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INVERSE SCATTERING BY A PENETRABLE CONDUCTIVE MEDIUM AND AN IMPENETRABLE OBSTACLE WITH GENERALIZED OBLIQUE DERIVATIVE BOUNDARY CONDITION*

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Abstract

We consider an inverse problem of scattering by a mixed-type scatterer consisting of an inhomogeneous penetrable conductive medium and an impenetrable obstacle with generalized oblique derivative boundary condition induced by incident plane waves scattering. Relying on the well-posedness of the direct problem which can be proved directly by a variational method, we are interested in studying the inverse problem of developing a modified factorization method to simultaneously reconstruct the shape and location of the mixed-type scatterer. The complex refractive index and the generalized oblique derivative boundary condition may bring new challenges since the factorization method is closely related to the refractive index and the mixed boundary conditions. Finally, some numerical examples are given to show the effectiveness and feasibility of the inversion algorithm.

Mathematics subject classification: 35R30, 35Q60, 35P25, 78A46.
Key words: Inverse scattering, Generalized oblique derivative boundary condition, Modified factorization method, Inhomogeneous medium.

1. Introduction

In this paper, we consider the inverse scattering problem of reconstructing a mixed-type scatterer in $\mathbb{R}^2$ from the far-field data induced by incident time-harmonic plane waves. The mixed-scatterer is assumed to be a combination of a penetrable inhomogeneous conductive medium and an impenetrable obstacle with generalized oblique derivative boundary condition. This kind of inverse problem arises in many fields, for example, radar and sonar, medical imaging and determining the gravitational fields of celestial bodies. More specifically, a bounded region obstacle D_1 represents an inhomogeneous penetrable conductive medium with C^2-smooth boundary ∂D_1, and an impenetrable obstacle D_2 represents a bounded simply connected region with C^2-smooth boundary ∂D_2. We further assume that $D_1 \cap D_2 = \emptyset$ (the geometry of the mixed scatterer is shown in Fig. 1.1).

Suppose that the refractive index $n(x)$ describes the inhomogeneity of the conductive medium D_1 satisfying that $n(x) \in L^\infty(\mathbb{R}^2)$ with $\mathrm{Re}[n(x)] < 1$ and $\mathrm{Im}[n(x)] \geq c_0 > 0$ with a positive constant c_0, and the external domain $\mathbb{R}^2 \backslash (\overline{D}_1 \cup \overline{D}_2)$ is homogeneous with the refractive index $n(x) = 1$. In this paper, a combination boundary consisting of the oblique derivative boundary condition and the impedance boundary condition is imposed on the boundary of the impenetrable obstacle D_2, which represents the influence of daily rotation of the earth on the ocean

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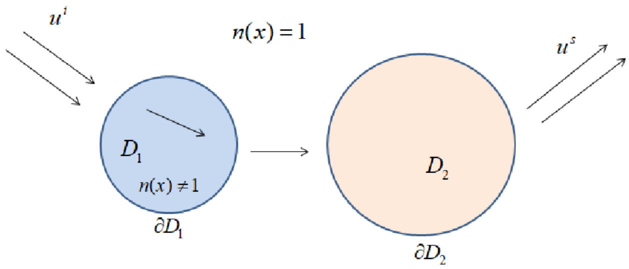


Fig. 1.1. Geometric representation of mixed scattering problem.

waves. We assume further that the incident waves are time-harmonic plane waves $u^i = e^{ikx\cdot d}$ with the incident directions $d \in \mathbb{S}^1$ and the wave number $k > 0$. Then the scattering of time-harmonic plane waves $u^i = e^{ikx\cdot d}$ by the mixed-type scatterer can be established by the following mathematical model:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^2 \backslash (\overline{D}_1 \cup \overline{D}_2), \\ \Delta v + k^2 n(x) v = 0 & \text{in } D_1, \\ \dfrac{\partial u}{\partial \nu} + i\lambda \dfrac{\partial u}{\partial \tau} + i\mu u = 0 & \text{on } \partial D_2. \end{cases} \tag{1.1}$$

Here ν is the unit outward normal on ∂D_2 directed into $\mathbb{R}^2\backslash\overline{D}_2$ and τ is unit tangent vector, while μ is the impedance coefficient satisfying that $\mu \geq 0$ and the real parameter $\lambda < 1$. In (1.1), $u = u^i + u^s$ denotes the total field in $\mathbb{R}^2\backslash(\overline{D}_1 \cup \overline{D}_2)$, where u^s is the associated scattered wave. Similarly, $v = u^i + v^s$ denotes the total field with v^s being the internal scattering field. Furthermore, the scattered field u^s satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0, \quad r = |x|, \tag{1.2}$$

which holds uniformly with respect to $\widehat{x} = x/|x| \in \mathbb{S}^1$. It is well-known that u^s has the asymptotic behavior [10]

$$u^s(x,d) = \frac{e^{ik|x| + i\pi/4}}{\sqrt{8\pi k|x|}} u_\infty(\widehat{x}, d) + \mathcal{O}\left(\frac{1}{|x|^{3/2}}\right) \quad \text{as} \quad |x| \to \infty, \tag{1.3}$$

uniformly for all $\widehat{x}$, where u_∞ is known as the far-field pattern of the scattered field u^s.

In this paper, we consider the inverse problem of simultaneously reconstructing the shape and location of the penetrable inhomogeneous conductive medium and the impenetrable obstacle with a generalized oblique derivative boundary condition, based on the knowledge of the far-field pattern u_∞ corresponding to the fixed wave number k and the time-harmonic incident plane waves in all directions.

The well-posedness of problems (1.1)-(1.2) can be verified by means of the variational method (see also [21, 39]). The generalized oblique derivative boundary condition investigated in this paper is a combination of an oblique derivative boundary condition and an impedance boundary condition, and it has fundamental differences from the so-called Poincaré condition or the generalized impedance boundary condition. Specifically, the Poincaré condition requires that the tangential derivative part has only real coefficients. This requirement results in the loss of the ellipticity of the associated solution. For the well-posedness results of the electromagnetic scattering problem under the Poincaré condition, we refer readers to [26–28, 30, 31, 37]. In contrast, the tangential derivative of the boundary condition in this paper has complex coefficients. This characteristic can introduce more accurate and intricate reflecting effects for our inverse scattering problem. Although our boundary conditions may seem similar to the generalized impedance boundary condition, they have distinct connotations. Besides including the normal derivative, the generalized impedance boundary condition also contains the surface divergence and the gradient term, making it a third-order boundary condition precisely. For the mathematical theory and reconstruction algorithms regarding both the direct and inverse problems of determining the shape and location of D, and its scattering coefficients under the generalized impedance boundary condition, readers are referred to [3, 4, 8, 13, 20, 29, 41].

For the inverse scattering by one obstacle with the generalized oblique derivative boundary condition, there are some related studies. For example, a boundary integral equation method was applied in [22] for solving the direct scattering problem. In [37], relying on the investigations of the associated Green function and the reciprocity relations, they obtained the corresponding uniqueness results for the inverse problem with a generalized oblique derivative boundary condition. Related numerical methods for the reconstruction of scattering obstacles with oblique derivative boundary conditions can also be found in [36] for a linear sampling method and in [38] for a classical factorization method.

The focus of our research lies in the inverse problem of simultaneously reconstructing the shape and location of the penetrable inhomogeneous conductive medium and the impenetrable obstacle with a generalized oblique derivative boundary condition by developing the modified factorization method. It is proposed in [16] that the factorization method is the most rigorous one of the qualitative methods, providing a fast and computable criterion for characterizing the scatterer position from the measured data. The generalization of scattering results of the Dirichlet boundary condition or the other boundary conditions was introduced in [17]. Recently, the factorization method has been widely extended to the inhomogeneous medium scattering problems [7, 25, 33, 34] or the mixed scattering problems. For instances, the scattering problem of time-harmonic acoustic plane waves by a mixed scatterer consisting of bounded impenetrable sound-soft obstacle and a penetrable inhomogeneous medium is verified in [18] by using the classical factorization method. Motivated by [18, 34], the authors in [40] justify the validity of the classical factorization method for the reconstruction algorithm for the electromagnetic scattering of a combination of two different inhomogeneous media. In addition, we also refer to the inverse scattering problem with inhomogeneous cavities [23, 35] and the impedance boundary condition [41]. We also recommend that the readers read [5, 17] to understand the inverse scattering of obstacles under different boundary conditions and [14, 15, 19] for the convexification method for coefficient inverse problems.

In fact, the factorization method is closely related to the refractive index $n(x)$ and the mixed oblique derivative boundary condition, so the classical factorization methods in [1, 2] cannot be applied directly to our inverse problem. Unlike [18, 33, 38, 40], our inverse scattering problem has some new characterizations due to the complex refractive index and the generalized oblique derivative boundary condition, which is a mixed boundary condition of an oblique derivative and an impedance boundary. In fact, our goal is to establish a modified factorization method to reconstruct the mixed-type scatterer, the main ingredient for solving this difficulty is to construct a series of perturbation operators of the far-field operator, which satisfy the range identity in [17, Theorem 2.15]. Then the corresponding numerical reconstruction can be carried out. The related qualitative methods for instance, the linear sampling method and the reciprocity gap functional method are often used for the inverse scattering problem of inhomogeneous media [6, 24]. In addition, some other numerical reconstruction methods can also be found, including point source method [32] and the iteration method [12, 42].

The remaining structure of this paper is as follows. In Section 2, we present some related operators and the well-posedness result for our scattering problem, as well as the basic properties of data-to-pattern operator. In Section 3, we mainly developed a modified factorization method to simultaneously image the shape and location of the mixed-type scatterer. In Section 4, some numerical simulations will be introduced to illustrate the feasibility of the inversion algorithms.

