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PARAREAL ALGORITHMS FOR STOCHASTIC MAXWELL EQUATIONS WITH THE DAMPING TERM DRIVEN BY ADDITIVE NOISE*

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Abstract

In this paper, we propose the parareal algorithms for stochastic Maxwell equations with the damping term driven by additive noise. The proposed algorithms proceed as two-level temporal parallelizable integrators with the stochastic exponential integrator as the coarse $\mathscr{G}$-propagator and both the exact solution integrator and the stochastic exponential integrator as the fine $\mathscr{F}$-propagator. The mean-square convergence order of the proposed algorithms consistently increases to k, regardless of whether the exact solution integrator or the stochastic exponential integrator is chosen as the fine $\mathscr{F}$-propagator. Several numerical experiments are illustrated in order to verify our theoretical findings for different choices of the iteration number k and the damping coefficient σ.

Mathematics subject classification: 60H35, 35Q61, 65M12.
Key words: Stochastic Maxwell equations, Parareal algorithm, Stochastic exponential integrator, Strong convergence.

1. Introduction

When the electric and magnetic fluxes are perturbed by noise, the uncertainty and stochasticity can have a subtle but profound influence on the evolution of complex dynamical systems [25]. In order to model the thermal motion of electrically charged microparticles, we consider the stochastic Maxwell equations with damping term driven by additive noise as follows:

$$\begin{cases} \varepsilon\partial_t \boldsymbol{E}(t,\boldsymbol{x}) = \nabla\times\boldsymbol{H}(t,\boldsymbol{x}) - \sigma\boldsymbol{E}(t,\boldsymbol{x}) & \\ \qquad\qquad -J_e(t,x,\boldsymbol{E},\boldsymbol{H}) - J_e^r(t,\boldsymbol{x})\cdot\dot{W}, & (t,\boldsymbol{x})\in(0,T]\times D, \\ \mu\partial_t \boldsymbol{H}(t,\boldsymbol{x}) = -\nabla\times\boldsymbol{E}(t,\boldsymbol{x}) - \sigma\boldsymbol{H}(t,\boldsymbol{x}) & \\ \qquad\qquad -J_m(t,x,\boldsymbol{E},\boldsymbol{H}) - J_m^r(t,\boldsymbol{x})\cdot\dot{W}, & (t,\boldsymbol{x})\in(0,T]\times D, \\ \boldsymbol{E}(0,\boldsymbol{x}) = \boldsymbol{E}_0(\boldsymbol{x}), \quad \boldsymbol{H}(0,\boldsymbol{x}) = \boldsymbol{H}_0(\boldsymbol{x}), & \boldsymbol{x}\in D, \\ \boldsymbol{n}\times\boldsymbol{E} = 0, & (t,\boldsymbol{x})\in(0,T]\times\partial D, \end{cases} \tag{1.1}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are $L^2(D)^3$-valued functions whose domain D is an open, bounded and Lipschitz domain in $\mathbb{R}^3$ with smooth boundary ∂D. The unit outward normal vector to ∂D is denoted by $\boldsymbol{n}$. Here ε is the electric permittivity and μ is the magnetic permeability. The damping terms $\sigma\boldsymbol{E}$ and $\sigma\boldsymbol{H}$ are usually added to simulate the attenuation of electromagnetic waves in the medium, which can be caused by absorption, scattering or other non-ideal factors

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in the medium. The function J_e and J_e^r describe electric currents (or J_m and J_m^r describe magnetic currents). In particular, J_e^r and J_m^r do not depend on the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$. The authors in [22] proved the mild, strong and classical well-posedness for the Cauchy problem of stochastic Maxwell equations. Meanwhile, the authors in [20] investigated the approximate controllability of the stochastic Maxwell equations through a generalization of the Hilbert uniqueness method. Subsequently the work [24] combined the study of well-posedness, homogenization and controllability of Maxwell equations by outlining the structural relationships of intricate materials and tackling deterministic and stochastic concerns in the frequency and time domains.

Since stochastic Maxwell equations are a kind of stochastic Hamiltonian partial differential equations, constructing stochastic multi-symplectic numerical methods for problem (1.1) has been paid more and more attention. The stochastic multi-symplectic numerical method for stochastic Maxwell equations driven by additive noise was proposed in [17] based on the stochastic variational principle. Subsequently the authors in [10] used a straightforward approach to avoid the introduction of additional variables and obtained three effective stochastic multi-symplectic numerical methods. Then the authors in [18] used the wavelet collocation method in space and the stochastic symplectic method in time to construct the stochastic multi-symplectic energy-conserving method for three-dimensional stochastic Maxwell equations driven by multiplicative noise. The work [31] made a review on these stochastic multi-symplectic methods and summarised numerical methods for various stochastic Maxwell equations driven by additive and multiplicative noise. The general case of stochastic Hamiltonian partial differential equations was considered in [32], where the multi-symplecticity of stochastic Runge-Kutta methods was investigated. Recently, the authors in [26, 27] constructed multi-symplectic discontinuous Galerkin methods for stochastic Maxwell equations driven by additive noise and multiplicative noise. Furthermore, the work [16] employed the local radial basis function collocation method and the work [21] utilized the global radial basis function collocation method for stochastic Maxwell equations driven by multiplicative noise to preserve multi-symplectic structure. Additionally, [4] developed a symplectic discontinuous Galerkin full discretisation method for stochastic Maxwell equations driven by additive noise. Other efficient numerical methods for stochastic Maxwell equations also are investigated, see [30] for the finite element method, [1] for the numerical method based on the Wiener chaos expansion, [9] for the ergodic numerical method, [7] for the operator splitting method and [35] for the finite-difference time-domain method.

