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UNIFORM STABILIZATION AND NUMERICAL ANALYSIS OF A THERMOVISCOELASTIC SYSTEM WITH THE GUYER-KRUMHANSL MODEL*

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Abstract

In this work, we investigate a thermoviscoelastic system governed by the Guyer-Krumhansl model. The system is free from the paradox of infinite heat propagation speed and, furthermore, is more suitable for modeling complex problems involving heterogeneous materials on a macroscale and at room temperature. Firstly, we establish the well-posedness of the system using the theory of semigroups of linear operators, and then we prove uniform exponential decay with respect to a given physical parameter using the multiplier method. Subsequently, we discretize the system and propose a monotone and consistent numerical scheme using finite differences. The convergence of the numerical solution is proven by the Lax equivalence theorem. Finally, we present numerical experiments using MATLAB to demonstrate the accuracy and efficiency of the scheme that reproduce the theoretical results.

Mathematics subject classification: 93D20, 65M06, 74H40.
Key words: Wave equation, Viscoelasticity, Guyer-Krumhansl model, Uniform exponential decay, Finite differences.

1. Introduction

Conventional thermoelasticity theory is based, among other constitutive relations, on Fourier's law [11]. The foundations of this theory were likely laid in the first half of the 19th century by Duhamel [10] and subsequently reformulated by Biot [6], who introduced an elegant model to study the coupling effects between elastic and thermal fields. Because of this, this theory became known as classical coupled thermoelasticity theory. However, due to the dependence on Fourier's law, this theory also suffers from the deficiency of allowing infinite heat propagation

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speed. To address this issue, Cattaneo [8] derived a new equation to relate heat flux and temperature. He introduced the thermal relaxation time, which is intrinsic to the material, to the heat flux. In this way, the problem of infinite heat propagation speed was resolved in the context of thermodynamics. However, the issue persisted in the context of thermoelasticity until the emergence of Lord-Schulman's generalized thermoelasticity theory [15], which modified the classical theory of coupled thermoelasticity by considering the heat flux idealized by Cattaneo. Consequently, the finite speed of heat propagation was also extended to thermoelasticity.

In this context, several works [4, 5, 9, 20, 23] have studied the well-posedness and asymptotic behavior of solutions to thermoelastic systems with the Cattaneo model. For instance, Racke [20] proved, for two distinct boundary conditions, the exponential stabilization of a thermoelastic system governed by the Cattaneo model. On the other hand, Alves *et al.* [4] considered the problem of vibrations in a non-uniform flexible structure modeled by a viscoelastic equation with thermal effect provided by the Cattaneo model. They established the well-posedness of the system, exponential decay for a set of boundary conditions, and polynomial decay for another set of boundary conditions. Ávalos *et al.* [5] considered a thermoviscoelastic system with the Cattaneo model, modeling a string composed of three different materials: thermoelastic, viscoelastic, and elastic. They proved that exponential stability depends on the position of each material, i.e. the model is exponentially stable if and only if the viscoelastic material is not in the center of the string. Otherwise, there is no exponential stability, and the corresponding semigroup decays polynomially.

One of the reasons researchers have been studying thermoelastic systems with the Cattaneo law is to achieve finite speed for heat propagation. However, the Cattaneo model hyperbolizes the system and can, therefore, produce oscillatory solutions for temperature, known as second sound – a phenomenon first observed in liquid helium [19] and later in solid crystals through appropriately designed experiments [1, 2, 16, 17]. On the other hand, for a broader range of engineering applications, such as in heterogeneous materials at macroscopic scales and at room temperature, we highlight the Guyer-Krumhansl (GK) model [12, 13]. This model generalizes the Cattaneo model and, furthermore, results in the heat transport known as over-diffusive. Recently, Ramos *et al.* [21] demonstrated the well-posedness and uniform exponential stabilization of the GK model with respect to two physical parameters. They showed, through the energy functional, that the dynamics of the GK model can be reduced to the dynamics of limit systems determined by the Cattaneo and Fourier models. Additionally, they employed the finite difference method to discretize the problem and conducted numerical simulations illustrating the theoretical results.

Since Zhukovsky's work [27], it was believed that the GK model violated non-negativity and the maximum principle. However, Ramos *et al.* [22] proved that the exact solution of the GK model does not violate non-negativity and satisfy the maximum principle. Continuing with the GK model, recent experiments [7, 26] have underscored that the GK model is the most suitable for modeling complex materials, positioning it as the next standard model in engineering.

Although the GK model is more suitable in many practical applications, there is no existing literature that studies its behavior when coupled with mechanical equations. Therefore, this is the first work to consider a thermoviscoelastic wave equation, where thermal effects are governed by the GK model. More precisely, we have

$$\begin{cases} u_{tt} - u_{xx} - \beta u_{xxt} + m\theta_x = 0 \quad \text{in } (0,\ell)\times(0,\infty), & \text{(1.1a)} \\ \rho c\theta_t + q_x + m u_{xt} = 0 \quad \text{in } (0,\ell)\times(0,\infty), & \text{(1.1b)} \\ \tau_0 q_t + q - \mu^2 q_{xx} + k\theta_x = 0 \quad \text{in } (0,\ell)\times(0,\infty), & \text{(1.1c)} \end{cases}$$

where u denotes the displacement, θ is the difference of temperature, q is the heat flux and $\rho, \beta, c, k, \mu^2, m, \tau_0$ are positive constants representing mass density, viscoelasticity coefficient, specific heat, thermal conductivity, dissipation coefficient, coupling coefficient, and relaxation time, respectively. The system is subject to boundary conditions

$$u(0,t) = u(\ell,t) = q(0,t) = q(\ell,t) = 0, \quad \forall\, t \geq 0, \tag{1.2}$$

and initial conditions

$$u(x,0) = f_0(x), \quad u_t(x,0) = f_1(x), \quad \theta(x,0) = g_0(x), \quad q(x,0) = h_0(x), \quad x \in (0,\ell). \tag{1.3}$$

We introduce the functional energy of the system (1.1)-(1.3) given by

$$E(t) := \frac{1}{2}\int_0^\ell u_t^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell u_x^2 \mathrm{d}x + \frac{\rho c}{2}\int_0^\ell \theta^2 \mathrm{d}x + \frac{\tau_0}{2k}\int_0^\ell q^2 \mathrm{d}x, \tag{1.4}$$

which decreases to the equilibrium point with time tending to infinity, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\beta\int_0^\ell u_{xt}^2 \mathrm{d}x - \frac{1}{k}\int_0^\ell q^2 \mathrm{d}x - \frac{\mu^2}{k}\int_0^\ell q_x^2 \mathrm{d}x < 0, \quad \forall\, t \geq 0. \tag{1.5}$$

Consequently, $E(t)$ is a non-increasing monotone function of the time variable t.

To support the integration of the system (1.1)-(1.3) as the next standard model in engineering, we must thoroughly study its mathematical properties regarding well-posedness and the asymptotic behavior of solutions. It is also crucial to develop robust numerical methodologies capable of providing practical and efficient algorithms for numerical simulations. In this regard, we have developed a hybrid approach that combines explicit and implicit finite difference methods. To illustrate the type of difficulty encountered in this approach and how our methodology addresses it, we use a toy model.

Toy model. Consider a coupled finite difference numerical scheme of the form

$$\begin{cases} U^{n+1} = F(U^n, U^{n-1}, V^n), \qquad \forall\, n = 0,1,\ldots,N, & \text{(1.6a)} \\ V^n = G(U^{n+1}, U^{n-1}, V^{n-1}), \quad \forall\, n = 1,2,\ldots,N, & \text{(1.6b)} \\ U^0 = \big(f(x_1), f(x_2), \cdots, f(x_m)\big)^\top, \quad U^1 - U^{-1} = 2\Delta t\big(g(x_1), g(x_2), \cdots, g(x_m)\big)^\top, & \text{(1.6c)} \\ V^0 = \big(h(x_1), h(x_2), \cdots, h(x_m)\big)^\top & \text{(1.6d)} \end{cases}$$

with $F : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{m+1} \to \mathbb{R}^m$, $G : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ being multilinear transformations and $f, g, h : \mathbb{R} \to \mathbb{R}$ scalar functions. The solution to (1.6) is given by $(U^n, V^n) \in \mathbb{R}^m \times \mathbb{R}^{m+1}$, where $U^n = (u_1^n, u_2^n, \cdots, u_m^n)^\top$ and $V^n = (v_0^n, v_1^n, \cdots, v_m^n)^\top$ are vectors. Observe that the initial conditions determine (U^0, V^0) however, to find (U^1, V^1), i.e.

$$\begin{cases} U^1 = F(U^0, U^{-1}, V^0), \\ V^1 = G(U^2, U^0, V^0), \end{cases}$$

we need the vectors $\{U^{-1}, U^2\}$. The vector U^{-1} can be expressed in terms of U^1 due to the initial conditions and the linearity of F. However, the vector U^2 is unknown, as we are still calculating the solution in the previous step. Our methodology consists of making specific combinations between Eqs. (1.6a) and (1.6b) to eliminate the appearance of terms like U^{n+1} when we are still computing U^n.

Main contributions of the paper. The main contributions of this paper can be highlighted as follows:

- We studied a new thermoviscoelastic system free from the infinite heat propagation speed, which generalizes the limit system governed by the Cattaneo model.
- We proved the well-posedness and uniform exponential decay with respect to the parameter μ^2, which allows us to extend our results to the limit system.
- We developed a monotone and consistent numerical scheme for the numerical approximation of the system (1.1)-(1.3), which can also be used for simulations of the limit system.
- We validated the numerical scheme with tests that accurately demonstrate the theoretical results.

Organization of the paper. In Section 2, we used the theory of linear operator semigroups to prove well-posedness. In Section 3, we employed the multiplier method to establish uniform exponential stabilization with respect to the parameter μ^2. In Section 4, we presented a finite difference numerical scheme capable of reproducing qualitative properties of the solution. Additionally, we proved the convergence of the numerical solution using the Lax equivalence theorem. In Section 5, we present numerical simulations that accurately and efficiently illustrate the theoretical results of this work. Finally, in Section 6, we discussed the difficulty of obtaining monotonicity in the numerical scheme and the importance of correctly reproducing the multiplier method in the numerical context.

2. Well-posedness

In this section, we show that the system (1.1)-(1.3) is well-posedness using the semigroup technique. Let us start by introducing Hilbert spaces

$$L^2(0,\ell) = \left\{ f : (0,\ell) \to \mathbb{C};\ \int_0^\ell |f(x)|^2 \mathrm{d}x < \infty \right\},$$
$$L^2_*(0,\ell) = \left\{ f \in L^2(0,\ell);\ \int_0^\ell f(x) \mathrm{d}x = 0 \right\}$$

with the norm

$$\|f\|_2^2 = \int_0^\ell |f(x)|^2 \mathrm{d}x,$$

and the subsets of $L^2(0,\ell)$

$$H^1(0,\ell) = \big\{ f \in L^2(0,\ell);\ f_x \in L^2(0,\ell) \big\},$$
$$H^1_0(0,\ell) = \big\{ f \in H^1(0,\ell);\ f(0) = f(\ell) = 0 \big\}.$$

Now we define the phase space

$$\mathcal{H} := H^1_0(0,\ell) \times L^2(0,\ell) \times L^2(0,\ell) \times L^2(0,\ell),$$

which is a Hilbert space endowed with the norm

$$\|\Phi\|^2_{\mathcal{H}} = \|v\|_2^2 + \|u_x\|_2^2 + \rho c \|\theta\|_2^2 + \frac{\tau_0}{k}\|q\|_2^2, \tag{2.1}$$

and the corresponding inner product

$$\left\langle \Phi, \Phi^* \right\rangle_{\mathcal{H}} = \int_0^\ell v\overline{v^*}dx + \int_0^\ell u_x\overline{u_x^*}dx + \rho c\int_0^\ell \theta\overline{\theta^*}dx + \frac{\tau_0}{k}\int_0^\ell q\overline{q^*}dx$$

for all $\Phi = (u, v, \theta, q)^\top \in \mathcal{H}$. Thus, denoting $u := u(x,t)$, $v := u_t(x,t)$, $\theta := \theta(x,t)$ and $q := q(x,t)$ we can transform the system (1.1)-(1.3) into the first-order set of equations

$$\begin{cases} \Phi_t(t) = \mathcal{A}\Phi(t), \quad \forall\, t > 0, \\ \Phi(0) = \Phi_0(x), \end{cases} \tag{2.2}$$

where

$$\Phi_0(x) = \big(f_0(x),\, f_1(x),\, g_0(x),\, h_0(x)\big)^\top,$$

and $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is the differential operator given by

$$\mathcal{A} := \begin{pmatrix} 0 & I_d(\cdot) & 0 & 0 \\ \dfrac{\partial^2(\cdot)}{\partial x^2} & \beta\dfrac{\partial^2(\cdot)}{\partial x^2} & -m\dfrac{\partial(\cdot)}{\partial x} & 0 \\ 0 & -\dfrac{m}{\rho c}\dfrac{\partial(\cdot)}{\partial x} & 0 & -\dfrac{1}{\rho c}\dfrac{\partial(\cdot)}{\partial x} \\ 0 & 0 & -\dfrac{k}{\tau_0}\dfrac{\partial(\cdot)}{\partial x} & -\dfrac{1}{\tau_0}I_d(\cdot) + \dfrac{\mu^2}{\tau_0}\dfrac{\partial^2(\cdot)}{\partial x^2} \end{pmatrix},$$

where $I_d(\cdot)$ denotes the identity operator, furthermore

$$\begin{aligned} D(\mathcal{A}) := \big\{ & U = (u, v, \theta, q)^\top \in \mathcal{H};\ v,\, q \in H_0^1(0,\ell), \\ & u_x + \beta v_x - m\theta \in H^1(0,\ell),\ \mu^2 q_x - k\theta \in H^1(0,\ell) \big\}. \end{aligned} \tag{2.3}$$

Theorem 2.1. *The system* (2.2) *is well-posed, i.e. for any* $\Phi_0 \in \mathcal{H}$*, the system* (2.2) *has a unique mild solution* $\Phi(t) = e^{t\mathcal{A}}\Phi_0 \in C(\mathbb{R}^+; \mathcal{H})$*. Furthermore, if* $\Phi_0 \in D(\mathcal{A})$*,* $\Phi(t) \in C^1(\mathbb{R}^+; \mathcal{H}) \cap C(\mathbb{R}^+; D(\mathcal{A}))$ *becomes the unique classical solution for* (2.2).

