, taking $\beta \in (0,1)$, we deduce

$$
\begin{aligned}
&\sum_{j=1}^{\ell} \Big| \int_t^{t_n} (s-t) H_{\tau j}(s) ds \Big|^2\\
&\leq \sum_{j=1}^{\ell} \big| (t_n - t) H_{\tau j}^n \big|^2 \leq \tau^2 \sum_{j=1}^{\ell} \big| H_{\tau j}^n \big|^2\\
&\leq c\tau^4 \sum_{j=1}^{\ell} \Big| \int_{\Gamma_{\theta,\kappa}^{\tau}} e^{zt_n} \big(\delta_\tau^\alpha(e^{-z\tau}) + \lambda_j\big)^{-1} \delta_\tau^{2-\gamma}(e^{-z\tau}) dz \Big|^2\\
&\leq c\tau^4 \sum_{j=1}^{\ell} \int_{\Gamma_{\theta,\kappa}^{\tau}} |e^{zt_n}|\, |z|^{-\beta} |dz| \int_{\Gamma_{\theta,\kappa}^{\tau}} |e^{zt_n}| \Big| \frac{\delta_\tau^\alpha(e^{-z\tau})}{\delta_\tau^\alpha(e^{-z\tau}) + \lambda_j} \Big|^2 |z|^{-2\alpha-2\gamma+4+\beta} |dz|\\
&\leq c\tau^4 t_n^{\beta-1} \Big(\int_\kappa^{\frac{\pi}{\tau\sin\theta}} e^{rt_n\cos\theta} r^{-2\alpha-2\gamma+\frac{\alpha d}{2}+4+\beta} dr\\
&\qquad + \int_{-\theta}^{\theta} e^{\kappa t_n\cos\varphi} \kappa^{-2\alpha-2\gamma+\frac{\alpha d}{2}+5+\beta} d\varphi \Big)\\
&\leq c\tau^4 t_n^{2\alpha+2\gamma-\frac{\alpha d}{2}-6} \leq c\tau^{2\alpha+2\gamma-\frac{\alpha d}{2}-1-2\varepsilon} t^{-1+2\varepsilon}. \qquad (5.16)
\end{aligned}
$$

According to (5.12)-(5.16) and triangle inequality, we get

$$
\sum_{j=1}^{\ell} \big| \big((H_j - H_{\tau j}) * t\big)(t) \big|^2 \leq c\tau^{2\alpha+2\gamma-\frac{\alpha d}{2}-1-2\varepsilon} t^{-1+2\varepsilon}.
$$

The proof is complete. □

Remark 5.1. From Theorem 5.1, the sharp convergence rates (omitting the ε term in the exponent) can be obtained

$$
\mathbb{E}\big\| V_\ell(t_n) - V_\ell^n \big\|^2 \leq c\tau^{2(\alpha+\gamma-\frac{\alpha d}{4}-\frac{1}{2})},
$$

if the solution is close to the initial layer $t = 0$. However, it is not easy to extend the global domain $t \in (0, T]$.

Theorem 5.2. *Let $V_\ell(t_n)$ and V_ℓ^n be the solutions of* (3.4) *and* (3.6)*, respectively. Let $\upsilon, b \in L^2(\mathcal{O})$ and $\alpha+\gamma-\alpha d/4-1/2>0$. Then we have*

$$\left(\mathbb{E}\left\|V_\ell(t_n)-V_\ell^n\right\|^2\right)^{\frac{1}{2}} \leq c\tau^2 t_n^{-2}\|\upsilon\|_{L^2(\mathcal{O})}+c\tau^2 t_n^{-1}\|b\|_{L^2(\mathcal{O})}+c\tau^{\alpha+\gamma-\frac{\alpha d}{4}-\frac{1}{2}-\varepsilon} t_n^{\varepsilon}.$$

