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WEAK GALERKIN METHOD FOR COUPLING STOKES AND DARCY-FORCHHEIMER FLOWS*

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Abstract

In this paper, we introduce the weak Galerkin (WG) method for solving the coupled Stokes and Darcy-Forchheimer flows problem with the Beavers-Joseph-Saffman interface condition in bounded domains. We define the WG spaces in the polygonal meshes and construct corresponding discrete schemes. We prove the existence and uniqueness of the WG scheme by the discrete inf-sup condition and monotone operator theory. Then, we derive the optimal error estimates for the velocity and pressure. Numerical experiments are presented to verify the efficiency of the WG method.

Mathematics subject classification: 65N30, 65N15, 76S05, 65N12.
Key words: Weak Galerkin method, Coupled Stokes and Darcy-Forchheimer flows, Monotone operator.

1. Introduction

In recent years, the coupled model of free fluid and porous medium fluid has received more and more attention. Such problems have been widely applied in groundwater flows [9, 13, 17], environmental science [15], flow in vuggy porous media [2, 4] and so on. The classical coupled Stokes-Darcy model consists of the Stokes equation in fluid region, the Darcy's law in porous medium region. The interface conditions in the model are the flow continuity condition, the force equilibrium condition, and the Beavers-Joseph-Saffman condition [34]. However, Darcy's law describes the porous media flow at low speed. In the case of high-speed seepage, porous media will be turbulent. Forchheimer found through experiments that the pressure gradient and the Darcy velocity should meet the nonlinear relationship with a coefficient depending on the soil grain diameter and porosity for the larger values of Reynolds number, which requires the use of the modified Darcy equation in 1901, namely the Forchheimer equation. The Darcy-Forchheimer model in porous media explains the nonlinear behavior by adding an item representing the inertial effect, which is used in the exploration and production of oil and gas in petroleum reservoir [6].

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So far, there are many types of research on the coupling of Stokes flow and Darcy flow, various numerical methods have been developed for the Stokes-Darcy model [3, 10, 14, 16, 18, 20, 26, 28, 29, 32], and there is a lot of research on the individual Darcy-Forchheimer model [5, 25, 30, 31, 33]. However, there are relatively few researches on the coupled Stokes-Darcy-Forchheimer model. Zhang and Rui [42] used the Crouzeix-Raviart element to approximate the velocity and obtained the optimal-order error estimates when the velocity and pressure were in $[H^2]^N$ ($N = 2$ or 3) and H^1 spaces respectively. The authors also derived a discrete inf-sup condition and establish the existence and uniqueness of the problem. Zhao *et al.* [45] proposed the staggered discontinuous Galerkin method for the model problems, which could be flexibly applied to rough grids such as the highly distorted grids and the polygonal grids. And they proved optimal convergence estimates by proposing some new discrete trace inequality and generalized Poincaré-Friedrichs inequality. The fully mixed finite element method was proposed by Almonacid *et al.* [1]. They considered the Darcy-Forchheimer model by means of a modified abstract theory for twofold saddle point problems, which could pose the variational formulation in terms of just Banach spaces, and the optimal-order error estimates were obtained.

The goal of this paper is to investigate the weak Galerkin method for solving the coupled Stokes and Darcy-Forchheimer flows problem. The WG method is first proposed for the second-order elliptic equations by Wang and Ye [36], and further developed in [23, 35, 37, 39, 43]. The key idea of the WG method is to use discontinuous piecewise polynomials as basis functions, and to replace the classical derivative operators by specifically defined weak derivative operators in the numerical scheme. In the past few years, the WG methods have been widely applied to [10,21,22,38,40,41,44]. The applications mentioned above are mostly linear problems and the theoretical analysis and numerical experiments are relatively more complicated for nonlinear problems.

In this paper, we first establish the WG numerical scheme on polygonal meshes. Due to the nonlinear and monotonicity of the Forchheimer term, we use Minty-Browder theorem and discrete inf-sup condition to prove the existence and uniqueness of the numerical scheme. Furthermore, by using the technical estimate for the Forchheimer term, we obtain the optimal orders of error estimates when the velocity and pressure are in the $[H^{k+1}]^2$ space and the H^k space, respectively. When conducting numerical experiments, we adopted triangular grids, and the numerical results are consistent with the theoretical estimates.

This paper is organized as follows. In Section 2, we introduce the model problem and some notations. In Section 3, we define some WG spaces and establish the numerical scheme. In Section 4, we prove the existence and uniqueness of the WG scheme. The error estimates for the WG approximations are given in Section 5. Some numerical experiments are presented in Section 6.

2. Model Problem

In this section, we describe in detail the coupled Stokes and Darcy-Forchheimer model [42]. We consider the coupled flow in a bounded domain $\Omega \in \mathbb{R}^2$, which consists of two subregions, a free region Ω_s and a porous medium region $\Omega_d = \Omega \backslash \overline{\Omega}_s$. Both Ω_s and Ω_d have Lipschitz continuous boundaries. Define $\Gamma_I = \partial\Omega_s \cap \partial\Omega_d, \Gamma_s = \partial\Omega_s \backslash \Gamma_I, \Gamma_d = \partial\Omega_d \backslash \Gamma_I$. $\boldsymbol{n}_s$ and $\boldsymbol{n}_d$ are the outward unit normal vectors on Ω_s and $\Omega_d, \boldsymbol{\tau}$ is the unit tangent vector on Γ_I, as shown in Fig. 2.1.