2. Properties of Operators Associated with the Factorization

In this section, we shall introduce some operators related to the conductive medium and the generalized oblique derivative boundary condition and its related properties for the factorization method for our inverse problem. So, we firstly introduce the following integral operators $\mathbf{V}_{D_1D_1}$, $\tilde{K}_{\partial D_2 D_1}, \tilde{H}_{\partial D_2 D_1}, \tilde{S}_{\partial D_2 D_1}, \tilde{V}_{D_1\partial D_2}, V_{D_1\partial D_2}, S_{\partial D_2\partial D_2}, K_{\partial D_2\partial D_2}, K^{'}_{\partial D_2\partial D_2}, T_{\partial D_2\partial D_2}$, $H_{\partial D_2\partial D_2}, H^{'}_{\partial D_2\partial D_2}$ defined as

$$
\begin{aligned}
(\mathbf{V}_{D_1D_1}\varphi)(x) &= \int_{D_1} \Phi(x,y)\varphi(y)dy, && x \in D_1,\\
(\tilde{K}_{\partial D_2 D_1}\varphi)(x) &= \int_{\partial D_2} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\varphi(y)ds, && x \in D_1,\\
(\tilde{H}_{\partial D_2 D_1}\varphi)(x) &= \int_{\partial D_2} \frac{\partial \Phi(x,y)}{\partial \tau(y)}\varphi(y)ds, && x \in D_1,\\
(\tilde{S}_{\partial D_2 D_1}\varphi)(x) &= \int_{\partial D_2} \Phi(x,y)\varphi(y)ds, && x \in D_1,\\
(\tilde{V}_{D_1\partial D_2}\varphi)(x) &= \frac{\partial}{\partial \tau(y)}\int_{D_1} \Phi(x,y)\varphi(y)dy, && x \in \partial D_2,\\
(V_{D_1\partial D_2}\varphi)(x) &= \int_{D_1} \Phi(x,y)\varphi(y)dy, && x \in \partial D_2,\\
(S_{\partial D_2\partial D_2}\varphi)(x) &= \int_{\partial D_2} \Phi(x,y)\varphi(y)ds, && x \in \partial D_2,\\
(K_{\partial D_2\partial D_2}\varphi)(x) &= \int_{\partial D_2} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\varphi(y)ds, && x \in \partial D_2,\\
(K^{'}{}_{\partial D_2\partial D_2}\varphi)(x) &= \int_{\partial D_2} \frac{\partial \Phi(x,y)}{\partial \nu(x)}\varphi(y)ds, && x \in \partial D_2,\\
(T_{\partial D_2\partial D_2}\varphi)(x) &= \frac{\partial}{\partial \nu(x)}\int_{\partial D_2} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\varphi(y)ds, && x \in \partial D_2,\\
(H_{\partial D_2\partial D_2}\varphi)(x) &= \int_{\partial D_2} \frac{\partial \Phi(x,y)}{\partial \tau(y)}\varphi(y)ds, && x \in \partial D_2,\\
(H^{'}{}_{\partial D_2\partial D_2}\varphi)(x) &= \int_{\partial D_2} \frac{\partial \Phi(x,y)}{\partial \tau(x)}\varphi(y)ds, && x \in \partial D_2,
\end{aligned}
$$

where

$$\Phi(x,y) = \frac{i}{4}H_0^{(1)}(k|x-y|), \quad x \neq y \quad \text{in } \mathbb{R}^2$$

is the fundamental solution of the Helmholtz equation. By the boundedness of the trace operator, we deduce that the operators

$$\begin{aligned}
\mathbf{V}_{D_1D_1}: L^2(D_1) &\to H^2(D_1),\\
\tilde{K}_{\partial D_2 D_1}: H^{\frac{1}{2}}(\partial D_2) &\to H^1(D_1),\\
\tilde{H}_{\partial D_2 D_1}: H^{\frac{1}{2}}(\partial D_2) &\to H^1(D_1),\\
\tilde{S}_{\partial D_2 D_1}: H^{-\frac{1}{2}}(\partial D_2) &\to H^1(D_1),\\
\tilde{V}_{D_1\partial D_2}: L^2(D_1) &\to H^{-\frac{1}{2}}(\partial D_2),\\
V_{D_1\partial D_2}: L^2(D_1) &\to H^2(\partial D_2),\\
S_{\partial D_2\partial D_2}: H^{-\frac{1}{2}}(\partial D_2) &\to H^{\frac{1}{2}}(\partial D_2),\\
K_{\partial D_2\partial D_2}: H^{\frac{1}{2}}(\partial D_2) &\to H^{\frac{1}{2}}(\partial D_2),\\
K'_{\partial D_2\partial D_2}: H^{-\frac{1}{2}}(\partial D_2) &\to H^{-\frac{1}{2}}(\partial D_2),\\
T_{\partial D_2\partial D_2}: H^{\frac{1}{2}}(\partial D_2) &\to H^{-\frac{1}{2}}(\partial D_2),\\
H_{\partial D_2\partial D_2}: H^{\frac{1}{2}}(\partial D_2) &\to H^{\frac{1}{2}}(\partial D_2),\\
H'_{\partial D_2\partial D_2}: H^{-\frac{1}{2}}(\partial D_2) &\to H^{-\frac{1}{2}}(\partial D_2)
\end{aligned}$$

are all bounded. The mapping properties and jump relations of the operators S, K, K' and T have been studied in [9]. Furthermore, it can be verified that there are no jumping relations between H and H', and we also define the gradient of the double-layer potential by

$$A(x) = \int_{\partial D_2} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \varphi(y) ds, \quad x \in \mathbb{R}^2 \backslash \partial D_2.$$

It then follows that

$$\nabla A(x) = k^2 \int_{\partial D_2} \Phi(x,y)\varphi(y)\nu(y) ds(y) + \left(\frac{\partial B(x)}{\partial x_2}, -\frac{\partial B(x)}{\partial x_1} \right),$$

where $x = (x_1, x_2) \in \mathbb{R}^2 \backslash \partial D_2$, and

$$B(x) = \int_{\partial D_2} \Phi(x,y) \frac{\partial \varphi(y)}{\partial \tau(y)} ds = -\int_{\partial D_2} \frac{\partial \Phi(x,y)}{\partial \tau(y)} \varphi(y) ds, \quad x \in \mathbb{R}^2 \backslash \partial D_2.$$

Then by employing the relation between $\nu = (s_2, -s_1)$ and $\tau = (s_1, s_2)$, we find that the normal derivative of $A(x)$ is given by

$$\begin{aligned}
\frac{\partial A_\pm}{\partial \nu}(x) &= \frac{\partial B(x)}{\partial \tau(x)} + k^2 \nu \cdot \big(S(\varphi\nu)\big)(x)\\
&= \left(H' \frac{\partial \varphi}{\partial \tau} \right)(x) + k^2 \nu \cdot \big(S(\varphi\nu)\big)(x),\\
&= \frac{\partial}{\partial \nu} \int_{\partial D_2} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \varphi(y) ds = T\varphi, \quad x \in \partial D_2,
\end{aligned} \tag{2.1}$$

and the tangential derivative is given by

$$\begin{aligned}
\frac{\partial A_\pm}{\partial \tau}(x) &= -\frac{\partial B(x)}{\partial \nu(x)} + k^2 \tau \cdot (S(\varphi\nu))(x)\\
&= \left(-K' \frac{\partial \varphi}{\partial \tau} \pm \frac{1}{2} \frac{\partial \varphi}{\partial \tau} \right)(x) + k^2 \tau \cdot \big(S(\varphi\nu)\big)(x), \quad x \in \partial D_2.
\end{aligned} \tag{2.2}$$

Here the notations "$-$" and "$+$" denote the limits on the boundary ∂D_2 from the interior and exterior of D_2, respectively. The properties of the operators $H, H^{'}$ and A can be found in [37, 38].

We next to show the properties of the data-to-pattern operator G associated with the well-posedness result of problem (1.1)-(1.2). So, we first briefly describe the well-posedness result of the scattering problem (1.1)-(1.2) (for a proof we refer the reader to [21, 39]). It is observed that the scattering field $(u^s, v^s) := (w, p)$ satisfies the following boundary value problem:

$$\begin{cases} \Delta w + k^2 w = 0 & \text{in } \mathbb{R}^2 \backslash (\overline{D}_1 \cup \overline{D}_2), \\ \Delta p + k^2 n(x) p = -q f_1 & \text{in } D_1, \\ \dfrac{\partial w}{\partial \nu} + i\lambda \dfrac{\partial w}{\partial \tau} + i\mu w = -f_2 & \text{on } \partial D_2, \\ \lim\limits_{r\to\infty} \sqrt{r}\left(\dfrac{\partial w}{\partial r} - ikw\right) = 0, & r = |x|, \end{cases} \tag{2.3}$$

where

$$\begin{aligned} & f_1 := u^i && \text{in } D_1, \\ & f_2 := \frac{\partial u^i}{\partial \nu} + i\lambda \frac{\partial u^i}{\partial \tau} + i\mu u^i && \text{on } \partial D_2, \\ & q := k^2[n(x) - 1] && \text{in } D_1. \end{aligned}$$

We now state the well-posedness result of problem (2.3) (for a similar proof we refer to [38]).

Theorem 2.1. *For any $f_1 \in L^2(D_1)$ and $f_2 \in H^{-1/2}(\partial D_2)$, there exists a unique solution $(w, p) \in H^1(B_R \backslash (\overline{D}_1 \cup \overline{D}_2)) \times H^1(D_1)$ to problem (2.3) satisfying that*

$$\|w\|_{H^1(B_R \backslash (\overline{D}_1 \cup \overline{D}_2))} + \|p\|_{H^1(D_1)} \leq C\big(\|f_1\|_{L^2(D_1)} + \|f_2\|_{H^{-1/2}(\partial D_2)}\big), \tag{2.4}$$

where B_R is a ball centered at the origin with the radius R large enough such that $\overline{D}_1 \cup \overline{D}_2 \subset B_R$ and $C > 0$ is a positive constant independent of f_1, f_2.

Based on Theorem 2.1, one can define the data-to-pattern operator $G : Y \to L^2(\mathbb{S}^1)$ by

$$G(f_1, f_2)^\top = w_\infty, \tag{2.5}$$

where w_∞ is the far-field pattern of the solution w to problem (2.3), and

$$Y := L^2(D_1) \times H^{-\frac{1}{2}}(\partial D_2).$$

For the solution operator G, we have the following lemma.

Lemma 2.1. *G is compact and has dense range in $L^2(\mathbb{S}^1)$.*

Proof. The compactness of the operator G can be obtained from the interior regularity results of the elliptic equations [11]. As long as we prove that the L^2-adjoint operator G^* of G is injective, we can obtain the denseness of the range of G in $L^2(\mathbb{S}^1)$. Let (u, v) be a solution to the following problem:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^2 \backslash (\overline{D}_1 \cup \overline{D}_2), \\ \Delta v + k^2 n(x) v = 0 & \text{in } D_1, \\ \frac{\partial u}{\partial \nu} - i\lambda \dfrac{\partial u}{\partial \tau} + i\mu u = 0 & \text{on } \partial D_2, \\ \lim\limits_{r\to\infty} \sqrt{r}\left(\dfrac{\partial u^s}{\partial r} - iku^s\right) = 0, & r = |x|, \end{cases} \tag{2.6}$$

corresponding to the incident field

$$u^i(y) = \int_{\mathbb{S}^2} e^{-ikd\cdot y}\overline{\varphi(d)}ds(d), \quad y \in \mathbb{R}^2, \quad \varphi \in L^2(\mathbb{S}^1), \tag{2.7}$$

where $u^s = u - u^i$ is the related scattered field.