Meanwhile, there are a lot of numerous works focused mainly on strong convergence analysis of the numerical methods for stochastic Maxwell equations. In the temporal discretization methods, the semi-implicit Euler method was proposed in [6] to prove mean-square convergence order is $1/2$ for stochastic Maxwell equations driven by multiplicative noise. Subsequently the work in [5] studied the stochastic Runge-Kutta method with mean-square convergence order 1 for stochastic Maxwell equations driven by additive noise. In addition, explicit exponential integrator was proposed in [11] for stochastic Maxwell equations with mean-square convergence order $1/2$ for multiplicative noise and convergence order 1 for additive noise. The work [4] developed discontinuous Galerkin full discretization method for stochastic Maxwell equations driven by additive noise with mean-square convergence order $\alpha/2$ in time and $\alpha - 1/2$ in space, where $\alpha = 1, 2$ represents H_α regularity. Another related work by authors of [9] showed the ergodic discontinuous Galerkin full discretization for stochastic Maxwell equations with mean-square convergence order $1/2$ both in time and in space. In recent works [26, 27], high order discontinuous Galerkin methods were designed for the stochastic Maxwell equations driven by additive noise and multiplicative noise with mean-square convergence order both $m+1$, where m represents the degree of the polynomial. Besides, the authors of [7] presented the operator splitting method for stochastic Maxwell equations driven by additive noise with mean-square convergence order 1.

In order to increase the convergence order and improve the computational efficiency on stochastic differential equations, the parareal algorithm has received attention. This algorithm we focus on is a two-stage time-parallel integrator originally proposed in [23] and further works studied on theoretical analysis and applications for differential model problems, see, for instance, [2, 12–15, 28]. In terms of stochastic model, the work [34] investigated the parareal algorithm that combines the projection method with stochastic differential equations possessing conserved quantities. Subsequently, in [33], the authors proposed a parareal algorithm for stochastic differential equations that utilizes the Milstein scheme as the coarse propagator and the exact solution as the fine propagator, while also analyzing its convergence order under specific regularity assumptions. Furthermore, the parareal algorithm for stochastic Schrödinger equations with a weak damping term driven by additive noise was examined in [19], where the fine propagator was the exact solver and the coarse propagator was the exponential θ-scheme. Notably, the proposed algorithm increases the convergence order to k in the linear case for $\theta \in [0,1]\backslash\{1/2\}$. The parareal algorithm for semilinear parabolic stochastic partial differential equations behaved differently in [3] depending on the choice of the coarse integrator. When the linear implicit Euler scheme was selected, the convergence order was limited by the regularity of the noise with the increase of iteration number, while for the stochastic exponential scheme, the convergence order always increased. To the best of our knowledge, there has been no reference considering the convergence analysis of the parareal algorithm for stochastic Maxwell equations till now.

Inspired by the pioneering works, we establish strong convergence analysis of the parareal algorithms for stochastic Maxwell equations with damping term driven by additive noise. Combining the benefits of the stochastic exponential integrator, we use this integrator as the coarse $\mathscr{G}$-propagator and for the fine $\mathscr{F}$-propagator, two choices are considered: the exact solution integrator as well as the stochastic exponential integrator. Taking advantage of the contraction semigroup generated by the Maxwell operator and the damping term, we derive the uniform mean-square convergence analysis of the proposed parareal algorithms with convergence order k. The key point of convergence analysis is that the error between the solution computed by the parareal algorithm and the reference solution generated by the fine propagator for the stochastic exponential integrator still maintains the consistent convergence results. Different from the exact solution integrator as the fine $\mathscr{F}$-propagator, we need to make use of the Lipschitz continuity of the residual operator rather than the integrability of the exact solution directly in this case, which requires us to make assumptions about the directional derivatives of the drift coefficient. We find that the selection of parameters have an impact on the convergence analysis results of the parareal algorithms. An appropriate damping coefficient ensures stability and accelerates the convergence results and the scale of noise induces a perturbation of the solution numerically.

The article is organized as follows. In the forthcoming section, we collect some preliminaries about stochastic Maxwell equations. In Section 3, we devote to introducing the parareal algorithms based on the exponential scheme as the coarse $\mathscr{G}$-propagator and both the exact solution integrator and the stochastic exponential integrator as the fine $\mathscr{F}$-propagator. In Section 4, two convergence results in the sense of mean-square are analyzed. In Section 5, numerical experiments are dedicated to illustrate the convergence analysis with the influences on the iteration number k and the damping coefficient σ and the effect of noise with different scale on the numerical solution.

To lighten notations, throughout this paper, C stands for a constant which might be dependent on T but is independent on ΔT and may vary from line to line.

2. Preliminaries

The basic Hilbert space is $\mathbb{H} := L^2(D)^3 \times L^2(D)^3$ with the inner product being defined by

$$\left\langle \begin{pmatrix} \boldsymbol{E}_1 \\ \boldsymbol{H}_1 \end{pmatrix}, \begin{pmatrix} \boldsymbol{E}_2 \\ \boldsymbol{H}_2 \end{pmatrix} \right\rangle_{\mathbb{H}} := \int_D (\varepsilon \boldsymbol{E}_1 \cdot \boldsymbol{E}_2 + \mu \boldsymbol{H}_1 \cdot \boldsymbol{H}_2) d\boldsymbol{x}$$

for all $\boldsymbol{E}_1, \boldsymbol{H}_1, \boldsymbol{E}_2, \boldsymbol{H}_2 \in L^2(D)^3$, where $L^2(D)$ is the space of real-valued square-integrable functions. The norm induced by this inner product corresponds to the electromagnetic energy of the physical system

$$\left\| \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} \right\|_{\mathbb{H}}^2 = \int_D (\varepsilon \|\boldsymbol{E}\|^2 + \mu \|\boldsymbol{H}\|^2) d\boldsymbol{x}, \quad \forall\, \boldsymbol{E}, \boldsymbol{H} \in L^2(D)^3.$$

In addition, assume that ε and μ are bounded and uniformly positive definite functions: $\varepsilon, \mu \in L^\infty(D)$ for any $\varepsilon, \mu > 0$.