Proof. Subtracting (3.4) from (3.6), we have the following split:

$$V_\ell^n - V_\ell(t_n) = I_1 - I_2 + I_3 - I_4 + I_5.$$

Here the related initial terms are given in

$$\begin{aligned}
I_1 &= \frac{1}{2\pi i}\int_{\Gamma_{\theta,\kappa}^{\tau}} e^{zt_n}\left(\left(\delta_\tau^\alpha(e^{-z\tau})-A\right)^{-1}\delta_\tau(e^{-z\tau})\rho_1(e^{-z\tau})\tau^2 A-(z^\alpha-A)^{-1}z^{-1}A\right)\upsilon dz,\\
I_2 &= \frac{1}{2\pi i}\int_{\Gamma_{\theta,\kappa}\setminus\Gamma_{\theta,\kappa}^{\tau}} e^{zt_n}(z^\alpha-A)^{-1}z^{-1}A\upsilon dz,\\
I_3 &= \frac{1}{2\pi i}\int_{\Gamma_{\theta,\kappa}^{\tau}} e^{zt_n}\left(\left(\delta_\tau^\alpha(e^{-z\tau})-A\right)^{-1}\delta_\tau(e^{-z\tau})\frac{\rho_2(e^{-z\tau})}{2}\tau^3 A-(z^\alpha-A)^{-1}z^{-2}A\right)bdz,\\
I_4 &= \frac{1}{2\pi i}\int_{\Gamma_{\theta,\kappa}\setminus\Gamma_{\theta,\kappa}^{\tau}} e^{zt_n}(z^\alpha-A)^{-1}z^{-2}Abdz.
\end{aligned}$$

The related noise term is

$$\begin{aligned}
I_5 = &\frac{\tau}{2\pi i}\int_{\Gamma_{\theta,\kappa}^{\tau}} e^{zt_n}\left(\delta_\tau^\alpha(e^{-z\tau})-A\right)^{-1}\delta_\tau^{2-\gamma}(e^{-z\tau})\widetilde{g}_\ell(e^{-z\tau})dz\\
&-\frac{1}{2\pi i}\int_{\Gamma_{\theta,\kappa}} e^{zt_n}(z^\gamma-A)^{-1}z^{2-\gamma}\widehat{g}_\ell(z)dz,
\end{aligned}$$

which is estimated in Theorem 5.1.

From Lemma 5.3, we estimate I_1 as following:

$$\begin{aligned}
\|I_1\|_{L^2(\mathcal{O})} &\leq c\tau^2\|\upsilon\|_{L^2(\mathcal{O})}\int_{\Gamma_{\theta,\kappa}^{\tau}} |e^{zt_n}|\,|z|\,|dz|\\
&\leq c\tau^2\|\upsilon\|_{L^2(\mathcal{O})}\left(\int_\kappa^{\frac{\pi}{\tau\sin\theta}} re^{rt_n\cos\theta}dr+\int_{-\theta}^{\theta}\kappa^2 e^{\kappa t_n\cos\varphi}d\varphi\right)\\
&\leq c\tau^2 t_n^{-2}\|\upsilon\|_{L^2(\mathcal{O})},\\
\|I_3\|_{L^2(\mathcal{O})} &\leq c\tau^2 t_n^{-1}\|b\|_{L^2(\mathcal{O})}.
\end{aligned}$$

Correspondingly, using the resolvent estimate (3.1), we estimate I_2 and I_4 as

$$\begin{aligned}
\|I_2\|_{L^2(\mathcal{O})} &\leq c\tau^2 t_n^{-2}\|\upsilon\|_{L^2(\mathcal{O})},\\
\|I_4\|_{L^2(\mathcal{O})} &\leq c\tau^2 t_n^{-1}\|b\|_{L^2(\mathcal{O})}.
\end{aligned}$$

The proof is complete. □

Remark 5.2. For ID1-BDF2 method,

$$\partial_\tau^\alpha V_l^n - AV_l^n = \partial_\tau\left(t_n A\upsilon+\frac{t_n^2}{2}Ab\right)+\partial_\tau^{1-\gamma}g_\ell(t_n),\quad \alpha\in(1,2) \tag{5.17}$$

with $g_\ell(\cdot,t)=1*\dot{W}_\ell(\cdot,t)$, the similar proof with order $O(\tau^{\min(\alpha+\gamma-\alpha d/4-1/2-\varepsilon,1)})$ can be obtained by Theorem 5.2.

6. Error Analysis for Subdiffusion

Consider the subdiffusion model with fractionally integrated white noise, for $0<\alpha,\gamma<1$,

$$\begin{cases} {}^C D_t^{\alpha} u(x,t) - Au(x,t) = I_t^{\gamma}\dot{W}(x,t), & (x,t)\in\mathcal{O}\times\mathbb{R}_+,\\ u(x,0)=\upsilon(x), & x\in\mathcal{O}\end{cases} \tag{6.1}$$

with the initial condition $\upsilon(x)\in L^2(\mathcal{O})$.