Assume further that (w, p) is a solution of problem (2.3) with the data $(f_1, f_2)^\top \in L^2(D_1) \times H^{-1/2}(\partial D_2)$ induced by the incident wave field $w^i = e^{ikx\cdot d}$. Let w_∞ be the far-field pattern corresponding to the scattered field w. It then follows from Green's representation theorem that

$$\begin{aligned} w_\infty(d) = \gamma \int_{\partial D_1} &\left[w(y)\frac{\partial e^{-ikd\cdot y}}{\partial \nu(y)} - \frac{\partial w}{\partial \nu}(y)e^{-ikd\cdot y}\right] ds(y) \\ + \gamma \int_{\partial D_2} &\left[w(y)\frac{\partial e^{-ikd\cdot y}}{\partial \nu(y)} - \frac{\partial w}{\partial \nu}(y)e^{-ikd\cdot y}\right] ds(y), \end{aligned} \tag{2.8}$$

where $\gamma = e^{ik/4}/\sqrt{8\pi k}$. Then, by the definition of the operator G, it can be verified that

$$\begin{aligned} &\left\langle G(f_1, f_2)^\top, \varphi \right\rangle_{L^2(\mathbb{S}^1)} \\ =& \int_{\mathbb{S}^2} w_\infty(d)\overline{\varphi(d)}ds(d) \\ =& \gamma \int_{\partial D_1} \left[w(y)\frac{\partial u^i}{\partial \nu}(y) - \frac{\partial w}{\partial \nu}(y)u^i(y)\right] ds(y) \\ &+ \gamma \int_{\partial D_2} \left[w(y)\frac{\partial u^i}{\partial \nu}(y) - \frac{\partial w}{\partial \nu}(y)u^i(y)\right] ds(y). \end{aligned} \tag{2.9}$$

Noting that both w and u^s satisfy the Sommerfeld radiation condition yields

$$\int_{\partial D_1} \left[w(y)\frac{\partial u^s}{\partial \nu}(y) - \frac{\partial w}{\partial \nu}(y)u^s(y)\right] ds(y) = 0.$$

Therefore, by applying Green's theorem, the conductive boundary condition and the generalized oblique derivative boundary condition, we deduce that

$$\begin{aligned} &\left\langle G(f_1, f_2)^\top, \varphi \right\rangle_{L^2(\mathbb{S}^1)} \\ =& \gamma \int_{\partial D_1} \left[w(y)\frac{\partial u}{\partial \nu}(y) - \frac{\partial w}{\partial \nu}(y)u(y)\right] ds(y) \\ &+ \gamma \int_{\partial D_2} \left[w(y)\frac{\partial u}{\partial \nu}(y) - \frac{\partial w}{\partial \nu}(y)u(y)\right] ds(y) \\ =& \gamma \int_{\partial D_1} \left[w(y)\frac{\partial u}{\partial \nu}(y) - \frac{\partial w}{\partial \nu}(y)u(y)\right] ds(y) \\ &+ \gamma \int_{\partial D_2} \left[w(y)\left(i\lambda\frac{\partial u}{\partial \tau}(y) - i\mu u\right) - \frac{\partial w}{\partial \nu}(y)u(y)\right] ds(y) \\ =& \gamma \int_{D_1} quf_1 dy + \gamma \int_{\partial D_2} \left\{ -\left(\frac{\partial w}{\partial \nu}(y) + i\lambda\frac{\partial w}{\partial \tau}(y) + i\mu w(y)\right) u(y)\right\} ds(y) \\ =& \gamma \int_{D_1} quf_1 dy + \gamma \int_{\partial D_2} f_2 u ds(y) \\ =& \left\langle (f_1, f_2)^\top, G^*\varphi \right\rangle_{L^2(\mathbb{S}^1)}. \end{aligned}$$

It is thus concluded that

$$G^*\varphi = \overline{\gamma}(\overline{qu}|_{D_1}, \overline{u}|_{\partial D_2})^\top, \quad \varphi \in L^2(\mathbb{S}^1). \tag{2.10}$$

Let $G^*\varphi = 0$. It follows from (2.10) that $u = 0$ in D_1 and $u = 0$ on ∂D_2, which further implies that $\partial u/\partial\nu = 0$ on ∂D_2 by the oblique derivative boundary condition. So we derive from Holmgren's uniqueness theorem that $u = u^i + u^s = 0$ in $\mathbb{R}^2\backslash\overline{D}_2$. One then obtains that $u^i = 0$ in $\mathbb{R}^2\backslash\overline{D}_2$ due to the fact that u^i does not satisfy the radiation condition. Hence, by [10, Theorem 3.19] we deduce that $\varphi = 0$. Thus the operator G^* is injective. This completes the proof of the lemma. □

3. A Modified Factorization Method for the Simultaneous Reconstruction

In this section, we intend to reconstruct the location and shape of the mixed-type scatterer. To this end, we first introduce the far-field operator $F : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ given by

$$(Fg)(\widehat{x}) = \int_{\mathbb{S}^1} w_\infty(\widehat{x}; d)g(d)ds(d), \quad g \in L^2(\mathbb{S}^1), \tag{3.1}$$

where w_∞ is the far-field pattern of problem (2.3) associated with the incident plane wave $u^i = e^{ikx\cdot d}$. Define an incident operator $H : L^2(\mathbb{S}^1) \to Y$ by $H = (H_1, H_2)^\top$ with

$$H_1g(x) = \gamma \int_{\mathbb{S}^1} e^{ikx\cdot d}g(d)ds(d), \qquad x \in D_1, \tag{3.2}$$

$$H_2g(x) = \gamma\left(\frac{\partial}{\partial\nu(x)} + i\lambda\frac{\partial}{\partial\tau(x)} + i\mu\right)\int_{\mathbb{S}^1} e^{ikx\cdot d}g(d)ds(d), \quad x \in \partial D_2. \tag{3.3}$$

Then $F = GH$ follows from the superposition principle and the definitions of the operators G and H. So we have the following factorization theorem.

Theorem 3.1. *F has the following factorization form:*

$$F = GM^*G^*, \tag{3.4}$$

where $M : Y^* \to Y$ is defined by

$$M = \begin{pmatrix} q^{-1}I - \boldsymbol{V}_{D_1D_1} & -\tilde{K}_{\partial D_2D_1} + i\lambda\tilde{H}_{\partial D_2D_1} + i\mu\tilde{S}_{\partial D_2D_1} \\ -\dfrac{\partial V_{D_1\partial D_2}}{\partial\nu} - i\lambda\tilde{V}_{D_1\partial D_2} - i\mu V_{D_1\partial D_2} & -T_{\partial D_2\partial D_2} + A_{22}^{(2)} \end{pmatrix} \tag{3.5}$$

with

$$\begin{aligned} A_{22}^{(2)} := {} & i\mu K'_{\partial D_2\partial D_2} - i\lambda k^2\tau\cdot S_{\partial D_2\partial D_2}(\nu) + \lambda^2 H'_{\partial D_2\partial D_2}\frac{\partial}{\partial\tau} - \mu\lambda H'_{\partial D_2\partial D_2} \\ & - i\mu K_{\partial D_2\partial D_2} - \mu\lambda H_{\partial D_2\partial D_2} - \mu^2 S_{\partial D_2\partial D_2} - i\mu I. \end{aligned}$$

Proof. For $\varphi = (\varphi_1, \varphi_2)$, from the definition of the operator H, it is easy to derive that the adjoint operator $H^* : Y^* \to L^2(\mathbb{S}^1)$ has the form

$$\begin{aligned} (H^*\varphi)(d) = {} & \gamma\int_{D_1} e^{-ikd\cdot y}\varphi_1(y)dy \\ & + \gamma\int_{\partial D_2}\left(\frac{\partial}{\partial\nu(y)} - i\lambda\frac{\partial}{\partial\tau(y)} - i\mu\right)e^{-ikd\cdot y}\varphi_2(y)ds(y), \end{aligned} \tag{3.6}$$

which is the far-field pattern of the function w defined by

$$
\begin{aligned}
w(x) = &\int_{D_1} \Phi(x,y)\varphi_1(y)dy \\
&+ \int_{\partial D_2} \left(\frac{\partial \Phi(x,y)}{\partial \nu(y)} - i\lambda \frac{\partial \Phi(x,y)}{\partial \tau(y)} - i\mu \Phi(x,y) \right) \varphi_2(y) ds(y), \quad x \in \mathbb{R}^2 \backslash \partial D_2. \qquad (3.7)
\end{aligned}
$$

When x approaches the boundary ∂D_2, we obtain from the jump relations of three different potentials in (3.7) that

$$
\begin{aligned}
\frac{\partial w}{\partial \nu} &= \frac{\partial V_{D_1 \partial D_2}}{\partial \nu}\varphi_1 + T\varphi_2 + i\lambda K' \frac{\partial \varphi_2}{\partial \tau} - \frac{i\lambda}{2}\frac{\partial \varphi_2}{\partial \tau} - i\mu K' \varphi_2 + \frac{i\mu}{2}\varphi_2, \qquad x \in \partial D_2, \\
\frac{\partial w}{\partial \tau} &= \tilde{V}\varphi_1 - K' \frac{\partial \varphi_2}{\partial \tau} + \frac{1}{2}\frac{\partial \varphi_2}{\partial \tau} + k^2 \tau \cdot \big(S(\varphi_2 \nu)\big) + i\lambda H' \frac{\partial \varphi_2}{\partial \tau} - i\mu H' \varphi_2, \quad x \in \partial D_2,
\end{aligned}
$$

and

$$
w = V\varphi_1 + K\varphi_2 + \frac{1}{2}\varphi_2 - i\lambda H\varphi_2 - i\mu S\varphi_2.
$$

It is easily found that w solves problem (2.3) with the following data $(f_1, f_2)^\top$:

$$
\begin{aligned}
f_1 &= q^{-1}\varphi_1 - \mathbf{V}_{D_1 D_1}\varphi_1 - \tilde{K}_{\partial D_2 D_1}\varphi_2 + i\lambda \tilde{H}_{\partial D_2 D_1}\varphi_2 + i\mu \tilde{S}_{\partial D_2 D_1}\varphi_2, \\
f_2 &= -\frac{\partial V_{D_1 \partial D_2}}{\partial \nu}\varphi_1 - i\lambda \tilde{V}_{D_1 \partial D_2}\varphi_1 - i\mu V_{D_1 \partial D_2}\varphi_1 - T_{\partial D_2 \partial D_2}\varphi_2 + A_{22}^{(2)}\varphi_2,
\end{aligned}
$$

where $A_{22}^{(2)}$ is defined above. Therefore,

$$
H^*\varphi = w_\infty = G(f_1, f_2)^\top = GM\varphi.
$$

Thus, $H = M^* G^*$. Recalling $F = GH$ yields that $F = GM^*G^*$. This completes the proof of the theorem. □

For the properties of the operator M, we have the following theorem.