The Q-Wiener process W is defined on a given probability space $(\Omega, \mathscr{F}, P, \{\mathscr{F}_t\}_{t\in[0,T]})$ and can be expanded in a Fourier series

$$W(t) = \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}} \beta_n(t) e_n, \quad t \in [0, T], \tag{2.1}$$

where $\{\beta_n(t)\}_{n=1}^\infty$ is a sequence of independent standard real-valued Wiener processes and $\{e_n\}_{n=1}^\infty$ is a complete orthonormal system of $\mathbb{H}$ consisting of eigenfunctions of a symmetric, nonnegative and finite trace operator Q, i.e. $\mathrm{Tr}(Q) < \infty$ and $Qe_n = \lambda_n e_n$ with corresponding eigenvalue $\lambda_n \geq 0$.

The Maxwell operator is defined by

$$M \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} := \begin{pmatrix} 0 & \varepsilon^{-1}\nabla\times \\ -\mu^{-1}\nabla\times & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} \tag{2.2}$$

with domain

$$\mathscr{D}(M) = \left\{ \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} \in \mathbb{H} : M \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} = \begin{pmatrix} \varepsilon^{-1}\nabla \times \boldsymbol{H} \\ -\mu^{-1}\nabla \times \boldsymbol{E} \end{pmatrix} \in \mathbb{H},\ \boldsymbol{n} \times \boldsymbol{E}\Big|_{\partial D} = 0 \right\}.$$

Based on the closedness of the operator $\nabla\times$, we have the following lemma.

Lemma 2.1 ([8]). *The Maxwell operator defined in* (2.2) *with domain $\mathscr{D}(M)$ is closed and skew-adjoint, and generates a C_0-semigroup $S(t) = e^{tM}$ on $\mathbb{H}$ for $t \in [0, T]$. Moreover, the frequently used property for Maxwell operator M is: $\langle Mu, u\rangle_{\mathbb{H}} = 0$.*

Let the drift term $F : [0,T]\times\mathbb{H} \to \mathbb{H}$ be a Nemytskij operator associated with $\boldsymbol{J}_e, \boldsymbol{J}_m$ defined by

$$F(t,u)(\boldsymbol{x}) = \begin{pmatrix} -\varepsilon^{-1}\boldsymbol{J}_e\big(t,\boldsymbol{x},\boldsymbol{E}(t,\boldsymbol{x}),\boldsymbol{H}(t,\boldsymbol{x})\big) \\ -\mu^{-1}\boldsymbol{J}_m\big(t,\boldsymbol{x},\boldsymbol{E}(t,\boldsymbol{x}),\boldsymbol{H}(t,\boldsymbol{x})\big) \end{pmatrix}, \quad \boldsymbol{x}\in D, \quad u = (\boldsymbol{E}^\top, \boldsymbol{H}^\top)^\top \in \mathbb{H}.$$

The diffusion term $B : [0,T] \to HS(U_0,\mathbb{H})$ is the Nemytskij operator defined by

$$\big(B(t)v\big)(\boldsymbol{x}) = \begin{pmatrix} -\varepsilon^{-1}\boldsymbol{J}_e(t,\boldsymbol{x})v(\boldsymbol{x}) \\ -\mu^{-1}\boldsymbol{J}_m(t,\boldsymbol{x})v(\boldsymbol{x}) \end{pmatrix}, \quad \boldsymbol{x}\in D, \quad v\in U_0 := Q^{\frac{1}{2}}\mathbb{H}.$$

We consider the abstract form of (1.1) in the infinite-dimensional space $\mathbb{H} := L^2(D)^3 \times L^2(D)^3$

$$\begin{cases} du(t) = [Mu(t) - \sigma u(t)]dt + F\big(t,u(t)\big)dt + B(t)dW, & t\in(0,T], \\ u(0) = u_0, \end{cases} \tag{2.3}$$

where the solution $u = (\boldsymbol{E}^\top, \boldsymbol{H}^\top)^\top$ is a stochastic process with values in $\mathbb{H}$.

Let $\widehat{S}(t) := e^{t(M-\sigma Id)}, \sigma \geq 0$ be the semigroup generated by operator $M - \sigma Id$, where Id is the identity operator. One can show that the damping stochastic Maxwell equations (2.3) possess the following lemma.

Lemma 2.2. *For the semigroup* $\{\widehat{S}(t) = e^{t(M-\sigma Id)},\, t\geq 0\}$ *on* $\mathbb{H}$*, we obtain*

$$\|\widehat{S}(t)\|_{\mathscr{L}(\mathbb{H})} \leq 1, \quad \forall\, t\geq 0.$$

Proof. Consider the deterministic system [8]

$$\begin{cases} \dfrac{du(t)}{dt} = Mu(t), & t\in(0,T], \\ u(0) = u_0. \end{cases}$$

Thus,

$$\frac{d}{dt}\|u(t)\|_{\mathbb{H}}^2 = 2\left\langle \frac{du(t)}{dt}, u(t)\right\rangle_{\mathbb{H}} = 2\langle Mu(t), u(t)\rangle_{\mathbb{H}} = 0,$$

which leads to

$$\|u(t)\|_{\mathbb{H}} = \|S(t)u_0\|_{\mathbb{H}} = \|u_0\|_{\mathbb{H}},$$

that is,

$$\|S(t)\|_{\mathscr{L}(\mathbb{H})} = 1.$$

Based on the semigroup $\{\widehat{S}(t) = e^{t(M-\sigma Id)},\, t\geq 0\}$ generated by the operator $M-\sigma Id$, we deduce

$$\|\widehat{S}(t)\|_{\mathscr{L}(\mathbb{H})} = \|e^{-\sigma t}S(t)\|_{\mathscr{L}(\mathbb{H})} \leq \|S(t)\|_{\mathscr{L}(\mathbb{H})}, \quad t\geq 0,$$

which concludes with the final result

$$\|\widehat{S}(t)\|_{\mathscr{L}(\mathbb{H})} \leq 1, \quad t\geq 0.$$

This completes the proof. □

To ensure the well-posedness of mild solution of the stochastic Maxwell equations (2.3), we need the following assumptions.