Proof. The proof is divided into two parts, where the title of each part indicates its main purpose.

(i) **The linear operator $\mathcal{A}$ is dissipative in $\mathcal{H}$.** It can be shown that $\mathcal{A}$ is a dissipative operator in the space $\mathcal{H}$. More precisely, we have

$$\begin{aligned} \langle \mathcal{A}U, U\rangle_{\mathcal{H}} = {} & i\mathrm{Im}\left\{ \int_0^\ell u_x\overline{v}_x \mathrm{d}x \right\} + im\mathrm{Im}\left\{ \int_0^\ell \theta_x\overline{v} \mathrm{d}x \right\} + ik\mathrm{Im}\left\{ \int_0^\ell \theta_x\overline{v} \mathrm{d}x \right\} \\ & - \beta\|v_x\|_2^2 - \frac{1}{k}\|q\|_2^2 - \frac{\mu^2}{k}\|q_x\|_2^2. \end{aligned}$$

Consequently,

$$\mathrm{Re}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\beta\|v_x\|_2^2 - \frac{1}{k}\|q\|_2^2 - \frac{\mu^2}{k}\|q_x\|_2^2.$$

This implies that $\mathcal{A}$ is dissipative in $\mathcal{H}$.

(ii) **The linear operator $\lambda I-\mathcal{A}$ is surjective in $\mathcal{H}$.** Next, we prove that $\lambda I-\mathcal{A}$ is surjective, for all $\lambda>0$. For this purpose, given $F=(f_1,f_2,f_3,f_4)^\top\in\mathcal{H}$, we seek $U=(u,v,\theta,q)^\top\in D(\mathcal{A})$ which is the solution of

$$(\lambda I-\mathcal{A})U=F,$$

i.e. the entries of U satisfy the system of equations

$$\begin{cases} \lambda u - v = f_1 \in H_0^1(0,\ell), & \text{(2.4a)}\\ \lambda v - u_{xx} - \beta v_{xx} + m\theta_x = f_2 \in L^2(0,\ell), & \text{(2.4b)}\\ \lambda\rho c\theta + q_x + mv_x = \rho c f_3 \in L^2(0,\ell), & \text{(2.4c)}\\ \lambda\tau_0 q + q - \mu^2 q_{xx} + k\theta_x = \tau_0 f_4 \in L^2(0,\ell). & \text{(2.4d)} \end{cases}$$

From (2.4a) we have that $v=\lambda u - f_1$, so we have the following system:

$$\begin{cases} \lambda^2 u - (1+\lambda\beta)u_{xx} + m\theta_x = \lambda f_1 - \beta f_{1,xx} + f_2 & \text{in } (0,\ell),\\ \lambda\rho c\theta + q_x + \lambda m u_x = m f_{1,x} + \rho c f_3 & \text{in } (0,\ell),\\ (\lambda\tau_0+1)q - \mu^2 q_{xx} + k\theta_x = \tau_0 f_4 & \text{in } (0,\ell), \end{cases} \tag{2.5}$$

where $f_{1,x}$ and $f_{1,xx}$ denote the first and second-order derivatives of f_1 with respect to x, respectively.

Solving the system (2.5) is equivalent to finding

$$(u,\theta,q)\in H_0^1(0,\ell)\times L^2(0,\ell)\times H_0^1(0,\ell),$$

such that

$$\begin{cases} \lambda^2\int_0^\ell u\overline{u}^*\mathrm{d}x + (1+\lambda\beta)\int_0^\ell u_x\overline{u}_x^*\mathrm{d}x - m\int_0^\ell \theta\overline{u}_x^*\mathrm{d}x\\ = \lambda\int_0^\ell f_1\overline{u}^*\mathrm{d}x + \beta\int_0^\ell f_{1,x}\overline{u}_x^*\mathrm{d}x + \int_0^\ell f_2\overline{u}^*\mathrm{d}x,\\ \rho c\int_0^\ell \theta\overline{\theta}^*\mathrm{d}x + \frac{1}{\lambda}\int_0^\ell q_x\overline{\theta}^*\mathrm{d}x + m\int_0^\ell u_x\overline{\theta}^*\mathrm{d}x = \frac{m}{\lambda}\int_0^\ell f_{1,x}\overline{\theta}^*\mathrm{d}x + \frac{\rho c}{\lambda}\int_0^\ell f_3\overline{\theta}^*\mathrm{d}x,\\ \frac{\lambda\tau_0+1}{\lambda k}\int_0^\ell q\overline{q}^*\mathrm{d}x + \frac{1}{\lambda k}\int_0^\ell (\mu^2 q_x - k\theta)\overline{q}_x^*\mathrm{d}x = \frac{\tau_0}{\lambda k}\int_0^\ell f_4\overline{q}^*\mathrm{d}x \end{cases}$$

for all $(u^*,\theta^*,q^*)\in H_0^1(0,\ell)\times L^2(0,\ell)\times H_0^1(0,\ell)$. Now we observe that solving the system (2.5) is equivalent to solve the problem

$$\mathcal{S}\big((u,\theta,q),\,(u^*,\theta^*,q^*)\big) = \mathcal{F}(u^*,\theta^*,q^*), \tag{2.6}$$

where $\mathcal{S}:[H_0^1(0,\ell)\times L^2(0,\ell)\times H_0^1(0,\ell)]^2\to\mathbb{C}$ is the sesquilinear form given by

$$\begin{aligned} &\mathcal{S}\big((u,\theta,q),(u^*,\theta^*,q^*)\big)\\ &:= \lambda^2\int_0^\ell u\overline{u}^*\mathrm{d}x + (1+\lambda\beta)\int_0^\ell u_x\overline{u}_x^*\mathrm{d}x - m\int_0^\ell \theta\overline{u}_x^*\mathrm{d}x\\ &\quad + \rho c\int_0^\ell \theta\overline{\theta}^*\mathrm{d}x + \frac{1}{\lambda}\int_0^\ell q_x\overline{\theta}^*\mathrm{d}x + m\int_0^\ell u_x\overline{\theta}^*\mathrm{d}x\\ &\quad + \frac{\lambda\tau_0+1}{\lambda k}\int_0^\ell q\overline{q}^*\mathrm{d}x + \frac{1}{\lambda k}\int_0^\ell \big(\mu^2 q_x - k\theta\big)\overline{q}_x^*\mathrm{d}x, \end{aligned}$$

and $\mathcal{F} : H_0^1(0,\ell) \times L^2(0,\ell) \times H_0^1(0,\ell) \to \mathbb{C}$ is the antilinear form given by

$$\mathcal{F}(u^*,\theta^*,q^*) := \lambda \int_0^\ell f_1 \overline{u}^* \mathrm{d}x + \beta \int_0^\ell f_{1,x} \overline{u}_x^* \mathrm{d}x + \int_0^\ell f_2 \overline{u}^* \mathrm{d}x + \frac{m}{\lambda} \int_0^\ell f_{1,x} \overline{\theta}^* \mathrm{d}x + \frac{\rho c}{\lambda} \int_0^\ell f_3 \overline{\theta}^* \mathrm{d}x + \frac{\tau_0}{\lambda k} \int_0^\ell f_4 \overline{q}^* \mathrm{d}x.$$

Now we introduce the Hilbert space $\mathcal{V} := H_0^1(0,\ell) \times L^2(0,\ell) \times H_0^1(0,\ell)$ equipped with the norm

$$\|(u,\theta,q)\|_{\mathcal{V}}^2 = \|u\|_2^2 + \|u_x\|_2^2 + \|\theta\|_2^2 + \|q\|_2^2 + \|q_x\|_2^2.$$

It is easy to see that $\mathcal{S}$ and $\mathcal{F}$ are bounded. Furthermore, there exists a positive constant C such that

$$\begin{aligned} &\mathrm{Re}\big\{\mathcal{S}\big((u,\theta,q),(u,\theta,q)\big)\big\} \\ &= \lambda^2 \|u\|_2^2 + (1+\lambda\beta)\|u_x\|_2^2 + \rho c\|\theta\|_2^2 + \frac{\lambda\tau_0 + 1}{\lambda k}\|q\|_2^2 + \frac{\mu^2}{\lambda k}\|q_x\|_2^2 \\ &\geq C\|(u,\theta,q)\|_{\mathcal{V}}^2, \end{aligned}$$

which implies that $\mathcal{S}$ is coercive.

Hence, we assert that $\mathcal{S}$ is continuous and coercive form on $\mathcal{V} \times \mathcal{V}$ and $\mathcal{F}$ is continuous form on $\mathcal{V}$. From Lax-Milgram's theorem, we conclude that there exists only one solution satisfying

$$(u,\theta,q) \in H_0^1(0,\ell) \times L^2(0,\ell) \times H_0^1(0,\ell).$$

Furthermore, it follows that $v = \lambda u - f_1 \in H_0^1(0,\ell)$, and from Eqs. (2.4b) and (2.4d) that $u_x + \beta v_x - m\theta \in H^1(0,\ell)$ and $\mu^2 q_x - k\theta \in H^1(0,\ell)$. Thus, the operator $\lambda I - \mathcal{A}$ is surjective for all $\lambda > 0$.

Since $D(\mathcal{A})$ is dense in $\mathcal{H}(\overline{D(\mathcal{A})} = \mathcal{H})$, $\mathcal{A}$ is a dissipative operator and $\lambda I - \mathcal{A}$ is surjective, then $\mathcal{A}$ is an infinitesimal generator of a C_0-semigroup of contractions (see Lumer-Phillips theorem [18]). It follows from the semigroup theory of linear operators (see Pazy [18]), that $\Phi(t) \in C^1(\mathbb{R}^+; \mathcal{H}) \cap C(\mathbb{R}^+; D(\mathcal{A}))$, i.e. $\Phi(t)$ is a global solution of (2.2). The proof is done. □

3. Uniform Exponential Decay

In this section, we prove that the energy functional of the system (1.1)-(1.3) exhibits uniform exponential decay with respect to the parameter μ^2 for an equilibrium point that depends on the space where the initial condition g_0 is embedded. More precisely, we demonstrate that the dynamics of the system (1.1)-(1.3) governed by the GK model can be reduced to the dynamics of the limit system determined by the Cattaneo model when $\mu^2 \to 0$, i.e.

$$\begin{cases} u_{tt} - u_{xx} - \beta u_{xxt} + m\theta_x = 0 & \text{in } (0,\ell) \times (0,\infty), \\ \rho c\theta_t + q_x + m u_{xt} = 0 & \text{in } (0,\ell) \times (0,\infty), \\ \tau_0 q_t + q + k\theta_x = 0 & \text{in } (0,\ell) \times (0,\infty). \end{cases} \tag{3.1}$$

In both cases, the energy functional exhibits exponential decay to an equilibrium point that depends on the space of g_0.

Before presenting the main result of this section, we state and prove two technical lemmas that will be used later.