Theorem 3.2. *The operator M defined in Theorem* 3.1 *is invertible and $M^{-1} = M_1^{-1} + M_3$, where*

$$
M_1^{-1} = \begin{pmatrix} qI & 0 \\ 0 & \big(-T_{\partial D_2 \partial D_2}(i)\big)^{-1} \end{pmatrix}, \qquad (3.8)
$$

and the operator $M_3 = -M_1^{-1}M_2M^{-1}$ is compact and M_2 is defined in Eq. (3.9) *below.*

Proof. We first decompose the operator M into $M = M_1 + M_2$ as follows:

$$
\begin{aligned}
M = &\begin{pmatrix} q^{-1}I & 0 \\ 0 & -T_{\partial D_2 \partial D_2}(i) \end{pmatrix} \\
&+ \begin{pmatrix} -\mathbf{V}_{D_1 D_1} & -\tilde{K}_{\partial D_2 D_1} + i\lambda \tilde{H}_{\partial D_2 D_1} + i\mu \tilde{S}_{\partial D_2 D_1} \\ -\dfrac{\partial V_{D_1 \partial D_2}}{\partial \nu} - i\lambda \tilde{V}_{D_1 \partial D_2} - i\mu V_{D_1 \partial D_2} & -\big(T_{\partial D_2 \partial D_2} - T_{\partial D_2 \partial D_2}(i)\big) + A_{22}^{(2)} \end{pmatrix} \\
=&: M_1 + M_2. \qquad (3.9)
\end{aligned}
$$

Clearly, M_1 is invertible on Y and the compactness of M_2 on Y can be derived from the compact embedding theorem and the compactness of the ingredient operator in M_2 (see e.g. [37, 38]). This ensures that $M = M_1 + M_2$ is Fredholm-type operator. We aim to prove the injectivity of the operator M. Let $M\varphi = 0$ for $\varphi = (\varphi_1, \varphi_2)^\top \in Y^*$. Based on Theorem 3.1, we conclude that (3.7) is a solution to problem (2.3) with the data $(f_1, f_2)^\top = (0, 0)^\top$. So

$$\frac{\partial w_+}{\partial \nu} + i\lambda \frac{\partial w_+}{\partial \tau} + i\mu w_+ = 0 \quad \text{on } \partial D_2.$$

Then the uniqueness of problem (2.3) leads to that $w(x) = 0$ in $\mathbb{R}^2 \backslash \overline{D_2}$. Since $\Delta w + k^2 w = -\varphi_1$ in D_1, we have that $\varphi_1 = 0$. Moreover, the jump relations of single and double layer operators imply that

$$\begin{aligned} &w_- = w_- - w_+ = -\varphi_2, \\ &\frac{\partial w_-}{\partial \nu} = \frac{\partial w_-}{\partial \nu} - \frac{\partial w_+}{\partial \nu} = i\lambda \frac{\partial \varphi_2}{\partial \tau} - i\mu\varphi_2 \quad \text{on } \partial D_2. \end{aligned} \tag{3.10}$$

Combining the two formulas yields that

$$\frac{\partial w_-}{\partial \nu} + i\lambda \frac{\partial w_-}{\partial \tau} - i\mu w_- = 0 \quad \text{on } \partial D_2.$$

Thus, we observe that w solves the following problem:

$$\begin{cases} \Delta w_- + k^2 w_- = 0 & \text{in } D_2, \\ \dfrac{\partial w_-}{\partial \nu} + i\lambda \dfrac{\partial w_-}{\partial \tau} - i\mu w_- = 0 & \text{on } \partial D_2. \end{cases} \tag{3.11}$$

Hence, we obtain that $w_- = 0$ in D_2 from the uniqueness of problem (3.11). This combines with (3.10) implies that $\varphi_2 = 0$. So the Fredholm alternative ensures that the operator M is invertible. Finally, applying a direct calculation leads to that $M^{-1} = M_1^{-1} + M_3$, where $M_3 := -M_1^{-1} M_2 M^{-1}$ is compact due to the compactness of M_2. This ends the proof. □

Next we will establish a modified factorization method to reconstruct the mixed-type scatterer. In the case of $\mathrm{Re}[n(x)] < 1$, it is observed from (3.9) that M_1 is no longer a coercive matrix since $\mathrm{Re}[q] < 0$ and

$$-\langle T(i)\varphi, \varphi\rangle_{H^{-1/2}(\partial D_2)\times H^{1/2}(\partial D_2)} \geq C\|\varphi\|^2_{H^{1/2}(\partial D_2)}.$$

This further implies that the operator M can not be decomposed into a coercive part and a compact part. So the classical factorization method can not be used directly. To overcome this difficult, we intend to construct a series of perturbation operators F_m of the far-field operator F in the sense that

$$\lim_{m\to\infty} \|F_m - F\|_{L^2(\mathbb{S}^1)} = 0.$$

We shall show that for any $m \in \mathbb{N}$, F_m satisfies the range identity as stated in [17, Theorem 2.15]. Consequently, the mixed-type scatterer under consideration are capable of approximately reconstructed by means of the knowledge of far-field data F_m. So that for some large enough $m_0 \in \mathbb{N}_+$, the operator F_{m_0} can be regraded as a sufficiently small perturbation of the exact far-field operator F, which further means that $F_{m_0;\#}$ can also be regarded as a sufficiently small perturbation of the noisy operator $F^\delta_\#$ with the noise level δ. Therefore, the shape and location of the mixed-type scatterer in the current paper can be numerically reconstructed by using the spectral data of $F_\#$ and $F^\delta_\#$.

Since $F = GH$ and $G = H^* M^{-1}$, whence $F = H^* M^{-1} H$ follows. We then define the perturbation operators F_m by

$$F_m := F + \rho_m \widetilde{H_2}^* N_{\partial\Omega}(i) \widetilde{H_2},$$

where $\rho_m > 0$ is a positive constant for each $m \in \mathbb{N}$ satisfying that $\rho_m \longrightarrow 0$ as $m \longrightarrow \infty$. Here $\overline{D}_2 \subset \Omega$, $\Omega \cap D_1 = \varnothing$ and $N_{\partial\Omega}(i)$ is defined by

$$\big(N_{\partial\Omega}(i)\varphi\big)(x) = \frac{\partial}{\partial\nu(x)}\int_{\partial\Omega}\frac{\partial\Phi(i;x,y)}{\partial\nu}\varphi(y)ds(y), \quad x \in \partial\Omega.$$

The bounded operator $\widetilde{H_2} : L^2(\mathbb{S}^1) \longmapsto H^{1/2}(\partial\Omega)$ is defined by

$$(\widetilde{H_2}\varphi)(x) = \int_{\mathbb{S}^1} e^{ikx\cdot d}\varphi(d)ds(d), \quad x \in \partial\Omega.$$

Then it follows that

$$\begin{aligned}\|F_m - F\|_{L^2(\mathbb{S}^1)} &= \big\|\rho_m \widetilde{H_2}^* N_{\partial\Omega}(i)\widetilde{H_2}\big\|_{L^2(\mathbb{S}^1)} \\ &= \rho_m \big\|\widetilde{H_2}^* N_{\partial\Omega}(i)\widetilde{H_2}\big\|_{L^2(\mathbb{S}^1)} \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty.\end{aligned}$$

For $h \in H^{1/2}(\partial\Omega)$, we define the compact operator $L : H^{1/2}(\partial\Omega) \longrightarrow L^2(\partial D_2)$ with

$$Lh = \left(\frac{\partial w}{\partial\nu} + i\lambda\frac{\partial w}{\partial\tau} + i\mu w\right)\bigg|_{\partial D_2},$$

where w satisfies that $\Delta w + k^2 w = 0$ in Ω with $w = h$ on $\partial\Omega$. It is noted that $L\widetilde{H_2} = H_2$, and

$$H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} I_{D_1} & 0 \\ 0 & L \end{pmatrix}\begin{pmatrix} H_1 \\ \widetilde{H_2} \end{pmatrix} := \boldsymbol{L}\widetilde{H}.$$

This further implies that

$$\rho_m \widetilde{H_2}^* N_{\partial\Omega}(i)\widetilde{H_2} = \widetilde{H}^* J_m \widetilde{H}, \quad J_m = \begin{pmatrix} 0 & 0 \\ 0 & \rho_m N_{\partial\Omega}(i)\end{pmatrix}.$$

Therefore, the perturbation operators F_m can be decomposed as

$$\begin{aligned}F_m &= F + \rho_m \widetilde{H}_2^* N_{\partial\Omega}(i)\widetilde{H}_2 \\ &= \widetilde{H}^*\big(\boldsymbol{L}^* M^{-1}\boldsymbol{L} + J_m\big)\widetilde{H} \\ &= \widetilde{H}^*\left[\boldsymbol{L}^*\left[\begin{pmatrix} qI & 0 \\ 0 & \big(-T_{\partial D_2\partial D_2}(i)\big)^{-1}\end{pmatrix} + M_3\right]\boldsymbol{L} + J_m\right]\widetilde{H} \\ &= \widetilde{H}^*\left[\begin{pmatrix} qI & 0 \\ 0 & \rho_m N_{\partial\Omega}(i)\end{pmatrix} + \left[\boldsymbol{L}^* M_3 \boldsymbol{L} + \begin{pmatrix} 0 & 0 \\ 0 & L^*\big(-T_{\partial D_2\partial D_2}(i)\big)^{-1}L\end{pmatrix}\right]\right]\widetilde{H} \\ &=: \widetilde{H}^*\big(M_m^{(1)} + M_{com}^{(2)}\big)\widetilde{H}. \end{aligned} \tag{3.12}$$

It is obvious to see that $M_{com}^{(2)}$ is compact on $\widetilde{Y} := L^2(D_1) \times H^{1/2}(\partial\Omega)$ and $-\mathrm{Re}(M_m^{(1)})$ is coercive on $\widetilde{Y}$. That is to say, there exists $C > 0$ with $-\langle \mathrm{Re}M_m^{(1)}\varphi, \varphi\rangle \geq C\|\varphi\|^2$ for all $\varphi \in \widetilde{Y}$ since $\mathrm{Re}(q) < 0$ and the operator $-N_{\partial\Omega}(i)$ is coercive on $H^{1/2}(\partial\Omega)$. By Theorem 3.2 and the factorization that $F = GH$, we have the following results.