Assumption 2.1 (Initial Value). The initial value u_0 satisfies

$$\|u_0\|^2_{L_2(\Omega,\mathbb{H})} < \infty.$$

Assumption 2.2 (Drift Nonlinearity). The drift operator F satisfies

$$\|F(t,u)\|_{\mathbb{H}} \le C(1+\|u\|_{\mathbb{H}}),$$
$$\|F(t,u)-F(s,v)\|_{\mathbb{H}} \le C(|t-s|+\|u-v\|_{\mathbb{H}})$$

for all $t,s \in [0,T]$, $u,v \in \mathbb{H}$.

Remark 2.1. Assumption 2.2 ensures that the drift operator F is Lipschitz continuous and grows linearly. These conditions are essential for the well-posedness of the solution and the stability of numerical methods. Additionally, the boundedness of the derivatives of F is crucial for higher-order convergence analysis, particularly when using numerical methods like the stochastic exponential integrator, which rely on the smoothness of the drift term.

Building on Assumption 2.2, we introduce a stronger condition.

Assumption 2.3 (Bounded Fréchet Derivative of the Drift Operator). The drift operator F has a bounded Fréchet derivative, i.e. there exists a constant $C>0$ such that

$$\|DF(u).h\|_{\mathbb{H}} \le C\|h\|_{\mathbb{H}}, \quad \forall\, u,h \in \mathbb{H},$$

where $DF(u)$ denotes the Fréchet derivative of F at u, and h is the directional vector.

Assumption 2.4 (Covariance Operator, [29]). To guarantee the existence of a mild solution, we further assume the covariance operator Q of $W(t)$ satisfies

$$\big\|M^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\big\|_{\mathscr{L}_2(\mathbb{H})} < \infty, \quad \beta \in [0,1],$$

where $\|\cdot\|_{\mathscr{L}_2(\mathbb{H})}$ denotes the Hilbert-Schmidt norm for operators from $\mathbb{H}$ to $\mathbb{H}$, $M^{(\beta-1)/2}$ is the $(\beta-1)/2$-th fractional powers of M and β is a parameter characterizing the regularity of noise. In this article, we primarily focus on the case where $\beta = 1$, under which Q is a trace class operator. Under this condition, we will establish the main results of the paper.

Lemma 2.3 ([8]). *Let Assumptions* 2.1*,* 2.2*, and* 2.4 *hold, there exists a unique mild solution to* (2.3)*, which satisfies*

$$u(t) = \widehat{S}(t)u_0 + \int_0^t \widehat{S}(t-s)F\big(s,u(s)\big)ds + \int_0^t \widehat{S}(t-s)B(s)dW(s), \quad \mathbb{P}-a.s.$$

for each $t \in [0,T]$*, where* $\widehat{S}(t) = e^{t(M-\sigma Id)}, t \ge 0$ *is a* C_0*-semigroup generated by* $M-\sigma Id$*.*

Moreover, there exists a constant $C \in (0,\infty)$ *such that*

$$\sup_{t\in[0,T]} \|u(t)\|_{L_2(\Omega,\mathbb{H})} \le C\big(1+\|u_0\|_{L_2(\Omega,\mathbb{H})}\big).$$

The following lemma is the stability of analytical solution, which will be used in the proof of the Theorem 4.1.

Lemma 2.4 ([29]). *If* $u(t)$ *and* $v(t)$ *are two solutions of* (2.3) *with different initial values* u_0 *and* v_0*, there exists a constant* $C \in (0,\infty)$ *such that*

$$\|u(t)-v(t)\|_{L_2(\Omega,\mathbb{H})} \le C\|u_0-v_0\|_{L_2(\Omega,\mathbb{H})}.$$

3. Parareal Algorithm for Stochastic Maxwell Equations

3.1. Parareal algorithm

To perform the parareal algorithm, the interval $[0,T]$ is initially divided into N time intervals $[t_{n-1},t_n]$ with a uniform coarse step-size $\Delta T = t_n - t_{n-1}$ for any $n=1,\ldots,N$. Each subinterval $[t_{n-1},t_n]$ is further divided into J small time intervals $[t_{n-1,j-1},t_{n-1,j}]$ with a uniform fine step-size $\Delta t = t_{n-1,j} - t_{n-1,j-1}$ for all $n=1,\ldots,N$ and $j=1,\ldots,J$. Here we use $\mathscr{G}(t_{n-1},t_n,u)$ to denote the solution computed by the coarse propagator $\mathscr{G}$ in subinterval $[t_{n-1},t_n]$ with u as initial value and $\mathscr{F}(t_{n-1,j-1},t_{n-1,j},u)$ to denote the solution computed by the fine propagator $\mathscr{F}$ in the subinterval $[t_{n-1,j-1},t_{n-1,j}]$ with u as the initial value. Denote k as the iteration number and K as the maximum iteration number. With notations introduced above, the parareal algorithm can be summarized as follows:

- Initialization. Use the coarse propagator $\mathscr{G}$ with the coarse step-size ΔT to compute the initial value $u_n^{(0)}$ starting from at time t_n starting from $u_0^{(0)} := u_0$ by

$$u_n^{(0)} = \mathscr{G}\big(t_{n-1},t_n,u_{n-1}^{(0)}\big), \quad n=1,\ldots,N.$$

- Time-parallel computation. Use the fine propagator $\mathscr{F}$ with the fine step-size Δt to compute $\widehat{u}_n$ within each subinterval $[t_{n-1},t_n]$ independently. By applying the one-step approximation $\mathscr{F}$ starting from $\hat{u}_{n-1,0} := u_{n-1}^{(k)}$ at time $t_{n-1,0} := t_{n-1}$ for $k=1,\ldots,K-1$, the numerical solution at subsequent time $t_{n-1,j}$ can be expressed as

$$\widehat{u}_{n-1,j} = \mathscr{F}(t_{n-1,j-1},t_{n-1,j},\widehat{u}_{n-1,j-1}), \quad j=1,\ldots,J.$$

  Hence, we derive $\widehat{u}_{n-1,J} = \mathscr{F}(t_{n-1},t_n,u_{n-1}^{(k)})$.