Lemma 3.1. *Let* (u, u_t, θ, q) *be a strong solution of the system* (1.1)-(1.3). *Then the functional*

$$F(t) := \int_0^\ell u_t \overline{u} \mathrm{d}x + \frac{\beta}{2} \|u_x\|_2^2$$

satisfies the following estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Re}\{F(t)\} \leq -\frac{1}{2}\|u_x\|_2^2 + \rho\|u_t\|_2^2 + \frac{m^2}{2}\|\theta\|_2^2, \quad \forall t \geq 0.$$

Proof. Multiplying Eq. (1.1a) by $\overline{u}$ and integrating over $(0, \ell)$ we obtain

$$\int_0^\ell u_{tt}\overline{u}\mathrm{d}x + \|u_x\|_2^2 + \frac{\beta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_x\|_2^2 - m\int_0^\ell \theta \overline{u}_x \mathrm{d}x = 0.$$

Using the identity

$$\int_0^\ell u_{tt}\overline{u}\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\ell u_t \overline{u}\mathrm{d}x - \|u_t\|_2^2,$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}F(t) = -\|u_x\|_2^2 + \|u_t\|_2^2 + m\int_0^\ell \theta\overline{u}_x\mathrm{d}x,$$

in which

$$F(t) := \int_0^\ell u_t\overline{u}\mathrm{d}x + \frac{\beta}{2}\|u_x\|_2^2.$$

Using Young's inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}\{F(t)\} \leq -\frac{1}{2}\|u_x\|_2^2 + \|u_t\|_2^2 + \frac{m^2}{2}\|\theta\|_2^2. \tag{3.2}$$

The proof is complete. □

Lemma 3.2. *Let* (u, u_t, θ, q) *be a strong solution of the system* (1.1)-(1.3). *Then the functional*

$$G(t) := \frac{\rho c}{2}\left\| \int_0^x \theta(s,t)\mathrm{d}s \right\|_2^2 + \frac{\rho c \mu^2}{2}\|\theta\|_2^2 - \tau_0 \int_0^\ell q \int_0^x \overline{\theta}(s,t)\mathrm{d}s\mathrm{d}x$$

satisfies the following estimate:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}\{G(t)\} \leq{}& -\frac{k}{2}\|\theta\|_2^2 + \frac{\tau_0}{\rho c}\left(1 + \frac{m}{2}\right)\|q\|_2^2 + m\left(\frac{\ell}{k} + \frac{\tau_0}{2\rho c}\right)\|u_t\|_2^2 \\ &+ \frac{m^2\mu^4}{k}\|u_{xt}\|_2^2 + |\Lambda(t)|, \quad \forall\, t \geq 0,\end{aligned}$$

where

$$\Lambda(t) := -\big(\mu^2 q_x(\ell,t) - k\theta(\ell,t)\big)\int_0^\ell \overline{\theta}(s,t)\mathrm{d}s.$$

Proof. Integrating Eq. (1.1b) over $(0,x) \subset (0,\ell)$, multiplying by $\int_0^x \overline{\theta}(s,t)\mathrm{d}s$ and integrating over $(0,\ell)$ we obtain

$$\rho c \int_0^\ell \left(\int_0^x \theta_t(s,t)\mathrm{d}s \int_0^x \overline{\theta}(s,t)\mathrm{d}s \right)\mathrm{d}x + \int_0^\ell q \int_0^x \overline{\theta}(s,t)\mathrm{d}s\mathrm{d}x + m\int_0^\ell u_t \int_0^x \overline{\theta}(s,t)\mathrm{d}s\mathrm{d}x = 0.$$

When using the identity

$$\int_0^\ell \left(\int_0^x \theta_t(s,t)\mathrm{d}s \int_0^x \theta(s,t)\mathrm{d}s \right)\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\| \int_0^x \theta(s,t)\mathrm{d}s \right\|_2^2,$$

we have

$$\frac{\rho c}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\| \int_0^x \theta(s,t)\mathrm{d}s \right\|_2^2 + \int_0^\ell q \int_0^x \overline{\theta}(s,t)\mathrm{d}s\mathrm{d}x + m\int_0^\ell u_t \int_0^x \overline{\theta}(s,t)\mathrm{d}s\mathrm{d}x = 0. \tag{3.3}$$

On the other hand, multiplying Eq. (1.1c) by $\int_0^x \overline{\theta}(s,t)\mathrm{d}s$ and integrating over $(0,\ell)$ we have

$$\tau_0 \int_0^\ell q_t \int_0^x \overline{\theta}(s,t)\mathrm{d}s\mathrm{d}x + \int_0^\ell q \int_0^x \overline{\theta}(s,t)\mathrm{d}s\mathrm{d}x + \Lambda(t) + \int_0^\ell (\mu^2 q_x - k\theta)\overline{\theta}\mathrm{d}x = 0, \tag{3.4}$$

where

$$\Lambda(t) := -\big(\mu^2 q_x(\ell,t) - k\theta(\ell,t)\big)\int_0^\ell \overline{\theta}(s,t)\mathrm{d}s.$$

Subtracting Eqs. (3.3) and (3.4), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\rho c}{2}\left\| \int_0^x \theta(s,t)\mathrm{d}s \right\|_2^2 &+ m\int_0^\ell u_t \int_0^x \overline{\theta}(s,t)\mathrm{d}s\mathrm{d}x \\ &- \underbrace{\left(\tau_0 \int_0^\ell q_t \int_0^x \overline{\theta}(s,t)\mathrm{d}s\mathrm{d}x + \mu^2 \int_0^\ell q_x\overline{\theta}\mathrm{d}x \right)}_{I:=} - \Lambda(t) + k\|\theta\|_2^2 = 0.\end{aligned} \tag{3.5}$$

Exploiting the identity

$$\int_0^\ell q_t \int_0^x \overline{\theta}(s,t)\mathrm{d}s\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\ell q \int_0^x \overline{\theta}(s,t)\mathrm{d}s\mathrm{d}x - \int_0^\ell q \int_0^x \overline{\theta}_t(s,t)\mathrm{d}s\mathrm{d}x$$

together with the Eq. (1.1b), it yield

$$\begin{aligned} I &:= \tau_0 \int_0^\ell q_t \int_0^x \overline{\theta}(s,t)\mathrm{d}s\mathrm{d}x + \mu^2 \int_0^\ell q_x \overline{\theta}\mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\tau_0 \int_0^\ell q \int_0^x \overline{\theta}(s,t)\mathrm{d}s\mathrm{d}x - \tau_0 \int_0^\ell q \int_0^x \overline{\theta}_t(s,t)\mathrm{d}s\mathrm{d}x + \mu^2 \int_0^\ell q_x\overline{\theta}\mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\tau_0 \int_0^\ell q \int_0^x \overline{\theta}(s,t)\mathrm{d}s\mathrm{d}x + \frac{\tau_0}{\rho c}\int_0^\ell q(\overline{q} + m\overline{u}_t)\mathrm{d}x - \mu^2 \int_0^\ell (\rho c\theta_t + m u_{xt})\overline{\theta}\mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\tau_0 \int_0^\ell q \int_0^x \overline{\theta}(s,t)\mathrm{d}s\mathrm{d}x - \frac{\rho c\mu^2}{2}\|\theta\|_2^2 \right) + \frac{\tau_0}{\rho c}\|q\|_2^2 + \frac{m\tau_0}{\rho c}\int_0^\ell q\overline{u}_t\mathrm{d}x - m\mu^2 \int_0^\ell u_{xt}\overline{\theta}\mathrm{d}x. \end{aligned}$$

Substituting I into (3.5) we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}G(t) = -k\|\theta\|_2^2 &+ \frac{\tau_0}{\rho c}\|q\|_2^2 - m\int_0^\ell u_t \int_0^x \overline{\theta}(s,t)\mathrm{d}s\mathrm{d}x \\ &+ \frac{m\tau_0}{\rho c}\int_0^\ell q\overline{u}_t\mathrm{d}x - m\mu^2 \int_0^\ell u_{xt}\overline{\theta}\mathrm{d}x + \Lambda(t),\end{aligned}$$

where

$$G(t) := \frac{\rho c}{2}\left\| \int_0^x \theta(s,t)\mathrm{d}s \right\|_2^2 + \frac{\rho c\mu^2}{2}\|\theta\|_2^2 - \tau_0 \int_0^\ell q \int_0^x \overline{\theta}(s,t)\mathrm{d}s\mathrm{d}x.$$

Using Cauchy-Schwarz, Young's and Poincaré inequalities we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}\{G(t)\} \leq -\frac{k}{2}\|\theta\|_2^2 + \frac{\tau_0}{\rho c}\left(1+\frac{m}{2}\right)\|q\|_2^2 + m\left(\frac{\ell}{k}+\frac{\tau_0}{2\rho c}\right)\|u_t\|_2^2 + \frac{m^2\mu^4}{k}\|u_{xt}\|_2^2 + |\Lambda(t)|. \tag{3.6}$$

The proof is complete. □

Now, we state and prove the main result of this section.

Theorem 3.1. *Let $E(t)$ be the energy functional of the system* (1.1)-(1.3). *Suppose the initial data $(f_0, f_1, g_0, h_0) \in D(\mathcal{A})$. Then, there exist uniformly bounded constants $M, C, \omega > 0$ such that $E(t)$ satisfies the following exponential decay estimates:*

(I) If $g_0 \in L^2(0,\ell)$ then

$$E(t) \leq ME(0)e^{-\omega t} + C\|\Lambda\|_\infty, \quad \forall\, t \geq 0, \tag{3.7}$$

where

$$\Lambda(t) := -\big(\mu^2 q_x(\ell,t) - k\theta(\ell,t)\big)\int_0^\ell \overline{\theta}(s,t)\mathrm{d}s,$$
$$\|\Lambda\|_\infty := \sup\{|\Lambda(t)|;\ t\in\mathbb{R}^+\}.$$

(II) If $g_0 \in L^2_*(0,\ell)$ then

$$E(t) \leq ME(0)e^{-\omega t}, \quad \forall\, t \geq 0. \tag{3.8}$$

Proof. We start the proof by defining the following Lyapunov functional:

$$\mathcal{L}(t) := N_1E(t) + \mathrm{Re}\{F(t)\} + N_2\mathrm{Re}\{G(t)\}, \quad \forall\, t\geq 0, \tag{3.9}$$

where F and G are defined in Lemmas 3.1 and 3.2 and $N_i, i = 1, 2$, will be chosen later. Using the Cauchy-Schwarz, Young and Poincaré inequalities we obtain

$$\big|\mathcal{L}(t) - N_1E(t)\big| \leq \left|\int_0^\ell u_t\overline{u}\mathrm{d}x\right| + \frac{\beta}{2}\|u_x\|_2^2 + N_2\frac{\rho c}{2}\left\|\int_0^x \theta(s,t)\mathrm{d}s\right\|_2^2 + N_2\frac{\rho c\mu^2}{2}\|\theta\|_2^2 + N_2\tau_0\left|\int_0^\ell q\int_0^x \overline{\theta}(s,t)\mathrm{d}s\mathrm{d}x\right| \leq \eta E(t),$$

where $\eta > 0$ is a positive constant. Consequently,

$$(N_1 - \eta)E(t) \leq \mathcal{L}(t) \leq (N_1+\eta)E(t), \quad \forall\, t\geq 0.$$

Taking $N_1 > \eta$ we find the positive constants c_1 and c_2 such that

$$c_1E(t) \leq \mathcal{L}(t) \leq c_2E(t), \quad \forall\, t \geq 0. \tag{3.10}$$

This proves that E and $\mathcal{L}$ are equivalent. Now using Eqs. (1.5), (3.2), (3.6) and (3.9) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) \leq -\left(N_1\beta - N_2\frac{m^2\mu^4}{k}\right)\|u_{tx}\|_2^2 + \left(1+N_2m\left(\frac{\ell}{k}+\frac{\tau_0}{2\rho c}\right)\right)\|u_t\|_2^2 - \frac{1}{2}\|u_x\|_2^2 - \left(N_2\frac{k}{2}-\frac{m^2}{2}\right)\|\theta\|_2^2 - \frac{1}{k}\left(N_1 - N_2\frac{\tau_0}{\rho c}k\left(1+\frac{m}{2}\right)\right)\|q\|_2^2 - N_1\frac{\mu^2}{k}\|q_x\|_2^2 + N_2|\Lambda(t)|.$$