Theorem 3.3. *$\widetilde{H}^*$ is compact with dense range in $L^2(\mathbb{S}^1)$. Moreover, $z \in D_1 \cup D_2$ if and only if $\phi_z(\widehat{x}) = e^{-ik\widehat{x}\cdot z} \in \mathcal{R}(\widetilde{H}^*)$ for $\widehat{x} \in \mathbb{S}^1$.*

Proof. It is known from the proof of Theorem 3.1 that $H^* = GM$ and thus $G = H^*M^{-1}$, which together with the fact that $H^* = \widetilde{H}^*\boldsymbol{L}^*$ implies that $\mathcal{R}(\widetilde{H}^*) = \mathcal{R}(G)$, and the compactness and denseness of $\widetilde{H}^*$ can be easily derived from the compactness and injectivity of the operator G. Let $z \in (D_1 \cup D_2)$, we next to prove that

$$\phi_z(\widehat{x}) = e^{-ik\widehat{x}\cdot z} \in \mathcal{R}(G), \quad \widehat{x} \in \mathbb{S}^1.$$

We now choose a ball $B_\varepsilon(z)$ with center z and the radius $\varepsilon > 0$ such that $\overline{B_\varepsilon(z)} \subseteq (D_1 \cup D_2)$. Then we define the function

$$w(x) = \chi(|x-z|)\Phi(x,z) = \chi(|x-z|)\frac{i}{4}H_0^{(1)}(k|x-z|), \quad x \neq z \quad \text{in } \mathbb{R}^2.$$

Here, $\chi(t) \in C^\infty(\mathbb{R}^2)$ is a cut-off function satisfying that $\chi(t) = 1$ for $|t| \geq \varepsilon$ and $\chi(t) = 0$ for $|t| \leq \varepsilon/2$. It is obviously found that $w(x) \in C^\infty(\mathbb{R}^2)$ and $w(x) = \Phi(x,z)$ for $|x-z| \geq \varepsilon$, and thus we derive

$$\Delta w + k^2 n w = \Phi\Delta\chi + \chi\Delta\Phi + 2\nabla\chi\nabla\Phi + k^2 n\chi\Phi =: -qf_1^{(0)} \quad \text{in } D_1,$$

and

$$\left(\frac{\partial w}{\partial \nu} + i\lambda\frac{\partial w}{\partial \tau} + i\mu w\right)\bigg|_{\partial D_2} = -f_2^{(0)}.$$

So $f_1^{(0)} \in L^2(D_1)$ and $f_2^{(0)} \in H^{-1/2}(\partial D_2)$ follows. The uniqueness of the solution ensures that the function w is the solution of problem (2.3) with the data $(f_1^{(0)}, f_2^{(0)})$. Thus,

$$G\big(f_1^{(0)}, f_2^{(0)}\big)^\top = w_\infty = \phi_z,$$

that is $\phi_z \in \mathcal{R}(G)$, which further gives that $\phi_z \in R(\widetilde{H}^*)$.

Now we let $z \notin (D_1 \cup D_2)$ and suppose that there exists $\varphi = (\varphi_1^z, \varphi_2^z)^\top \in \widetilde{Y}^*$ satisfying that $\widetilde{H}^*\varphi = \phi_z$. Hence applying Rellich's lemma and the unique continuation theorem leads to that

$$\int_{D_1}\Phi(\cdot,y)\varphi_1^z(y)dy + \int_{\partial\Omega}\left(\frac{\partial\Phi(\cdot,y)}{\partial\nu(y)} - i\lambda\frac{\partial\Phi(\cdot,y)}{\partial\tau(y)} - i\mu\Phi(\cdot,y)\right)\varphi_2^z(y)ds(y) = \Phi(\cdot,z)$$

for $x \in \mathbb{R}^2 \setminus (\overline{D}_1 \cup \overline{D}_2 \cup \{z\})$. However, there is a contradiction in the above equation since the left-hand belongs to $H^1(B_\varepsilon(z))$ while $\Phi(\cdot,z)$ does not belong to $H^1(B_\varepsilon(z))$, where $B_\varepsilon(z)$ is chosen to be a sufficiently small ball centered at z. So the theorem is proved. $\square$

Theorem 3.4. *Let* $\widetilde{M}_m = M_m^{(1)} + \underline{M_{com}^{(2)}}$. *Assume that* $\mathrm{Re}[n(x)] < 1$ *and* $\mathrm{Im}(\underline{n) \geq} c_0 > 0$. *Then we have that* $\mathrm{Im}\widetilde{M}_m$ *is positive on* $\mathcal{R}(\widetilde{\boldsymbol{H}})$, *i.e.* $\mathrm{Im}\langle\widetilde{M}_m\varphi,\varphi\rangle > 0$ *for all* $\varphi \in \mathcal{R}(\widetilde{\boldsymbol{H}})$ *with* $\varphi \neq 0$.

Proof. For any $\varphi \in L^2(D_1) \times H^{1/2}(\partial\Omega)$, define $\psi = (M^{-1})^*\boldsymbol{L}\varphi$, it follows that

$$\begin{aligned}\mathrm{Im}\langle\widetilde{M}_m\varphi,\varphi\rangle &= \mathrm{Im}\langle\boldsymbol{L}^*M^{-1}\boldsymbol{L}\varphi,\varphi\rangle = \mathrm{Im}\langle M^{-1}\boldsymbol{L}\varphi,\boldsymbol{L}\varphi\rangle \\ &= \mathrm{Im}\langle\boldsymbol{L}\varphi,(M^{-1})^*\boldsymbol{L}\varphi\rangle = \mathrm{Im}\langle M^*\psi,\psi\rangle = \mathrm{Im}\langle\psi,M\psi\rangle.\end{aligned}$$

To prove $\mathrm{Im}\langle\widetilde{M}_m\varphi,\varphi\rangle \geq 0$, we need firstly prove that $\mathrm{Im}\langle M\psi,\psi\rangle \leq 0$. Define a function $w(x)$ by

$$\begin{aligned}w(x) &= \int_{D_1}\Phi(x,y)\psi_1(y)dy \\ &\quad + \int_{\partial D_2}\left(\frac{\partial\Phi(x,y)}{\partial\nu(y)} - i\lambda\frac{\partial\Phi(x,y)}{\partial\tau(y)} - i\mu\Phi(x,y)\right)\psi_2(y)ds(y), \\ &=: w_1 + w_2, \quad x \in \mathbb{R}^2\backslash\partial D_2.\end{aligned}$$

It follows from the definition of the operator M that

$$
\begin{aligned}
\langle M\psi,\psi\rangle &= \big(q^{-1}\psi_1,\psi_1\big)_{D_1} - (w_1,\psi_1)_{D_1} - (w_2,\psi_1)_{D_1} \\
&\quad - \Big\langle \frac{\partial w_1}{\partial\nu} + i\lambda\frac{\partial w_1}{\partial\tau} + i\mu w_1, \psi_2\Big\rangle_{\partial D_2} - \Big\langle \frac{\partial w_2}{\partial\nu} + i\lambda\frac{\partial w_2}{\partial\tau} + i\mu w_2, \psi_2\Big\rangle_{\partial D_2} \\
&=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
$$

It is easily found that

$$
\mathrm{Im}(I_1) = \int_{D_1} \mathrm{Im}(q^{-1})|\psi_1|^2 dx.
$$

Clearly, $\mathrm{Im}(I_1) = 0$ if $\mathrm{Im}[n(x)] = 0, \mathrm{Im}(I_1) \le 0$ if $\mathrm{Im}[n(x)] \ge c_0 > 0$. Applying Green's theorem and the jump relations for layer potentials yields that

$$
\begin{aligned}
I_2 &= -(w_1,\psi_1)_{D_1} = \int_{D_1} w_1\big(\Delta\overline{w}_1 + k^2\overline{w}_1\big)dx \\
&= \int_{\partial D_1} w_1|_- \frac{\partial\overline{w}_1}{\partial\nu}\Big|_- ds - \int_{D_1}\big(|\nabla w_1|^2 - k^2|w_1|^2\big)dx \\
&= \int_{\partial B_R} w_1|_+ \frac{\partial\overline{w}_1}{\partial\nu}\Big|_+ ds - \int_{\partial D_2} w_1|_+ \frac{\partial\overline{w}_1}{\partial\nu}\Big|_+ ds \\
&\quad - \int_{B_R\backslash(\overline{D}_1\cup\overline{D}_2)}\big(|\nabla w_1|^2 - k^2|w_1|^2\big)dx - \int_{D_1}\big(|\nabla w_1|^2 - k^2|w_1|^2\big)dx \\
&= \int_{\partial B_R} w_1|_+ \frac{\partial\overline{w}_1}{\partial\nu}\Big|_+ ds - \int_{D_2}\big(|\nabla w_1|^2 - k^2|w_1|^2\big)dx \\
&\quad - \int_{B_R\backslash(\overline{D}_1\cup\overline{D}_2)}\big(|\nabla w_1|^2 - k^2|w_1|^2\big)dx - \int_{D_1}\big(|\nabla w_1|^2 - k^2|w_1|^2\big)dx,
\end{aligned}
$$

and the radiation condition ensures that

$$
\mathrm{Im}(I_2) = \mathrm{Im}\Big(\lim_{R\to\infty}\int_{\partial B_R} w_1|_+ \frac{\partial\overline{w}_1}{\partial\nu}\Big|_+ ds\Big) = -\frac{k}{|\gamma|^2}\int_{\mathbf{S}^1}\big|w_1^\infty\big|^2 ds.
$$

Similarly, applying Green's theorem again leads to that

$$
\begin{aligned}
I_3 &= -(w_2,\psi_1)_{D_1} = \int_{D_1} w_2\big(\Delta\overline{w}_1 + k^2\overline{w}_1\big)dx \\
&= \int_{\partial D_1} w_2|_- \frac{\partial\overline{w}_1}{\partial\nu}\Big|_- ds - \int_{D_1}\big(\nabla w_2\nabla\overline{w}_1 - k^2 w_2\overline{w}_1\big)dx \\
&= \int_{\partial D_1} w_2|_+ \frac{\partial\overline{w}_1}{\partial\nu}\Big|_+ ds - \int_{D_1}\big(\nabla w_2\nabla\overline{w}_1 - k^2 w_2\overline{w}_1\big)dx.
\end{aligned}
$$

By the jump relations for single, double and the gradient of double-layer potentials, we have

$$
w_2|_+ - w_2|_- = \psi_2, \quad \frac{\partial w_2}{\partial\nu}\Big|_+ - \frac{\partial w_2}{\partial\nu}\Big|_- = -i\lambda\frac{\partial\psi_2}{\partial\tau} + i\mu\psi_2 \quad \text{on } \partial D_2. \tag{3.13}
$$