- Prediction and correction. To obtain a family of more accurate numerical solutions, use the two numerical solutions $u_n^{(0)}$ and $\widehat{u}_{n-1,J}$ at time t_n obtained through initialization and parallelization to perform some corrections for $k=0,1,\ldots,K-1$,

$$\begin{aligned} u_n^{(0)} &= \mathscr{G}\big(t_{n-1},t_n,u_{n-1}^{(0)}\big), \\ u_n^{(k+1)} &= \mathscr{G}\big(t_{n-1},t_n,u_{n-1}^{(k+1)}\big) + \mathscr{F}\big(t_{n-1},t_n,u_{n-1}^{(k)}\big) - \mathscr{G}\big(t_{n-1},t_n,u_{n-1}^{(k)}\big). \end{aligned} \tag{3.1}$$

Remark 3.1. The coarse integrator $\mathscr{G}$ is required to be easy to calculate and enjoys a less computational cost, but need not to be of high accuracy. On the other hand, the fine integrator $\mathscr{F}$ defined on each subinterval is assumed to be more accurate but more costly than $\mathscr{G}$. Note that $\mathscr{G}$ and $\mathscr{F}$ can be the same numerical method or different numerical methods. In the article, the exponential integrator is chosen as the coarse integrator $\mathscr{G}$ and both the exact integrator and the exponential integrator are chosen as the fine integrator $\mathscr{F}$.

3.2. Stochastic exponential scheme

Consider the mild solution of the stochastic Maxwell equations (2.3) on the time interval $[t_{n-1},t_n]$

$$u(t_n) = \widehat{S}(\Delta T)u(t_{n-1}) + \int_{t_{n-1}}^{t_n} \widehat{S}(t_n - s)F\big(u(s)\big)ds + \int_{t_{n-1}}^{t_n} \widehat{S}(t_n - s)B(s)dW(s), \tag{3.2}$$

where C_0-semigroup $\widehat{S}(\Delta T) = e^{\Delta T(M-\sigma Id)}$.

By approximating the integrals in above mild solution (3.2) at the left endpoints, we can obtain the stochastic exponential scheme

$$u_n = \widehat{S}(\Delta T)u(t_{n-1}) + \widehat{S}(\Delta T)F\big(u(t_{n-1})\big)\Delta T + \widehat{S}(\Delta T)B(t_{n-1})\Delta W_n, \tag{3.3}$$

where $\Delta W_n = W(t_n) - W(t_{n-1})$.

3.3. Coarse and fine propagators

- Coarse propagator. The stochastic exponential scheme is chosen as the coarse propagator with time step-size ΔT by (3.3)

$$\mathscr{G}(t_{n-1}, t_n, u) = \widehat{S}(\Delta T)u + \widehat{S}(\Delta T)F(u)\Delta T + \widehat{S}(\Delta T)B(t_{n-1})\Delta W_n, \tag{3.4}$$

  where $\widehat{S}(\Delta T) = e^{\Delta T(M-\sigma Id)}$ and $\Delta W_n = W(t_n) - W(t_{n-1})$.

- Fine propagator. The exact solution as the fine propagator with time step-size Δt by (3.2)

$$\begin{aligned}&\mathscr{F}(t_{n-1,j-1}, t_{n-1,j}, u)\\ &= \widehat{S}(\Delta t)u + \int_0^{\Delta t} \widehat{S}(\Delta t - s)F\big(u(s)\big)ds\\ &\quad + \int_0^{\Delta t} \widehat{S}(\Delta t - s)B(t_{n-1,j-1})dW(s),\end{aligned} \tag{3.5}$$

  where $\widehat{S}(\Delta t) = e^{\Delta t(M-\sigma Id)}$.

  Besides, the other choice is to choose the stochastic exponential scheme as the fine propagator with time step-size Δt by (3.3)

$$\begin{aligned}&\mathscr{F}(t_{n-1,j-1}, t_{n-1,j}, u)\\ &= \widehat{S}(\Delta t)u + \widehat{S}(\Delta t)F(u)\Delta t + \widehat{S}(\Delta t)B(t_{n-1,j-1})\Delta W_{n-1,j},\end{aligned} \tag{3.6}$$

  where $\widehat{S}(\Delta t) = e^{\Delta t(M-\sigma Id)}$ and $\Delta W_{n-1,j} = W(t_{n-1,j}) - W(t_{n-1,j-1})$.

4. Main Results

In this section, two convergence analysis results will be given, i.e. we investigate the parareal algorithms obtained by choosing the stochastic exponential integrator as the coarse integrator and both the exact integrator and the stochastic exponential integrator as the fine integrator.

4.1. The exact integrator as the fine integrator $\mathscr{F}$

Theorem 4.1. *Let Assumptions* 2.1, 2.2 *and* 2.4 *hold, we apply the stochastic exponential integrator for coarse propagator $\mathscr{G}$ and exact solution integrator for fine propagator $\mathscr{F}$. Then we have the following convergence estimate for the fixed iteration number k:*

$$\sup_{1\le n\le N} \big\|u(t_n) - u_n^{(k)}\big\|_{L_2(\Omega,\mathbb{H})} \le \frac{C_k}{k!}\prod_{j=1}^{k}(N-j)\Delta T^k \sup_{1\le n\le N}\big\|u(t_n) - u_n^{(0)}\big\|_{L_2(\Omega,\mathbb{H})} \tag{4.1}$$

with a positive constant C independent on ΔT, where the parareal solution $u_n^{(k)}$ is defined in (3.1) *and the exact solution $u(t_n)$ is defined in* (3.2).