Using Poincaré inequality and eliminating unnecessary terms, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) \le -\left[\frac{N_1\beta}{c_p} - 1 - N_2 m\left(\frac{m\mu^4}{c_p k} + \frac{\ell}{k} + \frac{\tau_0}{2\rho c}\right)\right]\|u_t\|_2^2 - \frac{1}{2}\|u_x\|_2^2 - \frac{1}{2}\left(N_2 k - m^2\right)\|\theta\|_2^2 - \frac{1}{k}\left[N_1 - N_2\frac{\tau_0}{\rho c}k\left(1+\frac{m}{2}\right)\right]\|q\|_2^2 + N_2|\Lambda(t)|,$$

where $c_p > 0$. Choosing $N_2 > m^2/k$ and

$$N_1 > \max\left\{\frac{c_p}{\beta}\left[1 + N_2 m\left(\frac{m\mu^4}{c_p k} + \frac{\ell}{k} + \frac{\tau_0}{2\rho c}\right)\right],\ N_2\frac{\tau_0}{\rho c}k\left(1+\frac{m}{2}\right),\ \eta\right\},$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) \le -\xi_1\|u_t\|_2^2 - \xi_2\|u_x\|_2^2 - \xi_3\|\theta\|_2^2 - \xi_4\|q\|_2^2 + N_2|\Lambda(t)|,$$

where

$$\xi_1 := \frac{N_1\beta}{c_p} - 1 - N_2 m\left(\frac{m\mu^4}{c_p k} + \frac{\ell}{k} + \frac{\tau_0}{2\rho c}\right) > 0, \quad \xi_2 := \frac{1}{2} > 0,$$
$$\xi_3 := \frac{1}{2}\left(N_2 k - m^2\right) > 0, \quad \xi_4 := \frac{1}{k}\left[N_1 - N_2\frac{\tau_0}{\rho c}k\left(1+\frac{m}{2}\right)\right] > 0.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) \le -\delta E(t) + N_2|\Lambda(t)|,$$

where

$$\delta := \min\left\{2\xi_1,\ 2\xi_2,\ \frac{2\xi_3}{\rho c},\ \frac{2k\xi_4}{\tau_0}\right\}.$$

Using the second inequality of (3.10) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) \le -\omega\mathcal{L}(t) + N_2|\Lambda(t)|, \tag{3.11}$$

where $\omega := \delta/c_2$. Multiplying (3.11) by $e^{\omega t}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{L}(t)e^{\omega t}\right) \le N_2|\Lambda(t)|e^{\omega t}. \tag{3.12}$$

Integrating (3.12) over $[0,t]$, we get

$$\mathcal{L}(t) \le \mathcal{L}(0)e^{-\omega t} + N_2\int_0^t |\Lambda(r)|e^{-\omega(t-r)}\mathrm{d}r, \quad \forall\, t \ge 0,$$

where $r \in [0,t]$. Choosing $M := c_2/c_1$, $\gamma_0 := N_2/c_1$ and considering (3.10), we finally obtain

$$E(t) \le ME(0)e^{-\omega t} + \gamma_0\int_0^t |\Lambda(r)|e^{-\omega(t-r)}\mathrm{d}r, \quad \forall\, t \ge 0. \tag{3.13}$$

On the other hand, integrating the conjugate of Eq. (1.1b) over $(0,\ell)$, using the boundary conditions (1.2), and denoting $z(t) := \int_0^\ell \overline{\theta}(x,t)\mathrm{d}x$, we get the initial value problem

$$\begin{cases} z'(t) = 0, \quad t \ge 0, \\ z(0) = \displaystyle\int_0^\ell g_0(x)\,\mathrm{d}x, \end{cases}$$

whose solution is $z(t) = \int_0^\ell g_0(x)\,\mathrm{d}x$. Therefore, $\int_0^\ell \overline{\theta}(x,t)\mathrm{d}x = \int_0^\ell g_0(x)\,\mathrm{d}x$ is bounded over time t. Furthermore, noting that $\Phi(t)$ is a global solution, due to Sobolev immersion (continuous injection) $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and $\mu^2 q_x - k\theta \in H^1(0,\ell)$ we conclude that $|\Lambda(t)|$ is bounded on every $\mathbb{R}^+ := [0,\infty)$ and hence $\Lambda \in L^\infty(\mathbb{R}^+)$. Consequently if the initial condition $g_0 \notin L^2_*(0,\ell)$ then (3.13) implies

$$\begin{aligned} E(t) &\leq ME(0)e^{-\omega t} + (1-e^{-\omega t})\frac{\gamma_0}{\omega}\|\Lambda\|_\infty \\ &\leq ME(0)e^{-\omega t} + \frac{\gamma_0}{\omega}\|\Lambda\|_\infty, \quad \forall\, t \geq 0. \end{aligned}$$

This proves estimate (I). Otherwise, if $g_0 \in L^2_*(0,\ell)$, implies that $\Lambda(t) = 0$. Consequently from (3.13), we immediately obtain the proof of (II). □

4. Fully Discrete Finite Difference Model

In this section, we present a finite difference numerical scheme for the system (1.1)-(1.3) in $(0,\ell)\times(0,T)$ that combines explicit methods for the wave equation and implicit methods for the thermal equations. We prove the convergence of the numerical solution and propose a numerical algorithm to simulate the approximate solution of the system.

Considering $J, N \in \mathbb{N}$, we define the mesh parameters $\Delta x = \ell/(J+1), \Delta t = T/(N+1)$, the set $I_n = \{1, 2, \ldots, n\}$, and the mesh points

$$\begin{aligned} &0 = x_0 < x_1 < \cdots < x_j = j\Delta x < \cdots < x_J < x_{J+1} = \ell, \\ &0 = t_0 < t_1 < \cdots < t_n = n\Delta t < \cdots < t_N < t_{N+1} = T, \end{aligned}$$

followed by the discretization of (1.1)-(1.3)

$$\begin{cases} \overline{\partial}_t \circ \partial_t u_j^n - \nabla_x \circ \overline{\nabla}_x u_j^n \\ \quad -\beta \overline{\nabla}_x \circ \nabla_x \left(\dfrac{\overline{\partial}_t + \partial_t}{2} u_j^n \right) + m \overline{\nabla}_x \theta_j^n = 0, \quad j \in I_J, \quad n \in I_N, & \text{(4.1a)} \\ \rho c \overline{\partial}_t \theta_j^n + \nabla_x q_j^n + m \nabla_x \left(\dfrac{\overline{\partial}_t + \partial_t}{2} u_j^n \right) = 0, \quad j \in I_J \cup \{0\}, \quad n \in I_N, & \text{(4.1b)} \\ \tau_0 \overline{\partial}_t q_j^n + q_j^n - \mu^2 \nabla_x \circ \overline{\nabla}_x q_j^n + k \overline{\nabla}_x \theta_j^n = 0, \quad j \in I_J, \quad n \in I_N, & \text{(4.1c)} \end{cases}$$

that provides a consistent numerical scheme of order $\mathcal{O}(\Delta t, \Delta x)$, where

$$\begin{aligned} &\partial_t u_j^n := \frac{u_j^{n+1} - u_j^n}{\Delta t}, && \overline{\partial}_t u_j^n := \frac{u_j^n - u_j^{n-1}}{\Delta t}, \\ &\frac{\overline{\partial}_t + \partial_t}{2} u_j^n := \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t}, && \overline{\partial}_t \circ \partial_t u_j^n := \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} \end{aligned}$$

are finite difference approximations in time t_n, and

$$\begin{aligned} &\nabla_x u_j^n := \frac{u_{j+1}^n - u_j^n}{\Delta x}, && \overline{\nabla}_x u_j^n := \frac{u_j^n - u_{j-1}^n}{\Delta x}, \\ &\frac{\overline{\nabla}_x + \nabla_x}{2} u_j^n := \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}, && \overline{\nabla}_x \circ \nabla_x u_j^n := \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \end{aligned}$$

are finite difference approximations in space x_j. We denote the numerical approximations of (u, θ, q) at the mesh points (x_j, t_n) by $(u_j^n, \theta_j^n, q_j^n)$. For boundary conditions, we use

$$u_0^n = u_{J+1}^n = q_0^n = q_{J+1}^n = 0, \quad \forall\, n \in I_{N+1} \cup \{0\}, \tag{4.2}$$

and for the initial conditions, we adopted

$$u_j^0 = f_0(x_j), \quad \frac{u_j^1 - u_j^{-1}}{2\Delta t} = f_1(x_j), \quad \theta_j^0 = g_0(x_j), \quad q_j^0 = h_0(x_j), \quad \forall\, j \in I_{J+1} \cup \{0\}. \tag{4.3}$$

4.1. Solvability

Due to the combination of explicit and implicit methods in (4.1), we need to develop a uniquely solvable algorithm to plot the numerical solution in the system (1.1)-(1.3). For this, we rewrite the numerical scheme (4.1)-(4.3) in vector form given by

$$\begin{cases} \mathbb{U}^{n+1} = \mathbf{B}^{-1}\Big(2\mathbf{I}_J + \dfrac{\Delta t^2}{\Delta x^2}\mathbf{L}\Big)\mathbb{U}^n - \mathbf{B}^{-1}\Big(\mathbf{I}_J + \dfrac{\beta\Delta t}{2\Delta x^2}\mathbf{L}\Big)\mathbb{U}^{n-1} \\ \qquad - \dfrac{m\Delta t^2}{\Delta x}\mathbf{B}^{-1}\mathbf{A}\Theta^n, \qquad n \in I_N \cup \{0\}, & \text{(4.4a)} \\ \Theta^n = \Theta^{n-1} - \dfrac{\Delta t}{\rho c \Delta x}\mathbf{C}\mathbb{Q}^n - \dfrac{m}{2\rho c\Delta x}\mathbf{C}(\mathbb{U}^{n+1} - \mathbb{U}^{n-1}), \qquad n \in I_N, & \text{(4.4b)} \\ \mathbb{Q}^n = \dfrac{\tau_0}{\tau_0 + \Delta t}\mathbb{Q}^{n-1} + \dfrac{\mu^2 \Delta t}{(\tau_0 + \Delta t)\Delta x^2}\mathbf{L}\mathbb{Q}^n - \dfrac{k\Delta t}{(\tau_0+\Delta t)\Delta x}\mathbf{A}\Theta^n, \quad n \in I_N, & \text{(4.4c)} \\ \mathbb{U}^0 = \big(f_0(x_1), f_0(x_2), \cdots, f_0(x_J)\big)^\top, \quad \mathbb{U}^1 - \mathbb{U}^{-1} = 2\Delta t\big(f_1(x_1), f_1(x_2), \cdots, f_1(x_J)\big)^\top, & \text{(4.4d)} \\ \Theta^0 = \big(g_0(x_0), g_0(x_1), \cdots, g_0(x_J)\big)^\top, \quad \mathbb{Q}^0 = \big(h_0(x_1), h_0(x_2), \cdots, h_0(x_J)\big)^\top, & \text{(4.4e)} \end{cases}$$

where we denote

$$\mathbb{U}^n = \big(u_1^n, u_2^n, \cdots, u_J^n\big)^\top, \quad \Theta^n = \big(\theta_0^n, \theta_1^n, \cdots, \theta_J^n\big)^\top, \quad \mathbb{Q}^n = \big(q_1^n, q_2^n, \cdots, q_J^n\big)^\top,$$

and we define the mass matrix

$$\mathbf{B} := \mathbf{I}_J - \frac{\beta\Delta t}{2\Delta x^2}\mathbf{L}.$$

Furthermore we denote by $\mathbf{L}$ and $\mathbf{I}_p$ the tridiagonal matrix associated to the approximation of the Laplacian and $p \times p$ identity matrix respectively, i.e.

$$\mathbf{L} := \begin{pmatrix} -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \ddots & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 0 & 1 & -2 \end{pmatrix}_{J\times J}, \quad \mathbf{I}_p := \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix}_{p\times p}.$$

On the other hand, **A** and **C** are trapezoidal matrices given by

$$\mathbf{A} := \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \ddots & \ddots & \vdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 & -1 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & -1 & 1 \end{pmatrix}_{J\times(J+1)},$$

$$\mathbf{C} := \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \ddots & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & -1 & 1 & 0 \\ 0 & \cdots & 0 & 0 & -1 & 1 \\ 0 & \cdots & 0 & 0 & 0 & -1 \end{pmatrix}_{(J+1)\times J}.$$

This is a coupled linear system of equations in variables Θ^n and $\mathbb{Q}^n$.

