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ON UNCONDITIONAL STABILITY OF A VARIABLE TIME STEP SCHEME FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS*

Yalan Zhang and Pengzhan Huang[1)]
College of Mathematics and System Sciences, Xinjiang University, Urumqi 830017, China
Emails: zyl2024@stu.xju.edu.cn, hpzh@xju.edu.cn
Yinnian He
College of Mathematics and System Sciences, Xinjiang University, Urumqi 830017, China;
School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China
Email: heyn@mail.xjtu.edu.cn

Abstract

In this work, an unconditionally stable, decoupled, variable time step scheme is presented for the incompressible Navier-Stokes equations. Based on a scalar auxiliary variable in exponential function, this fully discrete scheme combines the backward Euler scheme for temporal discretization with variable time step and a mixed finite element method for spatial discretization, where the nonlinear term is treated explicitly. Moreover, without any restriction on the time step, stability of the proposed scheme is discussed. Besides, error estimate is provided. Finally, some numerical results are presented to illustrate the performances of the considered numerical scheme.

Mathematics subject classification: 65M60, 65M12.
Key words: Variable time step, Navier-Stokes equations, Scalar auxiliary variable, Unconditional stability, Numerical test.

1. Introduction

Time accuracy is critical for obtaining physically relevant solutions in the field of computational fluid dynamics. Many flow solvers use constant time step, but there has been an expanding interest in variable time step solvers [4, 19, 20]. In fact, if the solution changes rapidly in some intervals of the time region, and slowly in other intervals, then the variable time step idea can obtain computationally efficient and accurate results. In 1989, Lebedev [22] proposed four explicit stable difference schemes with time-varying step sizes to solve the Cauchy problem for a system of rigid differential equations, and studied the stability condition of the optimal algorithm for selecting the time step. Becker [3] used a second-order backward difference method with variable time step to discretize the parabolic problems in time, and proved the stability by the energy method under a certain limit of the ratio of continuous time step.

Recently, a series of implicit/explicit variable time step algorithms were proposed for the incompressible Navier-Stokes equations [8], where the stability of a first-order of time accuracy algorithm with variable time step was proved. Here a Courant-Friedrichs-Lewy-type (CFL-type) condition is needed for the time and space steps [8], i.e.

$$1-\frac{C_{stab}\Delta t_n\big(1+\omega_n^2\big)}{\nu h}\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|^2\geq 0,$$

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1) Corresponding author

where $C_{stab} > 0$ is a constant independent of mesh size h, variable time step Δt_n and variable time step ratio ω_n, and $E^{n+1}(\mathbf{u}_h)$ is extrapolation of $\mathbf{u}_h^{n+1}$. To the authors' best knowledge, for the incompressible Navier-Stokes equations, the Euler implicit/explicit scheme with constant time step is almost unconditionally stable [11, 13], i.e. the time step restriction is $\Delta t \leq C_{stab}$. Hence, it is natural to consider whether variable time step scheme for the Navier-Stokes equations can own almost unconditional stability, even unconditional stability. In fact, the unconditional stability is of great importance for the numerical method.

The scalar auxiliary variable method, which is based on the invariant energy quadratization method, has been initially studied by Shen *et al.* [32, 33]. It addresses the disadvantage of the invariant energy quadratization method, but it still retains unconditional stability of the invariant energy quadratization method. Based on unconditional stability of the scalar auxiliary variable method, several stable schemes have undergone some evolution and been well further developed [21, 23, 24, 26, 27, 30]. In particular, in [29], the authors proposed an equivalent system of the numerical scheme of the incompressible Navier-Stokes equation based on the auxiliary variable related to the total kinetic energy, which satisfied the discrete unconditional energy stability. In [25], some first-order and second-order pressure correction schemes for the Navier-Stokes equations were established by using the scalar auxiliary variable method. The strict error estimates of velocity and pressure without any conditional time step limitation were also established. In addition, based on the scalar auxiliary variable method, Zhang and Yuan [38] showed the unconditional stability and convergence analysis of the Euler implicit/explicit scheme with constant time step for the incompressible Navier-Stokes equations.

Inspired by [8, 33], for the incompressible Navier-Stokes equations, we design a decoupled, variable time step scheme. Unlike the stability in [8], we obtain unconditional stability of this scheme.

The main contents of this paper are listed as follows. In Section 2, we review some basic results and give some function spaces. In Section 3, we design a variable time step scheme for the incompressible Navier-Stokes equations and discuss the unconditional stability results. Besides, error estimates of numerical solution are given. Some numerical experiments are carried out to illustrate the theoretical results in Section 4. Finally, we draw a conclusion in final section.

2. A Variable Time Step Scheme

2.1. Notation and preliminaries

Let $\Omega \subset \mathbb{R}^2$ be an open domain with Lipschitz continuous boundary $\partial\Omega$. We consider the time-dependent incompressible Navier-Stokes equations

$$
\begin{aligned}
&\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} && \text{in } \Omega \times J, \\
&\nabla \cdot \mathbf{u} = 0 && \text{in } \Omega \times J, \\
&\mathbf{u} = \mathbf{0} && \text{on } \partial\Omega \times J, \\
&\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) && \text{in } \Omega
\end{aligned} \tag{2.1}
$$

for $(\mathbf{x}, t) \in \Omega \times J$ and $J = (0, T]$ with a fixed $T \in (0, \infty)$. Here, $\mathbf{u}(\mathbf{x}, t) : \Omega \times J \to \mathbb{R}^2$ and $p(\mathbf{x}, t) : \Omega \times J \to \mathbb{R}$ represent the unknown velocity field and the pressure of the flow in Ω, respectively. Besides, $\mathbf{f}(\mathbf{x}, t) : \Omega \times J \to \mathbb{R}^2$ is body force on the flow, and $\nu > 0$ is the kinematic viscosity.