To simplify the exposition, let us introduce the following notation.

Definition 4.1. *The residual operator*

$$\mathscr{R}(t_{n-1},t_n,u) := \mathscr{F}(t_{n-1},t_n,u) - \mathscr{G}(t_{n-1},t_n,u) \tag{4.2}$$

for $n = 1,\dots,N$.

Before the error analysis, the following two useful lemmas are introduced.

Lemma 4.1 ([15]). *Let* $M := M(\beta)_{N\times N}$ *be a strict lower triangular Toeplitz matrix and its elements are defined as*

$$M_{i1} = \begin{cases} 0, & i = 1, \\ \beta^{i-2}, & 2 \le i \le N. \end{cases}$$

The infinity norm of the k*-th power of* M *is bounded as follows:*

$$\|M^k(\beta)\|_\infty \le \begin{cases} \min\left\{ \left(\dfrac{1-|\beta|^{N-1}}{1-|\beta|} \right)^k, \begin{pmatrix} N-1 \\ k \end{pmatrix} \right\}, & |\beta| < 1, \\ |\beta|^{N-k-1} \begin{pmatrix} N-1 \\ k \end{pmatrix}, & |\beta| \ge 1. \end{cases}$$

Lemma 4.2 ([15]). *Let* $\gamma,\eta \ge 0$*, a double indexed sequence* $\{\delta_n^k\}$ *satisfies* $\delta_n^k \ge 0$*,* $\delta_0^k \ge 0$ *and*

$$\delta_n^k \le \gamma\delta_{n-1}^k + \eta\delta_{n-1}^{k-1}, \quad n = 1,\dots,N, \quad k = 0,1,\dots,K,$$

then vector $\zeta^k = (\delta_1^k, \delta_2^k, \cdots, \delta_N^k)^\top$ *satisfies* $\zeta^k \le \eta M(\gamma)\zeta^{k-1}$.

Subsequently, we will demonstrate the proof of the Theorem 4.1.

Proof. For all $n = 1,\dots,N$ and $k = 0,\dots,K$, denote the error

$$\varepsilon_n^{(k)} := \left\| u(t_n) - u_n^{(k)} \right\|_{L_2(\Omega,\mathbb{H})}.$$

Since the exact solution $u(t_n)$ is chosen as the fine propagator $\mathscr{F}$, it can be written as

$$\begin{aligned} u(t_n) &= \mathscr{F}\big(t_{n-1},t_n,u(t_{n-1})\big) \\ &= \mathscr{G}\big(t_{n-1},t_n,u(t_{n-1})\big) \\ &\quad + \mathscr{F}\big(t_{n-1},t_n,u(t_{n-1})\big) - \mathscr{G}\big(t_{n-1},t_n,u(t_{n-1})\big). \end{aligned} \tag{4.3}$$

Subtracting the (4.3) from (3.1) and using the notation of the residual operator (4.2), we obtain

$$\begin{aligned} \varepsilon_n^{(k)} &\le \left\| \mathscr{G}\big(t_{n-1},t_n,u(t_{n-1})\big) - \mathscr{G}\big(t_{n-1},t_n,u_{n-1}^{(k)}\big) \right\|_{L_2(\Omega,\mathbb{H})} \\ &\quad + \left\| \mathscr{R}\big(t_{n-1},t_n,u(t_{n-1})\big) - \mathscr{R}\big(t_{n-1},t_n,u_{n-1}^{(k-1)}\big) \right\|_{L_2(\Omega,\mathbb{H})} \\ &=: I_1 + I_2. \end{aligned}$$

Firstly, we estimate I_1. Applying the stochastic exponential integrator (3.4) for the coarse propagator $\mathscr{G}$, it holds that

$$\mathscr{G}\big(t_{n-1},t_n,u(t_{n-1})\big) = \widehat{S}(\Delta T)u(t_{n-1}) + \widehat{S}(\Delta T)F\big(u(t_{n-1})\big)\Delta T + \widehat{S}(\Delta T)B(t_{n-1})\Delta W_n, \tag{4.4}$$

$$\mathscr{G}\big(t_{n-1},t_n,u_{n-1}^{(k)}\big) = \widehat{S}(\Delta T)u_{n-1}^{(k)} + \widehat{S}(\Delta T)F\big(u_{n-1}^{(k)}\big)\Delta T + \widehat{S}(\Delta T)B(t_{n-1})\Delta W_n. \tag{4.5}$$

Subtracting the above two formulas leads to

$$\begin{aligned}
I_1 &= \big\|\widehat{S}(\Delta T)\big(u(t_{n-1})-u_{n-1}^{(k)}\big)+\widehat{S}(\Delta T)\big(F\big(u(t_{n-1})\big)-F\big(u_{n-1}^{(k)}\big)\big)\Delta T\big\|_{L_2(\Omega,\mathbb{H})}\\
&\le \|\widehat{S}(t)\|_{\mathscr{L}(\mathbb{H})}\big\|u(t_{n-1})-u_{n-1}^{(k)}\big\|_{L_2(\Omega,\mathbb{H})}+C\Delta T\|\widehat{S}(t)\|_{\mathscr{L}(\mathbb{H})}\big\|u(t_{n-1})-u_{n-1}^{(k)}\big\|_{L_2(\Omega,\mathbb{H})}\\
&= (1+C\Delta T)\big\|u(t_{n-1})-u_{n-1}^{(k)}\big\|_{L_2(\Omega,\mathbb{H})}=(1+C\Delta T)\varepsilon_{n-1}^{(k)}, \qquad (4.6)
\end{aligned}$$

which by the contraction property of semigroup and the global Lipschitz property of F.