4.1.1. Uncoupled system

To uncouple the linear system of Eqs. (4.4), we perform the following procedure: Replacing (4.4b) in (4.4c) we have

$$\begin{aligned} \mathbb{Q}^n = {} & \frac{\tau_0}{\tau_0 + \Delta t}\mathbf{D}^{-1}\mathbb{Q}^{n-1} - \frac{k\Delta t}{(\tau_0 + \Delta t)\Delta x}\mathbf{D}^{-1}\mathbf{A}\Theta^{n-1} \\ & + \frac{mk\Delta t}{2\rho c(\tau_0 + \Delta t)\Delta x}\mathbf{D}^{-1}\mathbf{A}\mathbf{C}(\mathbb{U}^{n+1} - \mathbb{U}^{n-1}), \end{aligned} \tag{4.5}$$

where

$$\mathbf{D} := \mathbf{I}_J - \frac{\mu^2\Delta t}{(\tau_0 + \Delta t)\Delta x^2}\mathbf{L} - \frac{k\Delta t^2}{\rho c(\tau_0 + \Delta t)\Delta x^2}\mathbf{A}\mathbf{C}.$$

Replacing, in turn, (4.5) in (4.4b) we obtain

$$\Theta^n = \mathbf{E}\Theta^{n-1} - \frac{\tau_0\Delta t}{\rho c(\tau_0 + \Delta t)\Delta x}\mathbf{C}\mathbf{D}^{-1}\mathbb{Q}^{n-1} - \frac{m}{2\rho c\Delta x}\widetilde{\mathbf{E}}\mathbf{C}(\mathbb{U}^{n+1} - \mathbb{U}^{n-1}), \tag{4.6}$$

where

$$\begin{aligned} \mathbf{E} &:= \mathbf{I}_{J+1} + \frac{k\Delta t^2}{\rho c(\tau_0 + \Delta t)\Delta x^2}\mathbf{C}\mathbf{D}^{-1}\mathbf{A}, \\ \widetilde{\mathbf{E}} &:= \mathbf{I}_{J+1} + \frac{k\Delta t^2}{\rho c(\tau_0 + \Delta t)\Delta x}\mathbf{C}\mathbf{D}^{-1}\mathbf{A}. \end{aligned}$$

Finally, substituting (4.4a) into (4.5) and (4.6) we have

$$\mathbb{Q}^n = \frac{\tau_0}{\tau_0 + \Delta t}\mathbf{D}^{-1}\mathbb{Q}^{n-1} - \frac{k\Delta t}{(\tau_0 + \Delta t)\Delta x}\mathbf{D}^{-1}\mathbf{A}\Theta^{n-1}$$

$$
\begin{aligned}
&- \frac{m^2 k \Delta t^3}{2\rho c(\tau_0+\Delta t)\Delta x^2}\mathbf{D}^{-1}\mathbf{A}\mathbf{C}\mathbf{B}^{-1}\mathbf{A}\Theta^n \\
&- \frac{mk\Delta t}{2\rho c(\tau_0+\Delta t)\Delta x}\mathbf{D}^{-1}\mathbf{A}\mathbf{C}\Big[\mathbf{I}_J+\mathbf{B}^{-1}\Big(\mathbf{I}_J+\frac{\beta\Delta t}{2\Delta x^2}\mathbf{L}\Big)\Big]\mathbb{U}^{n-1} \\
&+ \frac{mk\Delta t}{2\rho c(\tau_0+\Delta t)\Delta x}\mathbf{D}^{-1}\mathbf{A}\mathbf{C}\mathbf{B}^{-1}\Big(2\mathbf{I}_J+\frac{\Delta t^2}{\Delta x^2}\mathbf{L}\Big)\mathbb{U}^n,
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
\Theta^n = {} & \mathbf{F}^{-1}\mathbf{E}\Theta^{n-1} - \frac{\tau_0\Delta t}{\rho c(\tau_0+\Delta t)\Delta x}\mathbf{F}^{-1}\mathbf{C}\mathbf{D}^{-1}\mathbb{Q}^{n-1} \\
&+ \frac{m}{2\rho c\Delta x}\mathbf{F}^{-1}\widetilde{\mathbf{E}}\mathbf{C}\Big[\mathbf{I}_J+\mathbf{B}^{-1}\Big(\mathbf{I}_J+\frac{\beta\Delta t}{2\Delta x^2}\mathbf{L}\Big)\Big]\mathbb{U}^{n-1} \\
&- \frac{m}{2\rho c\Delta x}\mathbf{F}^{-1}\widetilde{\mathbf{E}}\mathbf{C}\mathbf{B}^{-1}\Big(2\mathbf{I}_J+\frac{\Delta t^2}{\Delta x^2}\mathbf{L}\Big)\mathbb{U}^n,
\end{aligned}
\tag{4.8}
$$

where

$$
\mathbf{F} := \mathbf{I}_{J+1} - \frac{m^2\Delta t^2}{2\rho c\Delta x^2}\widetilde{\mathbf{E}}\mathbf{C}\mathbf{B}^{-1}\mathbf{A}.
$$

Therefore, putting together Eqs. (4.4a), (4.7) and (4.8) we obtain the following algorithm:

$$
\begin{cases}
\mathbb{U}^{n+1} = \mathbf{X}_1\mathbb{U}^n - \mathbf{X}_2\mathbb{U}^{n-1} - \mathbf{X}_3\Theta^n, & n\in I_N\cup\{0\}, \\
\Theta^{n+1} = \mathbf{Y}_1\Theta^n - \mathbf{Y}_2\mathbb{Q}^n + \mathbf{Y}_3\mathbb{U}^n - \mathbf{Y}_4\mathbb{U}^{n+1}, & n\in I_N\cup\{0\}, \\
\mathbb{Q}^{n+1} = \mathbf{Z}_1\mathbb{Q}^n - \mathbf{Z}_2\Theta^n - \mathbf{Z}_3\Theta^{n+1} - \mathbf{Z}_4\mathbb{U}^n + \mathbf{Z}_5\mathbb{U}^{n+1}, & n\in I_N\cup\{0\}, \\
\mathbb{U}^0 = \big(f_0(x_1), f_0(x_2), \cdots, f_0(x_J)\big)^\top, \quad U^1 - U^{-1} = 2\Delta t\big(f_1(x_1), f_1(x_2), \cdots, f_1(x_J)\big)^\top, \\
\Theta^0 = \big(g_0(x_0), g_0(x_1), \cdots, g_0(x_J)\big)^\top, \quad \mathbb{Q}^0 = \big(h_0(x_1), h_0(x_2), \cdots, h_0(x_J)\big)^\top,
\end{cases}
\tag{4.9}
$$

where

$$
\mathbf{X}_1 := \mathbf{B}^{-1}\Big(2\mathbf{I}_J+\frac{\Delta t^2}{\Delta x^2}\mathbf{L}\Big), \quad \mathbf{X}_2 := \mathbf{B}^{-1}\Big(\mathbf{I}_J+\frac{\beta\Delta t}{2\Delta x^2}\mathbf{L}\Big), \quad \mathbf{X}_3 := \frac{m\Delta t^2}{\Delta x}\mathbf{B}^{-1}\mathbf{A},
$$

followed by

$$
\begin{aligned}
\mathbf{Y}_1 &:= \mathbf{F}^{-1}\mathbf{E}, \\
\mathbf{Y}_2 &:= \frac{\tau_0\Delta t}{\rho c(\tau_0+\Delta t)\Delta x}\mathbf{F}^{-1}\mathbf{C}\mathbf{D}^{-1}, \\
\mathbf{Y}_3 &:= \frac{m}{2\rho c\Delta x}\mathbf{F}^{-1}\widetilde{\mathbf{E}}\mathbf{C}\Big[\mathbf{I}_J+\mathbf{B}^{-1}\Big(\mathbf{I}_J+\frac{\beta\Delta t}{2\Delta x^2}\mathbf{L}\Big)\Big], \\
\mathbf{Y}_4 &:= \frac{m}{2\rho c\Delta x}\mathbf{F}^{-1}\widetilde{\mathbf{E}}\mathbf{C}\mathbf{B}^{-1}\Big(2\mathbf{I}_J+\frac{\Delta t^2}{\Delta x^2}\mathbf{L}\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{Z}_1 &:= \frac{\tau_0}{\tau_0+\Delta t}\mathbf{D}^{-1}, \\
\mathbf{Z}_2 &:= \frac{k\Delta t}{(\tau_0+\Delta t)\Delta x}\mathbf{D}^{-1}\mathbf{A},
\end{aligned}
$$

$$\mathbf{Z}_3 := \frac{m^2 k\Delta t^3}{2\rho c(\tau_0+\Delta t)\Delta x^2}\mathbf{D}^{-1}\mathbf{A}\mathbf{C}\mathbf{B}^{-1}\mathbf{A},$$
$$\mathbf{Z}_4 := \frac{mk\Delta t}{2\rho c(\tau_0+\Delta t)\Delta x}\mathbf{D}^{-1}\mathbf{A}\mathbf{C}\Big[\mathbf{I}_J + \mathbf{B}^{-1}\Big(\mathbf{I}_J + \frac{\beta\Delta t}{2\Delta x^2}\mathbf{L}g\Big)\Big],$$
$$\mathbf{Z}_5 := \frac{mk\Delta t}{2\rho c(\tau_0+\Delta t)\Delta x}\mathbf{D}^{-1}\mathbf{A}\mathbf{C}\mathbf{B}^{-1}\Big(2\mathbf{I}_J + \frac{\Delta t^2}{\Delta x^2}\mathbf{L}\Big).$$

Remark 4.1. The non-singularity of the matrices $\mathbf{B}, \mathbf{D}$ and $\mathbf{F}$ is guaranteed by their being strictly diagonally dominant (see Levy-Desplanques theorem [25]). Thus, we have the following result.

Theorem 4.1. *The difference scheme* (4.9) *is uniquely solvable.*

4.2. Convergence

Proving the convergence of the numerical solution in systems of partial differential equations is not a straightforward task; an alternative for this is to use the Lax equivalence theorem (see [24]). Knowing that the finite difference approximations used in (4.1) are consistent, we are left to prove the stability of the numerical scheme. Therefore, we define the discrete energy functional of (4.1)-(4.3) as

$$E^n := \frac{\Delta x}{2}\sum_{j=0}^{J}\left|\partial_t u_j^n\right|^2 + \frac{\Delta x}{2}\sum_{j=0}^{J}\left(\nabla_x u_j^{n+1}\cdot\nabla_x u_j^n\right) + \rho c\frac{\Delta x}{2}\sum_{j=0}^{J}\left|\theta_j^n\right|^2 + \frac{\tau_0}{k}\frac{\Delta x}{2}\sum_{j=0}^{J}\left|q_j^n\right|^2, \quad \forall\, n\in I_N, \tag{4.10}$$

and use the discrete Lyapunov stability theorem to prove the stability of the scheme (4.1). For this, we analyze in detail the following results regarding the dissipactivity and positivity of E^n.

Theorem 4.2 (Dissipactivity). *Let* $(u_j^n, \theta_j^n, q_j^n)$ *be the numerical solution of the system* (4.1)-(4.3)*. Thus, for all* $\Delta x, \Delta t \in (0,1)$ *the rate of change of the energy functional* E^n *in* (4.10) *at the instant* t_n *is given by*

$$\frac{E^n - E^{n-1}}{\Delta t} \le -\beta\Delta x\sum_{j=0}^{J}\left|\nabla_x\left(\frac{\overline{\partial}_t+\partial_t}{2}u_j^n\right)\right|^2 - \frac{1}{k}\Delta x\sum_{j=0}^{J}|q_j^n|^2 - \frac{\mu^2}{k}\Delta x\sum_{j=0}^{J}\left|\nabla_x q_j^n\right|^2 \le 0, \quad \forall\, n\in I_N. \tag{4.11}$$

In particular, $E^n \le E^0$ *for all* $n \in I_N \cup \{0\}$.

