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OPTIMAL ERROR ANALYSIS OF A HODGE-DECOMPOSITION BASED FINITE ELEMENT METHOD FOR THE GINZBURG-LANDAU EQUATIONS IN SUPERCONDUCTIVITY*

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Abstract

This paper is concerned with the new error analysis of a Hodge-decomposition based finite element method for the time-dependent Ginzburg-Landau equations in superconductivity. In this approach, the original equation of magnetic potential $\boldsymbol{A}$ is replaced by a new system consisting of four scalar variables. As a result, the conventional Lagrange finite element method (FEM) can be applied to problems defined on non-smooth domains. It is known that due to the low regularity of $\boldsymbol{A}$, conventional FEM, if applied to the original Ginzburg-Landau system directly, may converge to the unphysical solution. The main purpose of this paper is to establish an optimal error estimate for the order parameter in spatial direction, as previous analysis only gave a sub-optimal convergence rate analysis for all three variables due to coupling of variables. The analysis is based on a nonstandard quasi-projection for ψ and the corresponding negative-norm estimate for the classical Ritz projection. Our numerical experiments confirm the optimal convergence of ψ_h.

Mathematics subject classification: 65M60, 65N30, 65N12.
Key words: Ginzburg-Landau equation, Hodge decomposition, Optimal error estimate, Non-smooth domains, Superconductivity.

1. Introduction

In this paper, we consider the time-dependent Ginzburg-Landau (TDGL) equations under the Lorentz gauge

$$\begin{cases} \eta\dfrac{\partial\psi}{\partial t} - \mathrm{i}\kappa\eta\psi\nabla\cdot\boldsymbol{A} + \left(\dfrac{\mathrm{i}}{\kappa}\nabla + \boldsymbol{A}\right)^2\psi + (|\psi|^2-1)\psi = 0, & (1.1)\\ \dfrac{\partial\boldsymbol{A}}{\partial t} + \mathbf{curl}\,\mathrm{curl}\,\boldsymbol{A} - \nabla(\nabla\cdot\boldsymbol{A}) + \mathrm{Re}\left\{\left(\dfrac{\mathrm{i}}{\kappa}\nabla\psi + \boldsymbol{A}\psi\right)\psi^*\right\} = \mathbf{curl}\,H_e & (1.2)\end{cases}$$

for $t \in (0,T]$ and $\boldsymbol{x} \in \Omega$, where Ω is a simply-connected polygonal domain. The complex scalar function ψ denotes the order parameter and the real vector-valued function $\boldsymbol{A}$ is the magnetic potential. Here, $\mathrm{Re}\{\cdot\}$ denotes the real part in braces and ψ^* is the complex conjugate of ψ.

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In the model, $|\psi|^2$ indicates the density of the superconducting electron pairs. $|\psi|^2 = 1$ and $|\psi|^2 = 0$ represent the perfectly superconducting state and the normal state, respectively, while $0 < |\psi|^2 < 1$ represents a mixed state. The external applied magnetic field H_e is assumed here to be a constant for simplicity, κ is the Ginzburg-Landau parameter and η is a dimensionless constant. In the following, we set $\eta = 1$. For a vector function $\boldsymbol{A} = [A_1, A_2]^\top$ and a scalar function f, the two-dimensional **curl** and curl operators are defined by

$$\mathbf{curl}\, f = \left[\frac{\partial f}{\partial y}, -\frac{\partial f}{\partial x}\right]^\top, \quad \mathrm{curl}\, \boldsymbol{A} = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}.$$

The following boundary and initial conditions are supplemented to the above TDGL equations:

$$\begin{cases} \dfrac{\partial \psi}{\partial \mathbf{n}} = 0, \quad \mathrm{curl}\, \boldsymbol{A} = H_e, \quad \boldsymbol{A} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \times [0,T], & (1.3) \\ \psi(\boldsymbol{x}, 0) = \psi_0(\boldsymbol{x}), \quad \boldsymbol{A}(\boldsymbol{x}, 0) = \boldsymbol{A}_0(\boldsymbol{x}) \quad \text{in } \Omega, & (1.4) \end{cases}$$

where $\mathbf{n}$ denotes the unit outward normal vector of Ω.

The original TDGL system (1.1)-(1.4) was analyzed by several authors. We refer to [8, 9] for theoretical analysis for smooth domains. Numerical methods have been studied extensively, see [6,7,10,11,13,14,16,19,20,31–33,35]. It is well known that boundary defect has a significant influence on the superconductors [1, 4]. Therefore, numerical methods should be suitable for non-smooth domains. In this direction, we refer to the mixed methods proposed in [15–18, 23] and Nédélec edge element approximations in [12, 21]. It should be noted that face/edge elements usually admit a higher programming threshold and computational cost than the Lagrange element methods. However, due to the low regularity of $\boldsymbol{A}$ near the re-entrant corners, conventional lagrange FEM, if used directly to solve the TDGL system (1.1)-(1.2), may fail to compute the correct vortex pattern, see the numerical results in [1].

Indeed, there is a procedure to use conventional Lagrange FEM to solve the TDGL system on domains with re-entrant corners correctly. Based on the idea of Hodge decomposition, Li and Zhang [27,28] proposed and analysed an efficient FEM to solve the above TDGL equations (1.1)-(1.4), in which one only needs to solve several standard second order scalar elliptic boundary value problems at each time step. This approach was also used in [24] for solving the magneto-hydrodynamic equations. The main idea in [27,28] is first to rewrite the vectorial variables into the combination of gradients and rotations of scalars u, v, p and q

$$\boldsymbol{A} = \mathbf{curl}\, u + \nabla v, \quad \mathrm{Re}\left\{\left(\frac{\mathrm{i}}{\kappa}\nabla\psi + \boldsymbol{A}\psi\right)\psi^*\right\} = \mathbf{curl}\, p + \nabla q. \tag{1.5}$$

Then, the original TDGL equation (1.1)-(1.2) is reformulated into an alternative form with five scalar unknowns

$$\begin{cases} \dfrac{\partial \psi}{\partial t} - \mathrm{i}\kappa\psi\nabla\cdot\boldsymbol{A} + \left(\dfrac{\mathrm{i}}{\kappa}\nabla + \boldsymbol{A}\right)^2\psi + (|\psi|^2 - 1)\psi = 0, & (1.6) \\ \Delta p = -\mathrm{curl}\left(\mathrm{Re}\left(\psi^*\left(\dfrac{\mathrm{i}}{\kappa}\nabla + \boldsymbol{A}\right)\psi\right)\right), & (1.7) \\ \Delta q = \nabla\cdot\left(\mathrm{Re}\left(\psi^*\left(\dfrac{\mathrm{i}}{\kappa}\nabla + \boldsymbol{A}\right)\psi\right)\right), & (1.8) \\ \dfrac{\partial u}{\partial t} - \Delta u = H_e - p, & (1.9) \\ \dfrac{\partial v}{\partial t} - \Delta v = -q. & (1.10) \end{cases}$$