Now it remains to estimate I_2. Applying exact solution integrator (3.5) for fine propagator $\mathscr{F}$ leads to

$$\begin{aligned}
&\mathscr{F}\big(t_{n-1},t_n,u(t_{n-1})\big)\\
&=\widehat{S}(\Delta T)u(t_{n-1})+\int_0^{\Delta T}\widehat{S}(\Delta T-s)F\big(U\big(t_{n-1},t_{n-1}+s,u(t_{n-1})\big)\big)ds\\
&\quad+\int_0^{\Delta T}\widehat{S}(\Delta T-s)B(t_{n-1})dW(s), \qquad (4.7)\\
&\mathscr{F}\big(t_{n-1},t_n,u_{n-1}^{(k-1)}\big)\\
&=\widehat{S}(\Delta T)u_{n-1}^{(k-1)}+\int_0^{\Delta T}\widehat{S}(\Delta T-s)F\big(V\big(t_{n-1},t_{n-1}+s,u_{n-1}^{(k-1)}\big)\big)ds\\
&\quad+\int_0^{\Delta T}\widehat{S}(\Delta T-s)B(t_{n-1})dW(s), \qquad (4.8)
\end{aligned}$$

where $U(t_{n-1},t_{n-1}+s,u)$ and $V(t_{n-1},t_{n-1}+s,u)$ denote the exact solution of system (2.3) at time $t_{n-1}+s$ with the initial value u and the initial time t_{n-1}.

Substituting the above equations and Eqs. (4.4) and (4.5) into the residual operator (4.2), we obtain

$$\begin{aligned}
I_2 &= \big\|\mathscr{R}\big(t_{n-1},t_n,u(t_{n-1})\big)-\mathscr{R}\big(t_{n-1},t_n,u_{n-1}^{(k-1)}\big)\big\|_{L_2(\Omega,\mathbb{H})}\\
&\le \bigg\|\int_0^{\Delta T}\widehat{S}(\Delta T-s)\big[F\big(U\big(t_{n-1},t_{n-1}+s,u(t_{n-1})\big)\big)-F\big(V\big(t_{n-1},t_{n-1}+s,u_{n-1}^{(k-1)}\big)\big)\big]ds\bigg\|_{L_2(\Omega,\mathbb{H})}\\
&\quad+\big\|\widehat{S}(\Delta T)\big[F\big(u(t_{n-1})\big)-F\big(u_{n-1}^{(k-1)}\big)\big]\Delta T\big\|_{L_2(\Omega,\mathbb{H})}\\
&=: I_3+I_4.
\end{aligned}$$

To get the estimation of I_3, by Lipschitz continuity property for F and Lemma 2.4, we derive

$$\begin{aligned}
I_3 &\le \int_0^{\Delta T}\|\widehat{S}(\Delta T-s)\|_{\mathscr{L}(\mathbb{H})}\big\|F\big(U\big(t_{n-1},t_{n-1}+s,u(t_{n-1})\big)\big)\\
&\qquad\qquad -F\big(V\big(t_{n-1},t_{n-1}+s,u_{n-1}^{(k-1)}\big)\big)\big\|_{L_2(\Omega,\mathbb{H})}ds\\
&\le C\int_0^{\Delta T}\big\|U\big(t_{n-1},t_{n-1}+s,u(t_{n-1})\big)-V\big(t_{n-1},t_{n-1}+s,u_{n-1}^{(k-1)}\big)\big\|_{L_2(\Omega,\mathbb{H})}ds\\
&\le C\Delta T\big\|u(t_{n-1})-u_{n-1}^{(k-1)}\big\|_{L_2(\Omega,\mathbb{H})}. \qquad (4.9)
\end{aligned}$$

As for I_4, using the contraction property of semigroup and Lipschitz continuity property for F yields

$$I_4\le C\Delta T\big\|u(t_{n-1})-u_{n-1}^{(k-1)}\big\|_{L_2(\Omega,\mathbb{H})}. \qquad (4.10)$$

From (4.9) and (4.10), we know that

$$I_2 \le C\Delta T \big\| u(t_{n-1}) - u_{n-1}^{(k-1)} \big\|_{L_2(\Omega,\mathbb{H})} = C\Delta T \varepsilon_{n-1}^{(k-1)}. \tag{4.11}$$

Combining (4.6) and (4.11) enables us to derive

$$\varepsilon_n^{(k)} \le (1 + C\Delta T)\varepsilon_{n-1}^{(k)} + C\Delta T \varepsilon_{n-1}^{(k-1)}.$$

Let $\zeta^k = (\varepsilon_1^k, \varepsilon_2^k, \cdots, \varepsilon_N^k)^\top$. It follows from Lemma 4.2 that

$$\zeta^k \le C\Delta T M (1 + C\Delta T)\zeta^{k-1} \le C^k \Delta T^k M^k (1 + C\Delta T)\zeta^0.$$

Taking infinity norm and using Lemma 4.1 imply

$$\begin{aligned} \sup_{1\le n\le N} \varepsilon_n^{(k)} &\le C^k (1 + C\Delta T)^{N-k-1} \binom{N-1}{k} \Delta T^k \sup_{1\le n\le N} \varepsilon_n^{(0)} \\ &\le \frac{C_k}{k!} \prod_{j=1}^{k} (N-j)\Delta T^k \sup_{1\le n\le N} \varepsilon_n^{(0)}. \end{aligned}$$

This completes the proof. □

4.2. The stochastic exponential integrator as the fine propagator

In this section, the error we considered is the solution by the proposed algorithm and the reference solution generated by the fine propagator $\mathscr{F}$. To begin with, we define the reference solution as follows.