Proof. Multiplying the Eq. (4.1a) by $\Delta x((\overline{\partial}_t+\partial_t)u_j^n/2)$ and adding to $j\in I_J$ we have

$$\underbrace{\Delta x\sum_{j=1}^{J}\left(\overline{\partial}_t\circ\partial_t u_j^n\right)\left(\frac{\overline{\partial}_t+\partial_t}{2}u_j^n\right)}_{\mathcal{S}_1^n :=} \underbrace{-\Delta x\sum_{j=1}^{J}\left(\nabla_x\circ\overline{\nabla}_x u_j^n\right)\left(\frac{\overline{\partial}_t+\partial_t}{2}u_j^n\right)}_{\mathcal{S}_2^n :=}$$

$$- \beta \Delta x \underbrace{\sum_{j=1}^{J} \left[\overline{\nabla}_x \circ \nabla_x \left(\frac{\overline{\partial}_t + \partial_t}{2} u_j^n \right) \right] \left(\frac{\overline{\partial}_t + \partial_t}{2} u_j^n \right)}_{\mathcal{S}_3^n :=}$$

$$+ m \Delta x \sum_{j=1}^{J} \overline{\nabla}_x \theta_j^n \left(\frac{\overline{\partial}_t + \partial_t}{2} u_j^n \right) = 0. \tag{4.12}$$

After simplifications we obtain

$$\begin{aligned}
\mathcal{S}_1^n &:= \sum_{j=1}^{J} \left(\overline{\partial}_t \circ \partial_t u_j^n \right) \left(\frac{\overline{\partial}_t + \partial_t}{2} u_j^n \right) \\
&= \frac{1}{2\Delta t} \sum_{j=1}^{J} \left(\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} \right) \left(u_j^{n+1} - u_j^{n-1} \right) \\
&= \frac{1}{2\Delta t} \sum_{j=1}^{J} \left(\frac{u_j^{n+1} - u_j^n}{\Delta t} \frac{u_j^{n+1} - u_j^{n-1}}{\Delta t} \right) - \frac{1}{2\Delta t} \sum_{j=1}^{J} \left(\frac{u_j^n - u_j^{n-1}}{\Delta t} \frac{u_j^{n+1} - u_j^{n-1}}{\Delta t} \right) \\
&= \frac{1}{2\Delta t} \sum_{j=0}^{J} \left| \frac{u_j^{n+1} - u_j^n}{\Delta t} \right|^2 - \frac{1}{2\Delta t} \sum_{j=0}^{J} \left| \frac{u_j^n - u_j^{n-1}}{\Delta t} \right|^2 \\
&= \frac{1}{2\Delta t} \sum_{j=0}^{J} \left| \partial_t u_j^n \right|^2 - \frac{1}{2\Delta t} \sum_{j=0}^{J} \left| \partial_t u_j^{n-1} \right|^2,
\end{aligned} \tag{4.13}$$

followed by

$$\begin{aligned}
\mathcal{S}_2^n &:= \sum_{j=1}^{J} \left(\nabla_x \circ \overline{\nabla}_x u_j^n \right) \left(\frac{\overline{\partial}_t + \partial_t}{2} u_j^n \right) \\
&= \sum_{j=1}^{J} \left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \right) \left(u_j^{n+1} - u_j^{n-1} \right) \\
&= - \sum_{j=0}^{J} \left(\frac{u_{j+1}^{n+1} - u_j^{n+1}}{\Delta x} \frac{u_{j+1}^n - u_j^n}{\Delta x} \right) + \sum_{j=0}^{J} \left(\frac{u_{j+1}^n - u_j^n}{\Delta x} \frac{u_{j+1}^{n-1} - u_j^{n-1}}{\Delta x} \right) \\
&= - \sum_{j=0}^{J} \left(\nabla_x u_j^{n+1} \cdot \nabla_x u_j^n \right) + \sum_{j=0}^{J} \left(\nabla_x u_j^n \cdot \nabla_x u_j^{n-1} \right),
\end{aligned} \tag{4.14}$$

and finally we have

$$\begin{aligned}
\mathcal{S}_3^n &:= \sum_{j=1}^{J} \left[\overline{\nabla}_x \circ \nabla_x \left(\frac{\overline{\partial}_t + \partial_t}{2} u_j^n \right) \right] \left(\frac{\overline{\partial}_t + \partial_t}{2} u_j^n \right) \\
&= \sum_{j=1}^{J} \left[\left(\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{4\Delta t^2 \Delta x^2} \right) \left(u_j^{n+1} - u_j^{n-1} \right) \right. \\
&\qquad \left. - \left(\frac{u_{j+1}^{n-1} - 2u_j^{n-1} + u_{j-1}^{n-1}}{4\Delta t^2 \Delta x^2} \right) \left(u_j^{n+1} - u_j^{n-1} \right) \right] \\
&= - \frac{1}{4\Delta t^2 \Delta x^2} \sum_{j=0}^{J} \left| u_{j+1}^{n+1} - u_j^{n+1} \right|^2 - \frac{1}{4\Delta t^2 \Delta x^2} \sum_{j=0}^{J} \left| u_{j+1}^{n-1} - u_j^{n-1} \right|^2
\end{aligned}$$

$$-\sum_{j=1}^{J}\left(\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{4\Delta t^2\Delta x^2}\right)u_j^{n-1}-\sum_{j=1}^{J}\left(\frac{u_{j+1}^{n-1}-2u_j^{n-1}+u_{j-1}^{n-1}}{4\Delta t^2\Delta x^2}\right)u_j^{n+1}.$$

Making further simplification, we have

$$\begin{aligned}
\mathcal{S}_3^n &= -\frac{1}{4\Delta t^2\Delta x^2}\left[\sum_{j=0}^{J}\left|u_{j+1}^{n+1}-u_j^{n+1}\right|^2+\sum_{j=0}^{J}\left|u_{j+1}^{n-1}-u_j^{n-1}\right|^2+\sum_{j=1}^{J}\left(u_{j+1}^{n+1}-u_j^{n+1}\right)u_j^{n-1}\right]\\
&\quad+\frac{1}{4\Delta t^2\Delta x^2}\left[\sum_{j=1}^{J}\left(u_j^{n+1}-u_{j-1}^{n+1}\right)u_j^{n-1}-\sum_{j=1}^{J}\left(u_{j+1}^{n-1}-u_j^{n-1}\right)u_j^{n+1}+\sum_{j=1}^{J}\left(u_j^{n-1}-u_{j-1}^{n-1}\right)u_j^{n+1}\right]\\
&= -\frac{1}{4\Delta t^2\Delta x^2}\left[\sum_{j=0}^{J}\left|u_{j+1}^{n+1}-u_j^{n+1}\right|^2+\sum_{j=0}^{J}\left|u_{j+1}^{n-1}-u_j^{n-1}\right|^2+\sum_{j=0}^{J}\left(u_{j+1}^{n+1}-u_j^{n+1}\right)u_j^{n-1}\right]\\
&\quad+\frac{1}{4\Delta t^2\Delta x^2}\left[\sum_{j=0}^{J}\left(u_{j+1}^{n+1}-u_j^{n+1}\right)u_{j+1}^{n-1}-\sum_{j=0}^{J}\left(u_{j+1}^{n-1}-u_j^{n-1}\right)u_j^{n+1}+\sum_{j=0}^{J}\left(u_{j+1}^{n-1}-u_j^{n-1}\right)u_{j+1}^{n+1}\right]\\
&= -\frac{1}{4\Delta t^2\Delta x^2}\left[\sum_{j=0}^{J}\left|u_{j+1}^{n+1}-u_j^{n+1}\right|^2+\sum_{j=0}^{J}\left|u_{j+1}^{n-1}-u_j^{n-1}\right|^2-2\sum_{j=0}^{J}\left(u_{j+1}^{n+1}-u_j^{n+1}\right)\left(u_{j+1}^{n-1}-u_j^{n-1}\right)\right]\\
&= -\sum_{j=0}^{J}\left|\frac{u_{j+1}^{n+1}-u_j^{n+1}}{2\Delta t\Delta x}-\frac{u_{j+1}^{n-1}-u_j^{n-1}}{2\Delta t\Delta x}\right|^2=-\sum_{j=0}^{J}\left|\nabla_x\left(\frac{\overline{\partial}_t+\partial_t}{2}u_j^n\right)\right|^2. \qquad (4.15)
\end{aligned}$$

Replacing (4.13)-(4.15) in (4.12), we obtain

$$\begin{aligned}
&\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left|\partial_t u_j^n\right|^2-\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left|\partial_t u_j^{n-1}\right|^2+\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\nabla_x u_j^{n+1}\cdot\nabla_x u_j^n\right)\\
&-\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\nabla_x u_j^n\cdot\nabla_x u_j^{n-1}\right)+\beta\Delta x\sum_{j=0}^{J}\left|\nabla_x\left(\frac{\overline{\partial}_t+\partial_t}{2}u_j^n\right)\right|^2\\
&+\ m\Delta x\sum_{j=0}^{J}\overline{\nabla}_x\theta_j^n\left(\frac{\overline{\partial}_t+\partial_t}{2}u_j^n\right)=0. \qquad (4.16)
\end{aligned}$$

On the other hand, multiplying the Eq. (4.1b) by $\Delta x\theta_j^n$ and adding to $j\in I_J\cup\{0\}$, we have

$$\rho c\Delta x\underbrace{\sum_{j=0}^{J}\left(\overline{\partial}_t\theta_j^n\right)\theta_j^n}_{\mathcal{S}_4^n:=}+\Delta x\sum_{j=0}^{J}\left(\nabla_x q_j^n\right)\theta_j^n+m\Delta x\underbrace{\sum_{j=0}^{J}\left[\nabla_x\left(\frac{\overline{\partial}_t+\partial_t}{2}u_j^n\right)\theta_j^n\right]}_{\mathcal{S}_5^n:=}=0. \qquad (4.17)$$

Using Young's inequality in $\mathcal{S}_4^n$, we have

$$\mathcal{S}_4^n:=\sum_{j=0}^{J}\left(\overline{\partial}_t\theta_j^n\right)\theta_j^n=\sum_{j=0}^{J}\left(\frac{\theta_j^n-\theta_j^{n-1}}{\Delta t}\theta_j^n\right)\geq\frac{1}{2\Delta t}\sum_{j=0}^{J}\left|\theta_j^n\right|^2-\frac{1}{2\Delta t}\sum_{j=0}^{J}\left|\theta_j^{n-1}\right|^2. \qquad (4.18)$$

From $\mathcal{S}_5^n$ we have

$$\mathcal{S}_5^n:=\sum_{j=0}^{J}\left[\nabla_x\left(\frac{\overline{\partial}_t+\partial_t}{2}u_j^n\right)\theta_j^n\right]$$

$$= \sum_{j=0}^{J} \left[\left(\frac{u_{j+1}^{n+1} - u_j^{n+1}}{2\Delta t \Delta x} - \frac{u_{j+1}^{n-1} - u_j^{n-1}}{2\Delta t \Delta x} \right) \theta_j^n \right]$$

$$= \sum_{j=0}^{J} \left(\frac{u_{j+1}^{n+1} - u_{j+1}^{n-1}}{2\Delta t \Delta x} \right) \theta_j^n - \sum_{j=0}^{J} \left(\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t \Delta x} \right) \theta_j^n. \tag{4.19}$$

Using left shift assignment about j, we have

$$\mathcal{S}_5^n = \sum_{j=0}^{J} \left(\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t \Delta x} \right) \theta_{j-1}^n - \sum_{j=0}^{J} \left(\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t \Delta x} \right) \theta_j^n$$

$$= - \sum_{j=0}^{J} \left(\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} \right) \left(\frac{\theta_j^n - \theta_{j-1}^n}{\Delta x} \right)$$

$$= - \sum_{j=0}^{J} \left(\frac{\overline{\partial}_t + \partial_t}{2} u_j^n \right) \overline{\nabla}_x \theta_j^n. \tag{4.20}$$

Replacing (4.18) and (4.20) in (4.17), we obtain

$$\frac{\rho c \Delta x}{2\Delta t} \sum_{j=0}^{J} \left| \theta_j^n \right|^2 - \frac{c\Delta x}{2\Delta t} \sum_{j=0}^{J} \left| \theta_j^{n-1} \right|^2 + \Delta x \sum_{j=0}^{J} \left(\nabla_x q_j^n \cdot \theta_j^n \right)$$
$$- m\Delta x \sum_{j=0}^{J} \left(\frac{\overline{\partial}_t + \partial_t}{2} u_j^n \right) \overline{\nabla}_x \theta_j^n \leq 0. \tag{4.21}$$

Similarly, multiplying the Eq. (4.1c) by $\Delta x q_j^n / k$, adding to $j \in I_J$ and using the boundary condition $q_0^n = 0$, we have

$$\frac{\tau_0}{k} \frac{\Delta x}{2\Delta t} \sum_{j=0}^{J} \left| q_j^n \right|^2 - \frac{\tau_0}{k} \frac{\Delta x}{2\Delta t} \sum_{j=0}^{J} \left| q_j^{n-1} \right|^2 + \frac{1}{k} \Delta x \sum_{j=0}^{J} \left| q_j^n \right|^2$$
$$\underbrace{- \frac{\mu^2}{k} \Delta x \sum_{j=1}^{J} \left(\overline{\nabla}_x \circ \nabla_x q_j^n \cdot q_j^n \right) + \Delta x \sum_{j=1}^{J} \left(\overline{\nabla}_x \theta_j^n \cdot q_j^n \right)}_{\mathcal{S}_6^n :=} \leq 0. \tag{4.22}$$

Using the boundary conditions (4.2), we have

$$\mathcal{S}_6^n := - \frac{\mu^2}{k} \Delta x \sum_{j=1}^{J} \left(\overline{\nabla}_x \circ \nabla_x q_j^n \cdot q_j^n \right) + \Delta x \sum_{j=1}^{J} \left(\overline{\nabla}_x \theta_j^n \cdot q_j^n \right)$$
$$= \frac{\mu^2}{k} \Delta x \sum_{j=0}^{J} \left| \nabla_x q_j^n \right|^2 - \Delta x \sum_{j=0}^{J} \left(\nabla_x q_j^n \cdot \theta_j^n \right). \tag{4.23}$$