Using the relations (3.13) and the generalized oblique derivative boundary condition yields

$$
I_4 = -\Big\langle \frac{\partial w_1}{\partial\nu} + i\lambda\frac{\partial w_1}{\partial\tau} + i\mu w_1, \psi_2\Big\rangle_{\partial D_2}
$$

$$
\begin{aligned}
&= -\left\langle \frac{\partial w_1}{\partial \nu}, \psi_2 \right\rangle_{\partial D_2} - \left\langle i\lambda \frac{\partial w_1}{\partial \tau} + i\mu w_1, \psi_2 \right\rangle_{\partial D_2} \\
&= -\int_{\partial D_2} \frac{\partial w_1}{\partial \nu}\bigg|_+ (\overline{w}_2|_+ - \overline{w}_2|_-) ds - \int_{\partial D_2} \left(i\lambda \frac{\partial w_1}{\partial \tau} + i\mu w_1 \right) \overline{\psi}_2 ds \\
&= -\int_{\partial D_2} \frac{\partial w_1}{\partial \nu}\bigg|_+ (\overline{w}_2|_+ - \overline{w}_2|_-) ds + \int_{\partial D_2} \left(i\lambda \frac{\partial \overline{\psi}_2}{\partial \tau} - i\mu \overline{\psi}_2 \right) w_1|_+ ds \\
&= -\int_{\partial D_2} \frac{\partial w_1}{\partial \nu}\bigg|_+ (\overline{w}_2|_+ - \overline{w}_2|_-) ds + \int_{\partial D_2} \left(\frac{\partial \overline{w}_2}{\partial \nu}\bigg|_+ - \frac{\partial \overline{w}_2}{\partial \nu}\bigg|_- \right) w_1|_+ ds \\
&= \int_{\partial D_2} \frac{\partial \overline{w}_2}{\partial \nu}\bigg|_+ w_1|_+ - \frac{\partial w_1}{\partial \nu}\bigg|_+ \overline{w}_2|_+ ds + \int_{\partial D_2} \frac{\partial w_1}{\partial \nu}\bigg|_+ \overline{w}_2|_- - \frac{\partial \overline{w}_2}{\partial \nu}\bigg|_- w_1|_+ ds \\
&= I_7 + I_8.
\end{aligned}
$$

For the term I_7, it is found that

$$
\begin{aligned}
I_7 &= \int_{\partial D_2} \frac{\partial \overline{w}_2}{\partial \nu}\bigg|_+ w_1|_+ - \frac{\partial w_1}{\partial \nu}\bigg|_+ \overline{w}_2|_+ ds \\
&= \int_{\partial B_R} \frac{\partial \overline{w}_2}{\partial \nu}\bigg|_+ w_1|_+ - \frac{\partial w_1}{\partial \nu}\bigg|_+ \overline{w}_2|_+ ds - \int_{\partial D_1} \frac{\partial \overline{w}_2}{\partial \nu}\bigg|_+ w_1|_+ - \frac{\partial w_1}{\partial \nu}\bigg|_+ \overline{w}_2|_+ ds \\
&\quad - \int_{B_R \setminus (\overline{D}_1 \cup \overline{D}_2)} w_1 \Delta \overline{w}_2 - \overline{w}_2 \Delta w_1 ds \\
&= \int_{\partial B_R} \frac{\partial \overline{w}_2}{\partial \nu}\bigg|_+ w_1|_+ - \frac{\partial w_1}{\partial \nu}\bigg|_+ \overline{w}_2|_+ ds + \int_{D_1} k^2 \overline{w}_2 w_1 - \nabla \overline{w}_2 \nabla w_1 dx \\
&\quad + \int_{\partial D_1} \frac{\partial w_1}{\partial \nu}\bigg|_+ \overline{w}_2|_+ ds - \int_{B_R \setminus (\overline{D}_1 \cup \overline{D}_2)} w_1 \Delta \overline{w}_2 - \overline{w}_2 \Delta w_1 ds.
\end{aligned}
$$

For the term I_8, it is seen that

$$
I_8 = \int_{\partial D_2} \frac{\partial w_1}{\partial \nu} \overline{w}_2|_- - \frac{\partial \overline{w}_2}{\partial \nu}\bigg|_- w_1 ds = \int_{D_2} \overline{w}_2 \Delta w_1 - w_1 \Delta \overline{w}_2 ds.
$$

Making use of the fact that

$$
\begin{aligned}
&\mathrm{Im} \int_{\partial D_1} w_2|_+ \frac{\partial \overline{w}_1}{\partial \nu}\bigg|_+ ds + \mathrm{Im} \int_{\partial D_1} \frac{\partial w_1}{\partial \nu}\bigg|_+ \overline{w}_2|_+ ds = 0, \\
&\mathrm{Im} \int_{D_1} \left(k^2 w_2 \overline{w}_1 - \nabla w_2 \nabla \overline{w}_1 \right) dx + \mathrm{Im} \int_{D_1} k^2 \overline{w}_2 w_1 - \nabla \overline{w}_2 \nabla w_1 dx = 0,
\end{aligned}
$$

and applying the Green's formula and radiation condition, we conclude that

$$
\mathrm{Im}(I_3 + I_4) = \mathrm{Im} \left(\lim_{R \to \infty} \int_{\partial B_R} \frac{\partial \overline{w}_2}{\partial \nu}\bigg|_+ w_1|_+ - \frac{\partial w_1}{\partial \nu}\bigg|_+ \overline{w}_2|_+ ds \right) = -\frac{2k}{|\gamma|^2} \mathrm{Re} \int_{\mathbf{S}^1} w_1^\infty \overline{w_2^\infty} ds.
$$

With the aid of the generalized oblique derivative boundary condition and the relations (3.13) again, we deduce that

$$
\begin{aligned}
I_5 &= -\left\langle \frac{\partial w_2}{\partial \nu} + i\lambda \frac{\partial w_2}{\partial \tau} + i\mu w_2, \psi_2 \right\rangle_{\partial D_2} \\
&= -\left\langle \frac{\partial w_2}{\partial \nu}, \psi_2 \right\rangle_{\partial D_2} - \left\langle i\lambda \frac{\partial w_2}{\partial \tau} + i\mu w_2, \psi_2 \right\rangle_{\partial D_2}
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_{\partial D_2} \frac{\partial w_2}{\partial \nu}\bigg|_+ (\overline{w}_2|_+ - \overline{w}_2|_-)ds - \int_{\partial D_2} \left(i\lambda \frac{\partial w_2}{\partial \tau} + i\mu w_2\right)\overline{\psi}_2 ds \\
&= -\int_{\partial D_2} \frac{\partial w_2}{\partial \nu}\bigg|_+ (\overline{w}_2|_+ - \overline{w}_2|_-)ds + \int_{\partial D_2} \left(i\lambda \frac{\partial \overline{\psi}_2}{\partial \tau} - i\mu \overline{\psi}_2\right) w_2|_+ ds \\
&= -\int_{\partial D_2} \frac{\partial w_2}{\partial \nu}\bigg|_+ (\overline{w}_2|_+ - \overline{w}_2|_-)ds + \int_{\partial D_2} \left(\frac{\partial \overline{w}_2}{\partial \nu}\bigg|_+ - \frac{\partial \overline{w}_2}{\partial \nu}\bigg|_-\right) w_2|_+ ds \\
&= \int_{\partial D_2} \frac{\partial \overline{w}_2}{\partial \nu}\bigg|_+ w_2|_+ - \frac{\partial w_2}{\partial \nu}\bigg|_+ \overline{w}_2|_+ ds + \int_{\partial D_2} \frac{\partial w_2}{\partial \nu}\bigg|_+ \overline{w}_2|_- - \frac{\partial \overline{w}_2}{\partial \nu}\bigg|_- w_2|_+ ds \\
&= I_9 + I_{10}.
\end{aligned}
$$

Then applying Green's theorem and the Eq. (3.13) implies that

$$
\begin{aligned}
I_9 &= \int_{\partial D_2} \frac{\partial \overline{w}_2}{\partial \nu}\bigg|_+ w_2|_+ - \frac{\partial w_2}{\partial \nu}\bigg|_+ \overline{w}_2|_+ ds \\
&= \int_{\partial B_R} \frac{\partial \overline{w}_2}{\partial \nu}\bigg|_+ w_2|_+ - \frac{\partial w_2}{\partial \nu}\bigg|_+ \overline{w}_2|_+ ds - \int_{\partial D_1} \frac{\partial \overline{w}_2}{\partial \nu}\bigg|_+ w_2|_+ - \frac{\partial w_2}{\partial \nu}\bigg|_+ \overline{w}_2|_+ ds \\
&\quad - \int_{B_R \setminus (\overline{D}_1 \cup \overline{D}_2)} w_2 \Delta \overline{w}_2 - \overline{w}_2 \Delta w_2 ds \\
&= \int_{\partial B_R} \frac{\partial \overline{w}_2}{\partial \nu}\bigg|_+ w_2|_+ - \frac{\partial w_2}{\partial \nu}\bigg|_+ \overline{w}_2|_+ ds + \int_{D_1} \left(|\nabla w_2|^2 - k^2 |w_2|^2\right) dx \\
&\quad + \int_{\partial D_1} \frac{\partial w_2}{\partial \nu}\bigg|_+ \overline{w}_2|_+ ds - \int_{B_R \setminus (\overline{D}_1 \cup \overline{D}_2)} w_2 \Delta \overline{w}_2 - \overline{w}_2 \Delta w_2 ds, \\
I_{10} &= \int_{\partial D_2} \frac{\partial w_2}{\partial \nu}\bigg|_+ \overline{w}_2|_- - \frac{\partial \overline{w}_2}{\partial \nu}\bigg|_- w_2|_+ ds \\
&= \int_{\partial D_2} \frac{\partial w_2}{\partial \nu}\bigg|_+ (\overline{w}_2|_+ - \overline{\psi}_2) - \frac{\partial \overline{w}_2}{\partial \nu}\bigg|_- (w_2|_- + \psi_2) ds \\
&= \int_{\partial D_2} \frac{\partial w_2}{\partial \nu}\bigg|_+ \overline{w}_2|_+ ds - \int_{D_2} \frac{\partial \overline{w}_2}{\partial \nu}\bigg|_- w_2|_- ds - \int_{\partial D_2} \frac{\partial w_2}{\partial \nu}\bigg|_+ \overline{\psi}_2 ds - \int_{D_2} \frac{\partial \overline{w}_2}{\partial \nu}\bigg|_- \psi_2 ds \\
&= \int_{\partial B_R} \frac{\partial w_2}{\partial \nu}\bigg|_+ \overline{w}_2|_+ ds - \int_{\partial D_1} \frac{\partial w_2}{\partial \nu}\bigg|_+ \overline{w}_2|_+ ds - \int_{B_R \setminus (\overline{D}_1 \cup \overline{D}_2)} \left(|\nabla w_2|^2 - k^2 |w_2|^2\right) ds \\
&\quad - \int_{D_2} \left(|\nabla w_2|^2 - k^2 |w_2|^2\right) ds - \int_{\partial D_2} \frac{\partial w_2}{\partial \nu}\overline{\psi}_2 + \frac{\partial \overline{w}_2}{\partial \nu}\psi_2 ds + \int_{\partial D_2} \left(i\lambda \frac{\partial \overline{\psi}_2}{\partial \tau} - i\mu \overline{\psi}_2\right)\psi_2 ds.
\end{aligned}
$$