Let $V(t)=u(t)-\upsilon$. Then model (6.1) can be rewritten as

$$\begin{cases} \partial_t^{\alpha} V(x,t) - AV(x,t) = A\upsilon(x) + I_t^{\gamma}\dot{W}(x,t), & (x,t)\in\mathcal{O}\times\mathbb{R}_+,\\ V(x,0)=0, & x\in\mathcal{O}.\end{cases} \tag{6.2}$$

Substituting $\dot{W}_\ell(x,t)$ for $\dot{W}(x,t)$ in (6.2), we obtain

$$\begin{cases} \partial_t^{\alpha} V_\ell(x,t) - AV_\ell(x,t) = A\upsilon(x) + I_t^{\gamma}\dot{W}_\ell(x,t), & (x,t)\in\mathcal{O}\times\mathbb{R}_+,\\ V_\ell(x,0)=0, & x\in\mathcal{O}.\end{cases} \tag{6.3}$$

Then ID2-BDF2 method for (6.3) is designed by

$$\partial_\tau^{\alpha} V_\ell^n - AV_\ell^n = \partial_\tau\left(t_n A\upsilon\right) + \partial_\tau^{2-\gamma} g_\ell(t_n) \tag{6.4}$$

with $g_\ell(t)$ given in (2.5).

Using the same argument as in the proof of Theorems 5.1 and 5.2, we can easily carry out the proof of Theorems 6.1 and 6.2 below.

Theorem 6.1. *Let $V_\ell(t_n)$ and V_ℓ^n be the solutions of* (6.3) *and* (6.4)*, respectively. Let $\upsilon=0$ and $\alpha+\gamma-\alpha d/4-1/2>0$. Then we have*

$$\left(\mathbb{E}\left\|V_\ell(t_n)-V_\ell^n\right\|^2\right)^{\frac{1}{2}} \le c\tau^{\alpha+\gamma-\alpha d/4-1/2-\varepsilon} t_n^{\varepsilon}.$$

Theorem 6.2. *Let $V_\ell(t_n)$ and V_ℓ^n be the solutions of* (6.2) *and* (6.3)*, respectively. Let $\upsilon\in L^2(\mathcal{O})$ and $\alpha+\gamma-\alpha d/4-1/2>0$. Then we get*

$$\left(\mathbb{E}\left\|V_\ell(t_n)-V_\ell^n\right\|^2\right)^{\frac{1}{2}} \le c\tau^2 t_n^{-2}\|\upsilon\|_{L^2(\mathcal{O})} + c\tau^{\alpha+\gamma-\frac{\alpha d}{4}-\frac{1}{2}-\varepsilon} t_n^{\varepsilon}.$$

7. Numerical Results

We numerically verify the above theoretical results and the discrete L^2-norm ($\|\cdot\|_{l_2}$) is used to measure the numerical errors at the terminal time, e.g. $t=t_N=1$. We discretize the space direction by the Galerkin finite element method [31]. Here we mainly focus on the time direction convergence order, since the convergence rate of the spatial discretization is well understood [8]. The order of the convergence of the numerical results is computed by

$$\text{Convergence Rate} = \frac{1}{\ln 2}\ln\left(\frac{\|u^{N/2}-u^N\|}{\|u^N-u^{2N}\|}\right),$$

and

$$\left\|u^{\frac{N}{2}}-u^N\right\| := \sqrt{\mathbb{E}\left\|u^{\frac{N}{2}}-u^N\right\|_{l_2}^2}, \quad u^N = V^N + \upsilon.$$

Let

$$T = 1, \quad \mathcal{O} = (0,1), \quad v(x) = \sin(x)\sqrt{1-x^2}, \quad b(x) = \sqrt{x - x^2}.$$