From (4.22) and (4.23) we obtain

$$\frac{\tau_0}{k} \frac{\Delta x}{2\Delta t} \sum_{j=0}^{J} \left| q_j^n \right|^2 - \frac{\tau_0}{k} \frac{\Delta x}{2\Delta t} \sum_{j=0}^{J} \left| q_j^{n-1} \right|^2 + \frac{1}{k} \Delta x \sum_{j=0}^{J} \left| q_j^n \right|^2$$
$$+ \frac{\mu^2}{k} \Delta x \sum_{j=0}^{J} \left| \nabla_x q_j^n \right|^2 - \Delta x \sum_{j=0}^{J} \left(\nabla_x q_j^n \cdot \theta_j^n \right) \leq 0. \tag{4.24}$$

Adding (4.16), (4.21), (4.24) and using the energy E^n in (4.10), we obtain

$$\frac{E^n - E^{n-1}}{\Delta t} \leq -\beta \Delta x \sum_{j=0}^{J} \left| \nabla_x \left(\frac{\overline{\partial}_t + \partial_t}{2} u_j^n \right) \right|^2 - \frac{1}{k} \Delta x \sum_{j=0}^{J} \left| q_j^n \right|^2 - \frac{\mu^2}{k} \Delta x \sum_{j=0}^{J} \left| \nabla_x q_j^n \right|^2 \leq 0, \quad \forall\, n \in I_N.$$

Therefore $E^n \leq E^0$ for all $n \in I_N \cup \{0\}$. □

Theorem 4.3 (Positivity). *If $\Delta t \leq \Delta x$, then for every non-trivial solution of the discrete system* (4.1)-(4.3)*, we have*

$$\frac{E^n}{\Delta x} \geq \frac{1}{2} \left(1 - \frac{\Delta t^2}{\Delta x^2} \right) \sum_{j=0}^{J} \left| \partial_t u_j^n \right|^2 \geq 0, \quad \forall\, n \in I_N \cup \{0\}.$$

Proof. From (4.10) we have

$$\frac{E^n}{\Delta x} \geq \frac{1}{2\Delta t^2} \sum_{j=0}^{J} \left| u_j^{n+1} - u_j^n \right|^2 + \frac{1}{2\Delta x^2} \underbrace{\sum_{j=0}^{J} \left(u_{j+1}^{n+1} - u_j^{n+1} \right) \left(u_{j+1}^n - u_j^n \right)}_{\mathcal{I}^n :=} . \tag{4.25}$$

Because of boundary conditions (4.2) we know that

$$\sum_{j=0}^{J} u_{j+1}^{n+1} u_{j+1}^n = \sum_{j=0}^{J} u_j^{n+1} u_j^n, \quad \sum_{j=0}^{J} \left| u_{j+1}^n \right|^2 = \sum_{j=0}^{J} \left| u_j^n \right|^2. \tag{4.26}$$

Therefore we have

$$\begin{aligned}
\mathcal{I}^n &:= \sum_{j=0}^{J} \left(u_{j+1}^{n+1} - u_j^{n+1} \right) \left(u_{j+1}^n - u_j^n \right) \\
&= \sum_{j=0}^{J} \left(- u_j^{n+1} u_{j+1}^n + u_{j+1}^{n+1} u_{j+1}^n + u_j^{n+1} u_j^n - u_{j+1}^{n+1} u_j^n \right) \\
&= \sum_{j=0}^{J} \left(- u_j^{n+1} u_{j+1}^n + 2 u_j^{n+1} u_j^n - u_{j+1}^{n+1} u_j^n \right).
\end{aligned}$$

Then using (4.26) and Young's inequality, we obtain

$$\begin{aligned}
\mathcal{I}^n &\geq \sum_{j=0}^{J} \left(- \frac{1}{2} |u_j^{n+1}|^2 - \frac{1}{2} |u_{j+1}^n|^2 + 2 u_j^{n+1} u_j^n - \frac{1}{2} |u_{j+1}^{n+1}|^2 - \frac{1}{2} |u_j^n|^2 \right) \\
&= - \sum_{j=0}^{J} \left(\frac{1}{2} |u_j^{n+1}|^2 + \frac{1}{2} |u_j^n|^2 - 2 u_j^{n+1} u_j^n + \frac{1}{2} |u_j^{n+1}|^2 + \frac{1}{2} |u_j^n|^2 \right) \\
&= - \sum_{j=0}^{J} \left(|u_j^{n+1}|^2 - 2 u_j^{n+1} u_j^n + |u_j^n|^2 \right) = - \sum_{j=0}^{J} \left| u_j^{n+1} - u_j^n \right|^2.
\end{aligned} \tag{4.27}$$

Replacing (4.27) in (4.25), we obtain

$$\frac{E^n}{\Delta x} \geq \frac{1}{2}\left(\frac{1}{\Delta t^2} - \frac{1}{\Delta x^2}\right)\sum_{j=0}^{J}\left|u_j^{n+1} - u_j^n\right|^2 \geq 0, \tag{4.28}$$

since $\Delta t \leq \Delta x$. □

The following result is an immediate consequence of Theorems 4.2 and 4.3, and the discrete Lyapunov stability theorem (see [3, 14]).

Corollary 4.1 (Lyapunov Stability). *The scheme* (4.1)-(4.3) *is stable if and only if* $\Delta t \leq \Delta x$.

Therefore, the convergence of the numerical solution of the scheme (4.1) is obtained from the Lax Equivalence theorem [24]. In Fig. 4.1, we provide an illustration with the convergence summary.

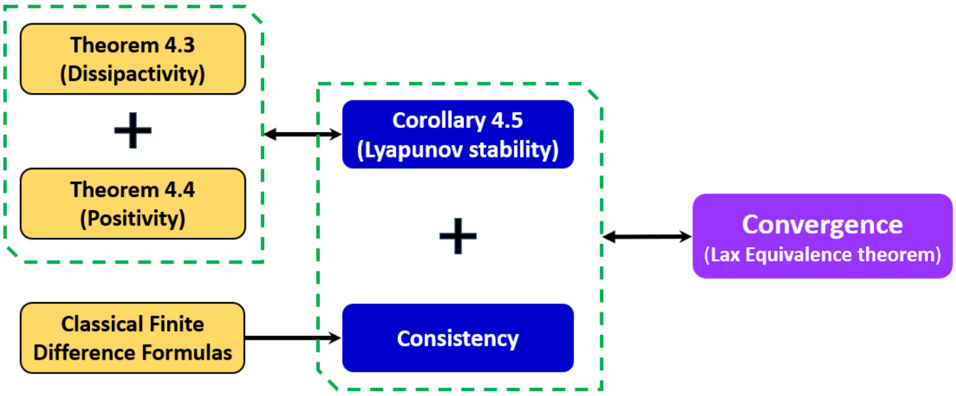


Fig. 4.1. Convergence summary.

5. Numerical Simulations

In this section, we present the numerical simulation of the thermoviscoelastic system with the GK model. We also compare it with the limit system governed by the Cattaneo model. The simulations demonstrate that the numerical scheme (4.9) is robust enough to reproduce the theoretical results from the previous sections. For this purpose, we choose the following numerical data:

$$\begin{array}{lllll} \ell = 1\cdot 10^{-1}, & \Delta x = 2.5\cdot 10^{-4}, & \Delta t = 2.4998\cdot 10^{-4}, & \rho = 2.7\cdot 10^{3}, & \beta = 1\cdot 10^{-3}, \\ c = 5\cdot 10^{2}, & \tau_0 = 8\cdot 10^{-3}, & \mu^2 = 2.8\cdot 10^{-3}, & k = 5\cdot 10^{3}, & m = 9\cdot 10^{-1}, \end{array}$$

followed by the initial conditions

$$\begin{aligned} f_0(x_j) &= 3\cdot 10^{-1}\sin\left(\frac{\pi x_j}{\ell}\right), \quad f_1(x_j) = -6\cdot 10^{-1}\sin\left(\frac{\pi x_j}{\ell}\right), \\ g_0(x_j) &= T_b + \frac{T_f}{2}\cos\left(\frac{\pi x_j}{\ell}\right), \\ h_0(x_j) &= \frac{kT_f\pi}{2\ell(1+\mu^2\pi^2/\ell^2)}\sin\left(\frac{\pi x_j}{\ell}\right), \quad j \in I_{J+1}\cup\{0\}, \end{aligned}$$

where T_b and T_f are determined later to ensure that the initial condition g_0 belongs to or not to the space $L^2_*(0,\ell)$. In the simulations below, we reproduce the results of Theorem 3.1.

Case I of Theorem 3.1: We adopt the initial condition $g_0 \notin L^2_*(0,\ell)$ by choosing $T_b = 1.5 \cdot 10^1$ °C and $T_f = 3 \cdot 10^1$ °C, which implies $\Lambda(t) \neq 0$. According to Theorem 3.1, the energy functional decays exponentially to a positive constant. In Fig. 5.1, we observe that Θ^n decays rapidly to a non-zero equilibrium point, while the other numerical solutions decay to zero. On the other hand, from Fig. 5.2(a), we can assume that the decay of Θ^n directly influences the asymptotic behavior of the energy functional, which, in turn, decays to a horizontal asymptote $y > 0$ instead of zero. From here, we want to know if $E(t)$ decays exponentially to y, as proven in Theorem 3.1. For this, we invoke item (I) and establish the following equivalence:

$$E^n \leq ME^0 e^{-\omega t_n} + C\|\Lambda\|_\infty \;\Leftrightarrow\; H^n \leq -\omega t_n + \gamma, \tag{5.1}$$

where

$$H^n = \ln\left(\frac{E^n}{Z^n}\right), \quad Z^n = 1 + \left(\frac{C\|\Lambda\|_\infty}{ME^0}\right)e^{\omega t_n}, \quad \gamma = \ln(ME^0).$$

Having this in mind, we look at Fig. 5.2(b) on a semi-log scale and observe a linear function with a negative slope. According to (5.1), this confirms the exponential decay of E^n to the asymptote $y > 0$, as we can see in Fig. 5.2(a).

Case II of Theorem 3.1: Now we adopt the initial condition $g_0 \in L^2_*(0,\ell)$, i.e. we choose the values $T_b = 0$ °C and $T_f = 3 \cdot 10^1$ °C, which implies $\Lambda(t) = 0$. According to Theorem 3.1, the energy functional decays exponentially to zero.

In Fig. 5.3, we clearly notice that Θ^n decays exponentially to zero, as well as the other numerical solutions. This is corroborated by the graphs in Fig. 5.4.

Remark 5.1. As we can see in the figures above, the space where the initial condition g_0 is embedded directly affects the asymptotic behavior of the system.

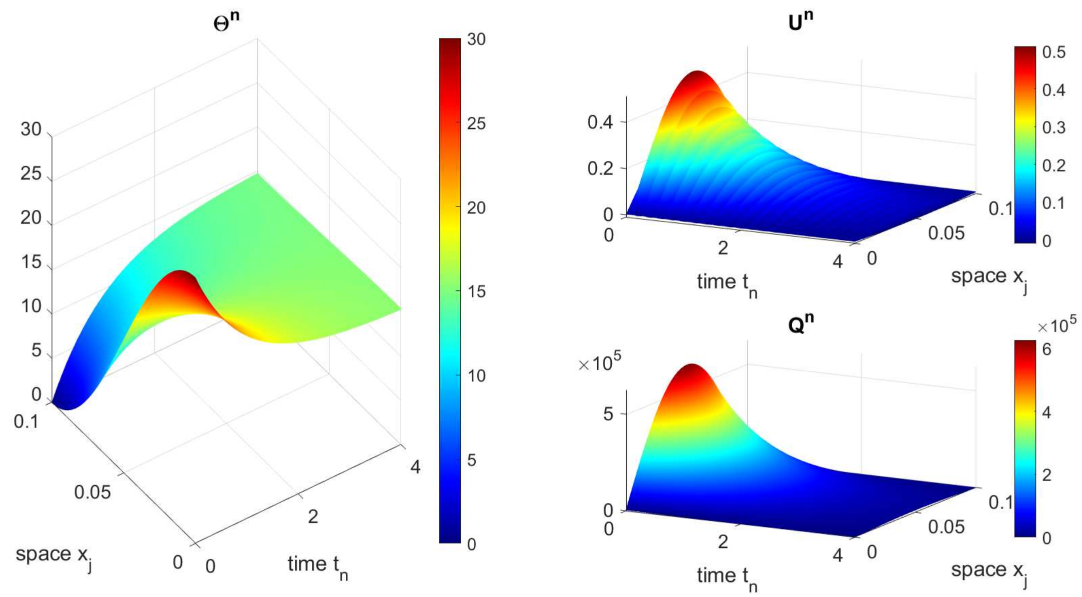
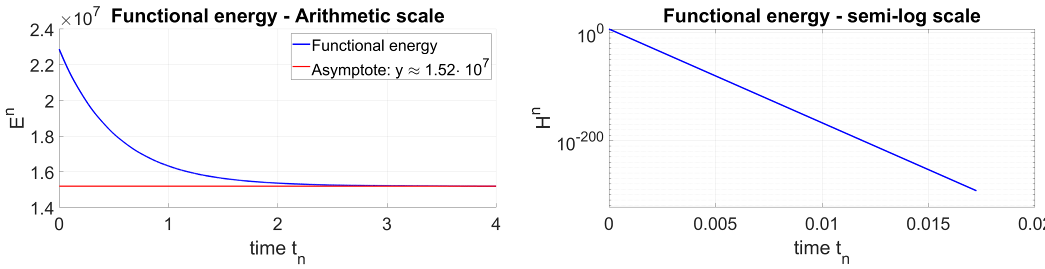


Fig. 5.1. Numerical solution of the thermoviscoelastic system in $[0, 0.1] \times [0, 4]$.