It is easily known that

$$
\mathrm{Im} \int_{\partial D_2} \left(\frac{\partial w_2}{\partial \nu}\overline{\psi}_2 + \frac{\partial \overline{w}_2}{\partial \nu}\psi_2\right) ds = 0, \quad \mathrm{Im} \int_{\partial D_2} i\lambda \frac{\partial \overline{\psi_2}}{\partial \tau}\psi_2 ds = 0.
$$

Therefore, we deduce that

$$
\begin{aligned}
\mathrm{Im}(I_5) &= \mathrm{Im}\left(\lim_{R\to\infty} \int_{\partial B_R} \frac{\partial \overline{w}_2}{\partial \nu}|_+ w_2|_+ ds\right) - \mu \int_{\partial D_2} |\psi_2|^2 ds \\
&= -\frac{k}{|\gamma|^2} \int_{\mathbf{S}^1} \left|w_2^\infty\right|^2 ds - \mu \int_{\partial D_2} |\psi_2|^2 ds.
\end{aligned}
$$

Combining the above analysis leads to that

$$\begin{aligned}\operatorname{Im}\langle M\psi,\psi\rangle = \operatorname{Im}(q^{-1})\int_{D_1}|\psi_1|^2dx - \frac{k}{|\gamma|^2}\int_{\mathbf{S}^1}\left|w_1^\infty\right|^2ds - \frac{2k}{|\gamma|^2}\operatorname{Re}\int_{\mathbf{S}^1} w_1^\infty\overline{w_2^\infty}ds \\ -\frac{k}{|\gamma|^2}\int_{\mathbf{S}^1}\left|w_2^\infty\right|^2ds - \mu\int_{\partial D_2}|\psi_2|^2ds \le 0, \end{aligned} \tag{3.14}$$

where $\gamma = e^{ikR}/4\pi R$ and the assumption $\mu \ge 0$ is used. Hence, we have

$$\operatorname{Im}\langle \tilde{M}_m\varphi,\varphi\rangle \ge 0. \tag{3.15}$$

Noting that $H^* = GM$, which further gives $G^* = (M^{-1})^*H$. Recalling $H = \boldsymbol{L}\widetilde{H}$ implies that $G^* = (M^{-1})^*\boldsymbol{L}\widetilde{H}$. It follows that for any $\varphi \in \overline{\mathcal{R}(\widetilde{H})}$, we have that $\psi = (M^{-1})^*\boldsymbol{L}\varphi \in \overline{\mathcal{R}(G^*)}$. Therefore, to prove this theorem, it is sufficient to show that

$$\operatorname{Im}\langle\psi, M\psi\rangle > 0, \quad \forall\,\psi\in\overline{\mathcal{R}(G^*)}, \quad \psi\neq 0. \tag{3.16}$$

It is known from (3.14) that $\operatorname{Im}\langle\psi, M\psi\rangle \ge 0$. Now we let $\operatorname{Im}\langle\psi, M\psi\rangle = 0$ for some $\psi \in \overline{\mathcal{R}(G^*)}$. Since $\operatorname{Im}[n(x)] \ge c_0 > 0$, $\operatorname{Im}(\mu) \ge \mu_0 > 0$ and $\lambda < 1$, it then follows from (3.14) that $\psi = (\psi_1,\psi_2) = 0$. The theorem is thus proved. $\square$

We now proceed to the characterization of the mixed scatterer by introducing the following range identity in [17, Theorem 2.15] as follows.

Theorem 3.5. *Let X be a reflexive Banach space and U be a Hilbert space satisfying that $X \subset U \subset X^*$, and the embeddings are dense. Assume that Y is a different Hilbert space. Let $F : Y \to Y, H : Y \to X$ and $M : X \to X^*$ be linear bounded operators satisfying that*

$$F = H^*MH. \tag{3.17}$$

Moreover, suppose the following assumptions hold true:

(a) H^* is compact with dense range.

(b) $\operatorname{Re}(M) = M_1 + M_2$ with a self-adjoint coercive operator M_1 and a compact operator M_2.

(c) $\operatorname{Im}(M)$ is positive on X, i.e.

$$\langle \operatorname{Im}(M)\varphi,\varphi\rangle > 0, \quad \forall\,\varphi\in X, \quad \varphi\neq 0.$$

Then the operator $F_\sharp := |\operatorname{Re}(F)| + |\operatorname{Im}(F)|$ is positive and the ranges of $H^* : X^* \to Y$ and $F_\sharp^{1/2} : Y \to Y$ coincide.

Therefore, we have the following characterization theorem.

Theorem 3.6. *For $z \in \mathbb{R}^2$, define $\phi_z \in L^2(\mathbb{S}^1)$ by $\phi_z(\widehat{x}) = e^{-ik\widehat{x}\cdot z}, \widehat{x} \in \mathbb{S}^1$. Assume that $\operatorname{Re}[n(x)] < 1, \operatorname{Im}(n) \ge c_0 > 0, \lambda < 1$ and $\operatorname{Im}(\mu) \ge \mu_0 > 0$. Then*

$$\begin{aligned} z\in D_1\cup D_2 &\iff \phi_z \in R\big(F_{m,\#}^{\frac{1}{2}}\big) \\ &\iff W_m(z) := \left[\sum_j \frac{|\langle\phi_z,\psi_j^{(m)}\rangle_{L^2(\mathbb{S}^1)}|^2}{\lambda_j^{(m)}}\right]^{-1} > 0 \end{aligned}$$

with $m \in \mathbb{N}$, where $\{\lambda_j^{(m)};\psi_j^{(m)}\}_{j\in\mathbb{N}}$ is an eigen-system of the self-adjoint operator $F_{m,\#}$: $L^2(\mathbb{S}^1) \longmapsto L^2(\mathbb{S}^1)$ given by $F_{m,\#} := |\operatorname{Re}F_m| + |\operatorname{Im}F_m|$.

Proof. It is seen from (3.12) that $F_M = \widetilde{H}^* \widetilde{M}_m \widetilde{H}$ with $\widetilde{M}_m = M_m^{(1)} + M_{com}^{(2)}$, and $M_{com}^{(2)}$ is compact on $\widetilde{Y} := L^2(D_1) \times H^{1/2}(\partial\Omega)$ and $-\mathrm{Re}(M_m^{(1)})$ is coercive on $\widetilde{Y}$ since $\mathrm{Re}(q) < 0$ and the operator $-N_{\partial\Omega}(i)$ is coercive on $H^{1/2}(\partial\Omega)$. By Theorem 3.3 we find that $\widetilde{H}^*$ is compact with dense range in $L^2(\mathbb{S}^1)$. It is also observed from Theorem 3.4 that $\mathrm{Im}\langle \widetilde{M}_m \varphi, \varphi \rangle > 0$ for all $\varphi \in \overline{\mathcal{R}(\widetilde{\boldsymbol{H}})}$ with $\varphi \neq 0$. Moreover, $z \in D_1 \cup D_2$ if and only if $\phi_z(\widehat{x}) = e^{-ik\widehat{x}\cdot z} \in \mathcal{R}(\widetilde{H}^*)$ for $\widehat{x} \in \mathbb{S}^1$, which coincides with $R(F_{m,\#}^{1/2})$ by Theorem 3.5. We then complete the proof of the characterization theorem. □

4. Numerical Examples

In this section, we will provide several numerical examples in $\mathbb{R}^2$ to illustrate the effectiveness and applicability of our modified factorization method for simultaneous imaging of the inhomogeneous penetrable conductive medium and the impenetrable obstacle with the generalized oblique derivative boundary condition.

Step 1. Solve the integral equation $M(\varphi_1, \varphi_2) = (f_1, f_2)^\top$ with M defined by (3.5) and

$$(f_1, f_2) = \left(e^{ikx\cdot d}, \frac{\partial e^{ikx\cdot d}}{\partial \nu} + i\lambda \frac{\partial e^{ikx\cdot d}}{\partial \tau} + i\mu e^{ikx\cdot d} \right).$$

Then relying on the solution (φ_1, φ_2), we derive the far-field data w_∞. Hence, we have the far-field operator

$$(Fg)(\widehat{x}) = \int_{\mathbb{S}^1} w_\infty(\widehat{x}; d) g(d) ds(d), \quad g \in L^2(\mathbb{S}^1).$$

Our data set is represented by a matrix

$$F_M = w_\infty(\widehat{x}_r; d_s)_{1\le r,s\le M} \in \mathbb{C}^{M\times M},$$

where $w_\infty(\widehat{x}_r; d_s)$ is the far-field data with the finite observation direction $\widehat{x} = \widehat{x}_r \in \mathbb{S}$ and the incident direction $d = d_s \in \mathbb{S}$ with $r, s = 1, 2, \ldots, M$, which are equally spaced on the unit circle $\mathbb{S}$.

