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A NEW MIXED FINITE ELEMENT FOR THE LINEAR ELASTICITY PROBLEM IN 3D*

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Abstract

This paper constructs the first mixed finite element for the linear elasticity problem in 3D using P_3 polynomials for the stress and discontinuous P_2 polynomials for the displacement on tetrahedral meshes under some mild mesh conditions. The degrees of freedom of the stress space as well as the corresponding nodal basis are established by characterizing a space of certain piecewise constant symmetric matrices on a patch around each edge. Macro-element techniques are used to define a stable interpolation to prove the discrete inf-sup condition. Optimal convergence is obtained theoretically.

Mathematics subject classification: 65N30, 74B05.
Key words: Linear elasticity, Lower order mixed elements, Macro-element techniques, Discrete inf-sup condition.

1. Introduction

It is a challenge to design stable discretizations for the linear elasticity equations based on the Hellinger-Reissner variational principle, due to the additional symmetry constraint on the stress tensor. For the 3D problem, the first mixed finite element on tetrahedral meshes was proposed in [1] as the generalization of the lowest order case of the two dimensional symmetric mixed elements in [4]. This element was extended in [2] to higher order cases, where the displacement is approximated by discontinuous P_{k-1} polynomials, and the stress is approximated by P_{k+2} polynomials whose divergence is P_{k-1} with $k \geqslant 2$. Later, Hu and Zhang [20] designed symmetric mixed finite elements using P_k polynomials for the stress with $k \geqslant 4$, see [19] for analogous elements in 2D and [14] in any dimension. Since there are not sufficient degrees of freedom (DoFs) on faces, the analysis of the discrete inf-sup conditions in [1, 2, 20, 21] needs P_4 polynomials in stress spaces to define some stable commuting interpolations. Lower order mixed elements merely using P_k ($k < 4$) polynomials for the stress were constructed on macro-element meshes, such as $k \geqslant 2$ for Alfeld splits [10] and $k \geqslant 1$ for Worsey-Farin splits [11], and each macro-element therein consists of four and twelve elements, respectively. Other attempts to lower the polynomial order include nonconforming finite element methods [3, 7, 13, 17, 25]

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and stabilized methods [8]. Interested readers can refer to [5, 15] for mixed finite elements on cubic meshes and [16] for those on triangular prism meshes.

This paper proposes the first mixed finite element using P_3 polynomials for the stress without any higher order polynomials on tetrahedral meshes under some mild mesh conditions. The displacement space is the space of discontinuous P_2 polynomials. The newly constructed stress element is $H(\mathrm{div})$-conforming and continuous at vertices. The DoFs of this stress element cannot be defined in Ciarlet's convention. Motivated by the C^1 continuous finite elements in 2D in [22, 23], this paper establishes the DoFs of the new stress element and the corresponding basis functions by geometric analysis at edges. The parity of the number of elements in a patch around an interior edge and the singularity of the edge play an essential role and determine the choices of some face DoFs. The main ingredient of the analysis is to introduce a space $\mathbb{T}_e$ consisting of piecewise constant symmetric matrices with normal continuity on the edge patch ω_e of an edge e and vanishing tangential components along e. Such a space can be characterized by the normal-normal components and normal-tangential components of matrices on the faces sharing e. More precisely, for a boundary edge e as well as a singular edge e, the matrices in $\mathbb{T}_e$ can be uniquely determined by the normal-normal component on each face plus the normal-tangential component on one face. For an interior edge e with odd numbers of elements in ω_e, the matrices in $\mathbb{T}_e$ can be uniquely determined by the normal-normal component on each face. While for a non-singular interior edge e with even elements in ω_e, the normal-normal components of matrices in $\mathbb{T}_e$ are linearly dependent, and the matrices in $\mathbb{T}_e$ can be uniquely determined by the normal-normal components on all faces except one plus the normal-tangential component on one face. The corresponding basis of $\mathbb{T}_e$ can be computed by explicit expressions. With the characterization of $\mathbb{T}_e$, some DoFs on faces can be obtained for the new stress space, which are similar to the second order derivative DoFs on edges in [23]. The multiplications of the basis of $\mathbb{T}_e$ with scalar Lagrange basis functions lead to the corresponding basis for the new stress space. The other DoFs of the new stress space are analogous to the DoFs of the stress spaces in [9, 20]. Some estimates of the basis of $\mathbb{T}_e$ are obtained in this paper under the mesh conditions in Assumption 2.1. Analogous ideas have been applied in two dimensions to relax the continuity of the stress at vertices in [12, 18]. Instead of presenting the details of the nodal basis, a hybridized method was considered in [12]. In [18], this idea was used to deal with non-consistent traction boundary conditions.

This paper adopts the two-step method in [14, 20, 21] and some macro-element techniques to analyze the discrete inf-sup condition of the new mixed element. The first step of the two-step method therein requires to define a stable interpolation satisfying the partial commuting diagram property with respect to piecewise rigid motions. However, for a non-singular interior edge e with even number of elements in ω_e, matrices in $\mathbb{T}_e$ have linearly dependent normal-normal components on the interior faces of ω_e. As a result, it is difficult to construct a stable commuting interpolation as the higher order case in [14, Lemma 3.1]. To circumvent this, under some mild mesh conditions, this paper proposes proper linear combinations of the basis functions of the stress space for each face. These combinations are supported on macro-elements and are exactly the basis functions corresponding to the constant and linear moments of the normal-normal component of the stress on each face. This and the estimates of the basis of $\mathbb{T}_e$ enable to define a stable interpolation for the new stress space using P_3 polynomials as the shape function space to deal with the rigid motion as in [14, Lemma 3.1]. The discrete inf-sup condition is then established by using the two-step method in [14, 20, 21]. Some error estimates are provided as well.