(a) (b)

Fig. 5.2. Asymptotic behavior of the energy functional.

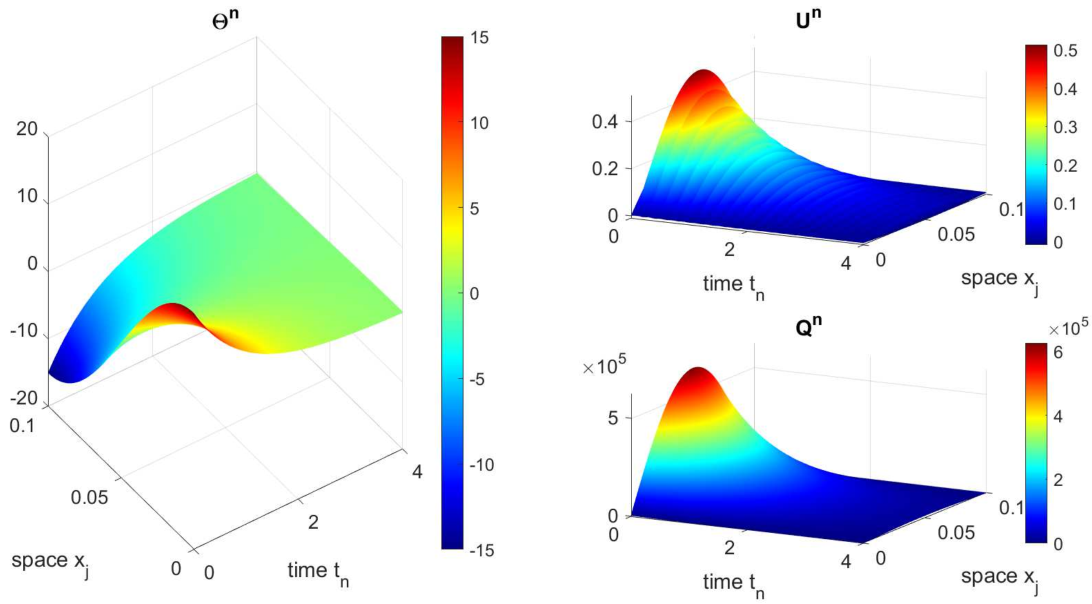
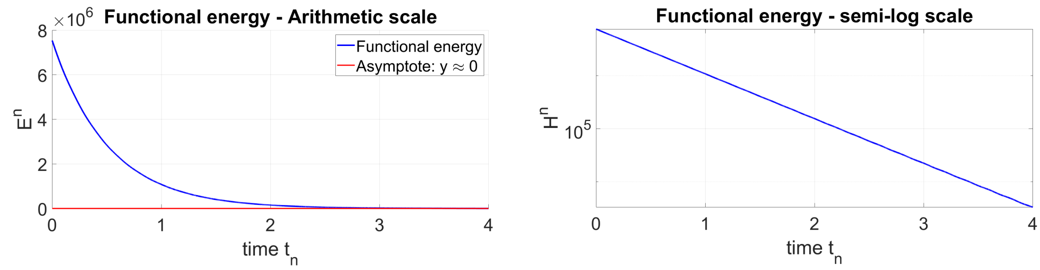


Fig. 5.3. Numerical solution of the thermoviscoelastic system in $[0, 0.1] \times [0, 4]$.



(a) (b)

Fig. 5.4. Asymptotic behavior of the energy functional.

5.1. Comparison between the thermoviscoelastic systems

Here, we compare the solutions of the thermoviscoelastic system (1.1)-(1.3) subject to the thermal effects of the GK and Cattaneo models.

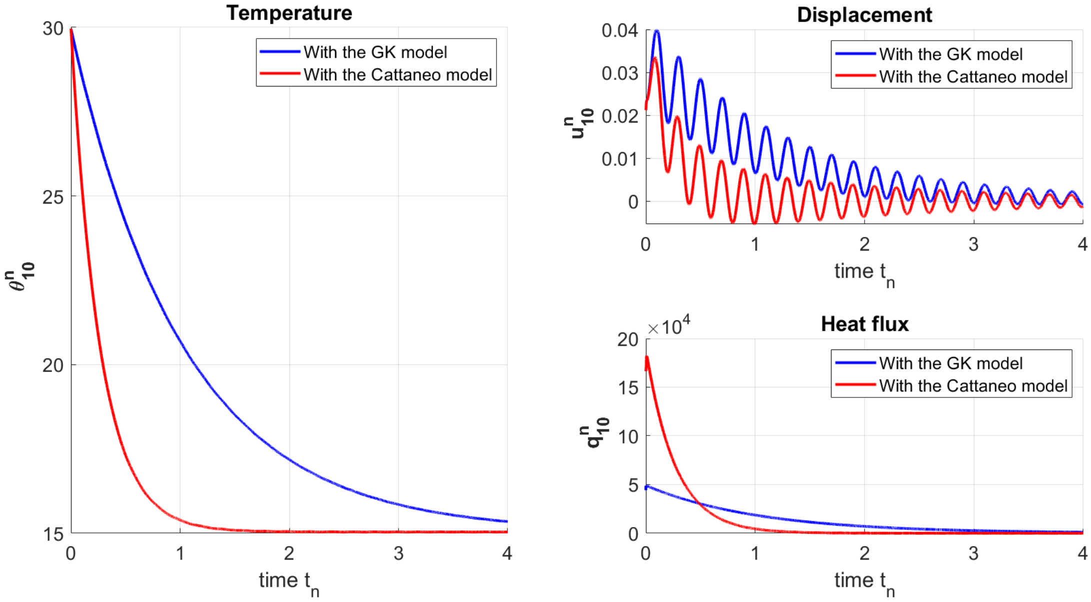


Fig. 5.5. Temporal evolution of the solutions from the point x_{10} in the discretized space.

Fig. 5.5 (left side) clearly shows the over-diffusive heat conduction provided by the GK model. It is also possible to observe an interference in the solution of the transverse displacement u^n_{10}, causing the approximation to zero to be slower than in the case governed by the Cattaneo model.

6. Final Comments

In the first part of this work, we present a thermoviscoelastic system governed by the GK model that is free from the issue of infinite heat propagation speed. Furthermore, it is more suitable for modeling heat transfer at room temperature. Here, we prove well-posedness and exponential decay uniformly. In the second part, we study the fully discrete problem using the finite difference method and propose a monotone and consistent numerical scheme to approximate the exact solution of the system. The difficulty in achieving monotonicity is due to the discretizations of the thermal equations and the Kelvin-Voigt damping. To overcome this challenge, we use a precise combination of explicit and implicit methods, with suitable choices for the discretization of viscoelastic damping. This seems to be the first finite difference numerical scheme in the literature to solve a system with Kelvin-Voigt damping while preserving monotonicity. Finally, we emphasize the importance of the correct algebraic manipulation of discrete terms to reproduce the multiplier method in the numerical context. This care is essential to produce robust finite difference numerical schemes.

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References

[1] C.C. Ackerman and R.A. Guyer, Temperature pulses in dielectric solids, *Ann. Physics*, **50** (1968), 128–185.

[2] C.C. Ackerman and W.C. Overton, Second sound in solid helium-3, *Phys. Rev. Lett.*, **22** (1969), 764–767.

[3] A.A. Ahmadi and P.A. Parrilo, Non-monotonic Lyapunov functions for stability of discrete time nonlinear and switched systems, in: *2008 47th IEEE Conference on Decision and Control*, IEEE, (2008), 614–621.

[4] M.S. Alves, P. Gamboa, G.C. Gorain, A. Rambaudd, and O. Vera, Asymptotic behavior of a flexible structure with Cattaneo type of thermal effect, *Indag. Math.* **27** (2016), 821–834.

[5] G.G. Ávalos, J. Muñoz Rivera, and O.V. Villagran, Stability in thermoviscoelasticity with second sound, *Appl. Math. Optim.*, **82** (2020), 135–150.

[6] M.A. Biot, Thermoelasticity and irreversible thermodynamics, *J. Appl. Phys.*, **27**:3 (1956), 240–253.

[7] S. Both, B. Czél, T. Fülöp, G. Gróf, Á. Gyenis, R. Kovács, P. Ván, and J. Verhás, Deviation from the Fourier law in room-temperature heat pulse experiments, *J. Non-Equilib. Thermodyn.*, **41**:1 (2016), 41–48.

[8] C. Cattaneo, Sulla conduzione del calore, *Atti Sem. Mat. Fis. Modena*, **3** (1948), 83–101.

[9] D.S. Chandrasekharaiah, Hyperbolic thermoelasticity: A review of recent literature, *Appl. Mech. Rev.*, **51** (1998), 705–729.

[10] J. Duhamel, Second memoire sur les phenomenon thermo-mechanique, *J. Ec. Polytech.*, **15** (1937), 1–15.

[11] J.B.J. Fourier, Théorie de la propagation de la chaleur dans les solides, *Mém. Acad. R. Sci.*, (1824), 185–555.

[12] R.A. Guyer and J.A. Krumhansl, Dispersion relation for second sound in solids, *Phys. Rev.*, **133**:5A (1964), A1411.

[13] R.A. Guyer and J.A. Krumhansl, Thermal conductivity, second sound, and phonon hydrodynamic phenomena in nonmetallic crystals, *Phys. Rev.*, **148** (1966), 778–788.

[14] H.K. Khalil, *Nonlinear Systems*, Prentice Hall, 2002.

[15] H.W. Lord and Y. Shulman, A generalized dynamical theory of thermoelasticity, *J. Mech. Phys. Solids*, **15**:5 (1967), 299–309.

[16] T.F. McNelly, S.J. Rogers, D.J. Channin, R.J. Rollefson, W.M. Goubau, G.E. Schmidt, J.A. Krumhansl, and R.O. Pohl, Heat pulses in NaF: Onset of second sound, *Phys. Rev. Lett.*, **24** (1970), 100–102.

[17] V. Narayanamurti and R.D. Dynes, Observation of second sound in bismuth, *Phys. Rev. Lett.*, **26** (1972), 1461–1465.

[18] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, 1983.

[19] V. Peshkov, Second sound in helium II, *J. Phys. (Moscow)*, **8** (1944), 381–389.

[20] R. Racke, Thermoelasticity with second sound – exponential stability in linear and non-linear 1-d, *Math. Methods Appl. Sci.*, **25** (2002), 409–441.

[21] A.J.A. Ramos, R. Kovács, M.M. Freitas, and D.S. Almeida Júnior, Mathematical analysis and numerical simulation of the Guyer-Krumhansl heat equation, *Appl. Math. Model.*, **115** (2023), 191–202.

[22] A.J.A. Ramos, L.G.R. Miranda, M.M. Freitas, and R. Kovács, Non-negativity and maximum principle: Revisiting the Guyer-Krumhansl heat equation, *Int. J. Heat Mass Transf.*, **211** (2023), 124288.

[23] K.K. Tamma and R.R. Namburu, Computational approaches with applications to non-classical and classical thermomechanical problems, *Appl. Mech. Rev.*, **50** (1997), 514–551.

[24] J.W. Thomas, *Numerical Partial Differential Equations: Finite Difference Methods*, Springer, 1995.

[25] E.E. Tyrtyshnikov, *A Brief Introduction to Numerical Analysis*, Birkhäuser, 2012.

[26] P. Ván, A. Berezovski, T. Fülöp, G. Gróf, R. Kovács, Á. Lovas, and J. Verhás, Guyer-Krumhansl-type heat conduction at room temperature, *EPL*, **118**:5 (2017), 50005.

[27] K. Zhukovsky, Violation of the maximum principle and negative solutions for pulse propagation in Guyer-Krumhansl model, *Int. J. Heat Mass Transf.*, **211** (2023), 124288.