Next, we briefly discuss the implementation of the term involving the noise defined in (2.2), that is

$$\dot{W}_\ell(x,t) = \sum_{j=1}^{\ell} \dot{\beta}_j(t)\varphi_j(x),$$

where $\beta_j, j = 1, 2, \ldots$, are the independently identically distributed Brownian motions, and φ_j are the L^2-norm normalized eigenfunctions of the operator $-\Delta$. In particular, in the one-dimensional case, we have

$$\varphi_j(x) = \sqrt{2}\sin(j\pi x), \quad j = 1, 2, \ldots .$$

Using BDF2 integrals convolution quadrature formula [2, 20], it yields

$$I_t^2 \dot{W}_\ell(x,t) = I_t^1 W_\ell(x,t) \approx \overline{\tau} \sum_{k=1}^{\overline{n}} w_{\overline{n}-k}^{(-1)} \sum_{j=1}^{\ell} \beta_j(\overline{t}_k)\varphi_j$$

with $\ell = M = 100$, where M is the dimension of the finite element space. Here Brownian motions $\{\beta_j\}_{j=1}^{l}$ can be generated by MATLAB code, see [18, p. 395]. Since we do not have an explicit representation of the exact value $I_t^1 W_\ell(t_n, x)$, we compare the numerical integrals with a reference obtained on very fine grids with time step size $\overline{\tau} = 2^{-20}$ and

$$t_n = n\overline{\tau} = n\frac{1}{N} = n\frac{\overline{N}}{N}\overline{\tau} = \overline{n\tau} = \overline{t}_{\overline{n}}, \quad \overline{n} = n\frac{\overline{N}}{N}, \quad \overline{N} = 2^{20}.$$

All the expected values are computed with 1000 trajectories.

For the linear stochastic fractional PDEs with integrated white noise, many predominant time-stepping methods lead to low-order error estimates with $O(\tau^{\min\{\alpha+\gamma-\alpha d/4-1/2-\varepsilon,1\}})$, see [32, Theorem 2.8], also see Table 7.1 by ID1-BDF2 method. Here the white noise (1.2) is regularized by using a one-fold integral-differential (ID1) calculus.

To break the first-order barrier in stochastic fractional PDEs, we establish the convergence properties of the ID2-BDF2 method, demonstrating a convergence order of $O(\tau^{\alpha+\gamma-\alpha d/4-1/2-\varepsilon})$, which is characterized by Theorem 5.2, see Table 7.2.

Furthermore, numerical experiments for the subdiffusion case have been included, as shown in Table 7.3.

Table 7.1: Convergent order of ID1-BDF2 method for superdiffusion (5.17).

γ	$\alpha = 1.2$			$\alpha = 1.6$		
	$N = 128$	$N = 256$	$N = 512$	$N = 128$	$N = 256$	$N = 512$
0.1	1.0880e-02	7.8454e-03	5.6015e-03	5.0279e-03	2.9634e-03	1.7312e-03
		0.4718	0.4860		0.7627	0.7755
0.5	1.8930e-03	1.1055e-03	6.1176e-04	1.2391e-03	6.1293e-04	3.0882e-04
		0.7761	0.8536		1.0155	0.9890
0.9	5.6659e-04	2.8282e-04	1.4199e-04	5.3070e-04	2.5254e-04	1.2482e-04
		1.0024	0.9941		1.0714	1.0166

Table 7.2: Convergent order of ID2-BDF2 method for superdiffusion (2.7).

γ	$\alpha = 1.2$			$\alpha = 1.6$		
	$N = 128$	$N = 256$	$N = 512$	$N = 128$	$N = 256$	$N = 512$
0.1	2.4834e-02	1.7853e-02	1.2854e-02	5.1026e-03	2.9224e-03	1.5534e-03
		0.4762	0.4739		0.8041	0.9117
0.5	1.6463e-03	8.9387e-04	4.8413e-04	5.8338e-04	2.5348e-04	1.0896e-04
		0.8811	0.8847		1.2026	1.2180
0.9	1.2936e-04	5.2761e-05	2.1597e-05	1.3228e-04	4.0064e-05	1.2521e-05
		1.2938	1.2887		1.7232	1.6779

Table 7.3: Convergent order of ID2-BDF2 method for subdiffusion (6.4).

γ	$\alpha = 0.4$			$\alpha = 0.8$		
	$N = 128$	$N = 256$	$N = 512$	$N = 128$	$N = 256$	$N = 512$
0.5	6.2999e-02	5.4221e-02	4.6031e-02	1.0829e-02	7.3419e-03	4.9439e-03
		0.2165	0.2363		0.5607	0.5705
0.9	3.9614e-03	2.5893e-03	1.6829e-03	6.2984e-04	3.1978e-04	1.6417e-04
		0.6135	0.6216		0.9779	0.9619

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