Step 2. Add noise to the far-field operator F by

$$F_M^\delta := F_M + \delta \frac{X}{\|X\|_2} \|F_M\|_2, \quad \big(F_M^\delta\big)_\# := \big|\mathrm{Re}\big(F_M^\delta\big)\big| + \big|\mathrm{Im}\big(F_M^\delta\big)\big|.$$

Step 3. Based on Theorem 3.6, we define the translated indicator function $W_M(z)$ of the far field operator F as

$$W_M(z) = \left[\sum_{p=1}^{M} \frac{1}{\lambda_p} \left| \sum_{q=1}^{M} \phi_{z,q} \overline{\psi_{p,q}} \right|^2 \right]^{-1}, \quad z \in \mathbb{R}^2, \tag{4.1}$$

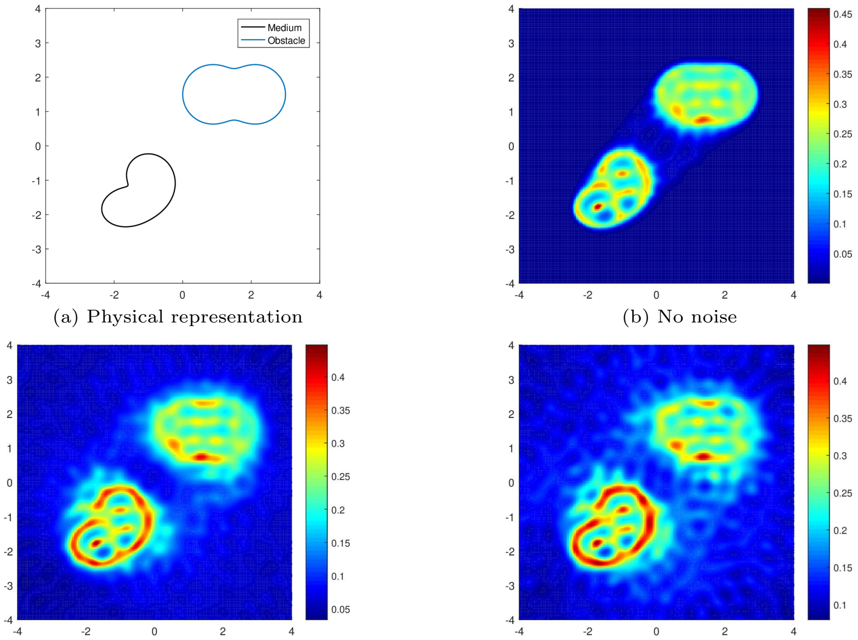
where $\{\phi_{z,q}\}_{q=1}^M$ is the discretization of the test function ϕ_z and $\{\lambda_p; \psi_p\}_{p=1}^M$ is the eigen-system of the self-adjoint matrix $F_{M,\#} = |\mathrm{Re}(F_M)| + |\mathrm{Im}(F_M)|$ with $\psi_p = \{\psi_{p,q}\}_{q=1}^M$. It is expected that the value of $W_M(z)$ to be much larger for $z \in D_1 \cup D_2$ than for the points that lie outside the domain $D_1 \cup D_2$.

Table 4.1: Parametrization of the Graph.

Graph type	Parametrization
Circle shaped	$x(t) = R(\cos t, \sin t),\ \ t \in [0, 2\pi],\ \ R > 0$
Ellipse shaped	$x(t) = (5\cos t, 4\sin t),\ \ t \in [0, 2\pi]$
Apple shaped	$x(t) = [(0.5 + 0.4\cos t + 0.1\sin 2t)/(1 + 0.7\cos t)](\cos t, \sin t), t \in [0, 2\pi]$
Kite shaped	$x(t) = (\cos t + 0.55\cos 2t - 0.55, 1.5\sin t),\ \ t \in [0, 2\pi]$
Peanut shaped	$x(t) = \sqrt{\cos^2 t + 0.25\sin^2 t}(\cos t, \sin t),\ \ t \in [0, 2\pi]$

Therefore, the indicator function $W_M(z)$ can be calculated directly from the eigen-system of the perturbation matrix $(F_M^\delta)_\#$ to simultaneously reconstruct the shape and location of the mixed-type scatterer consisting of the inhomogeneous penetrable medium D_1 and the impenetrable obstacle D_2 with the generalized oblique derivative boundary condition. For simplicity, we shall use the test curves shown in Table 4.1 and we let $k_1^2 = k^2 n(x)$ be a constant to show that D_1 is homogeneous, which is different from the background medium in $\mathbb{R}^2 \backslash (\overline{D}_1 \cup \overline{D}_2)$. In these numerical examples, we take $k = 6, k_1 = 2 + 5i, \mu = 1, \lambda = 0.5$ and $M = 64$.

Example 4.1. In this example, we take the far-field data without noise, with 2% noise and with 5% noise to reconstruct the results of penetrable Circle shaped medium and the impenetrable Kite shaped obstacle. See Fig. 4.1.

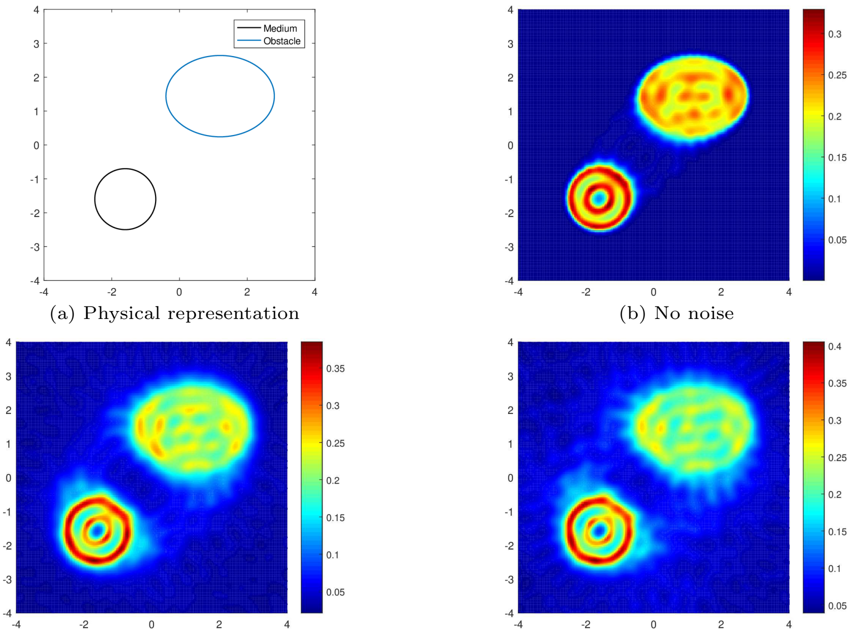


(a) Physical representation (b) No noise

(c) 2% noise (d) 5% noise

Fig. 4.1. Reconstruction of Circle shaped and Kite shaped.

Example 4.2. In this example, we take the far-field data without noise, with 2% noise and with 5% noise to reconstruct the results of penetrable Circle shaped medium and the impenetrable Ellipse shaped obstacle. See Fig. 4.2.

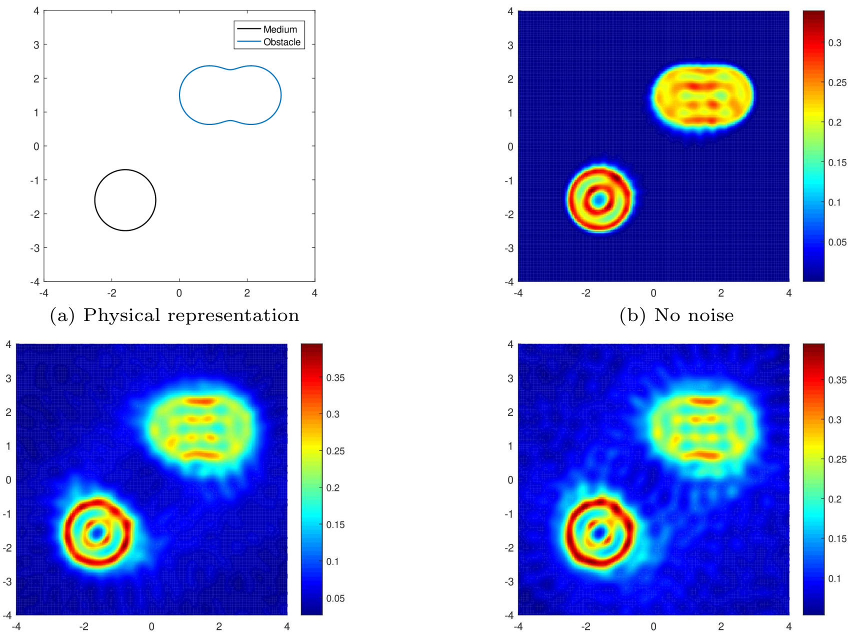


(a) Physical representation (b) No noise

(c) 2% noise (d) 5% noise

Fig. 4.2. Reconstruction of Circle shaped and Ellipse shaped.

Example 4.3. In this example, we take the far-field data without noise, with 2% noise and with 5% noise to reconstruct the results of penetrable Circle shaped medium and the impenetrable Peanut shaped obstacle. See Fig. 4.3.

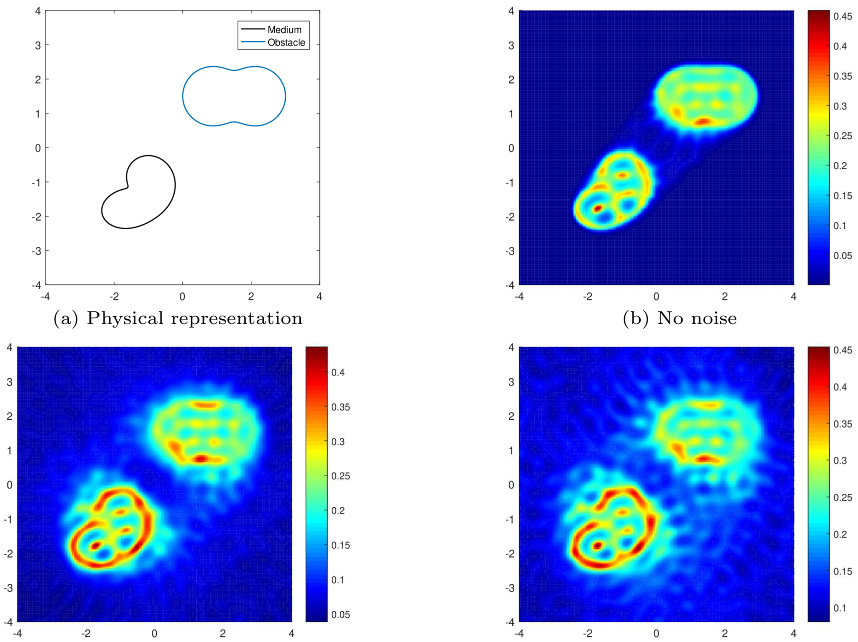


(a) Physical representation (b) No noise

(c) 2% noise (d) 5% noise

Fig. 4.3. Reconstruction of Circle shaped and Peanut shaped.

Example 4.4. In this example, we take the far-field data without noise, with 2% noise and with 5% noise to reconstruct the results of penetrable Apple shaped medium and the impenetrable Peanut shaped obstacle. See Fig. 4.4.



(a) Physical representation (b) No noise (c) 2% noise (d) 5% noise

Fig. 4.4. Reconstruction of Apple shaped and Peanut shaped.

5. Conclusion

From the above examples and others that have been verified but not given in detail in this paper, it can be seen that the shape and location of the mixed-type scatterer studied in this paper can be simultaneously numerically reconstructed from the spectral data of the far-field operator. This also shows that the modified factorization method used in this paper is feasible. In addition, we plan to extend our results to the electromagnetic scattering problem.

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