Definition 4.2. *For all $n = 1, \ldots, N$, the reference solution is defined by the fine propagator on each subinterval $[t_{n-1}, t_n]$*

$$\begin{aligned} u_n^{ref} &= \mathscr{F}\big(t_{n-1}, t_n, u_{n-1}^{ref}\big), \\ u_0^{ref} &= u_0. \end{aligned} \tag{4.12}$$

Precisely,

$$\begin{aligned} u_{n-1,j}^{ref} &= \mathscr{F}\big(t_{n-1,j-1}, t_{n-1,j}, u_{n-1,j-1}^{ref}\big), \quad j = 1, \ldots, J, \\ u_{n-1,0}^{ref} &= u_{n-1}^{ref}. \end{aligned} \tag{4.13}$$

Theorem 4.2. *Let Assumptions 2.1-2.4 hold, we apply the stochastic exponential integrator for coarse propagator $\mathscr{G}$ and the stochastic exponential integrator for fine propagator $\mathscr{F}$. Then we have the following convergence estimate for the fixed iteration number k:*

$$\sup_{1\le n\le N} \big\| u_n^{(k)} - u_n^{ref} \big\|_{L_2(\Omega,\mathbb{H})} \le \frac{C_k}{k!} \prod_{j=1}^{k} (N-j)\Delta T^k \sup_{1\le n\le N} \big\| u_n^{(0)} - u_n^{ref} \big\|_{L_2(\Omega,\mathbb{H})} \tag{4.14}$$

with a positive constant C independent on ΔT, where the parareal solution $u_n^{(k)}$ is defined in (3.1) and the reference solution u_n^{ref} is defined in (4.12).

Proof. For all $n = 1, \ldots, N$ and $k = 0, \ldots, K$. Let the error be defined by

$$\varepsilon_n^{(k)} := \big\|u_n^k - u_n^{ref}\big\|_{L_2(\Omega,\mathbb{H})}.$$

Observe that the reference solution (4.12) can be rewritten

$$u_n^{ref} = \mathscr{F}\big(t_{n-1}, t_n, u_{n-1}^{ref}\big) + \mathscr{G}\big(t_{n-1}, t_n, u_{n-1}^{ref}\big) - \mathscr{G}\big(t_{n-1}, t_n, u_{n-1}^{ref}\big). \tag{4.15}$$

Combining the parareal algorithm form (3.1) and the reference solution (4.15) and using the notation of the residual operator (4.2), the error can be written as

$$\begin{aligned}\varepsilon_n^{(k)} &= \big\|\mathscr{G}\big(t_{n-1}, t_n, u_{n-1}^{(k)}\big) - \mathscr{G}\big(t_{n-1}, t_n, u_{n-1}^{ref}\big) + \mathscr{R}\big(t_{n-1}, t_n, u_{n-1}^{(k-1)}\big) - \mathscr{R}\big(t_{n-1}, t_n, u_{n-1}^{ref}\big)\big\|_{L_2(\Omega,\mathbb{H})}\\ &\le \big\|\mathscr{G}\big(t_{n-1}, t_n, u_{n-1}^{(k)}\big) - \mathscr{G}\big(t_{n-1}, t_n, u_{n-1}^{ref}\big)\big\|_{L_2(\Omega,\mathbb{H})}\\ &\quad + \big\|\mathscr{R}\big(t_{n-1}, t_n, u_{n-1}^{(k-1)}\big) - \mathscr{R}\big(t_{n-1}, t_n, u_{n-1}^{ref}\big)\big\|_{L_2(\Omega,\mathbb{H})}\\ &=: I_1 + I_2.\end{aligned}$$

Now we estimate I_1. Applying the stochastic exponential integrator (3.4) for the coarse propagator $\mathscr{G}$, we obtain

$$\mathscr{G}\big(t_{n-1}, t_n, u_{n-1}^{ref}\big) = \widehat{S}(\Delta T)u_{n-1}^{ref} + \widehat{S}(\Delta T)F\big(u_{n-1}^{ref}\big)\Delta T + \widehat{S}(\Delta T)B(t_{n-1})\Delta W_n. \tag{4.16}$$

Subtracting the above formula (4.16) from (4.5), we have

$$I_1 = \big\|\widehat{S}(\Delta T)\big(u_{n-1}^{(k)} - u_{n-1}^{ref}\big) + \widehat{S}(\Delta T)\big[F\big(u_{n-1}^{(k)}\big) - F\big(u_{n-1}^{ref}\big)\big]\Delta T\big\|_{L_2(\Omega,\mathbb{H})}.$$

Armed with contraction property of semigroup and Lipschitz continuity property of F yield

$$\begin{aligned}I_1 &\le \|\widehat{S}(t)\|_{\mathscr{L}(\mathbb{H})}\big\|u_{n-1}^{(k)} - u_{n-1}^{ref}\big\|_{L_2(\Omega,\mathbb{H})} + C\Delta T\|\widehat{S}(t)\|_{\mathscr{L}(\mathbb{H})}\big\|u_{n-1}^{(k)} - u_{n-1}^{ref}\big\|_{L_2(\Omega,\mathbb{H})}\\ &\le \big\|u_{n-1}^{(k)} - u_{n-1}^{ref}\big\|_{L_2(\Omega,\mathbb{H})} + \Delta TC\big\|u_{n-1}^{(k)} - u_{n-1}^{ref}\big\|_{L_2(\Omega,\mathbb{H})}\\ &\le (1 + C\Delta T)\big\|u_{n-1}^{(k)} - u_{n-1}^{ref}\big\|_{L_2(\Omega,\mathbb{H})}.\end{aligned} \tag{4.17}$$

As for I_2, regarding the estimation of the residual operator, we need to resort to its directional derivatives. Due to formula (4.2), the derivatives can be given by

$$D\mathscr{R}(t_{n-1}, t_n, u).h := D\mathscr{F}(t_{n-1}, t_n, u).h - D\mathscr{G}(t_{n-1}, t_n, u).h. \tag{4.18}$$

On the one hand, since the stochastic exponential scheme is chosen as the fine propagator (3.6) with time step-size Δt, we obtain

$$\begin{cases} u_{n,j+1} = \widehat{S}(\Delta t)u_{n,j} + \Delta t\widehat{S}(\Delta t)F(u_{n,j}) + \widehat{S}(\Delta t)B(t_{n,j})\Delta W_{n,j}, & j \in 1, \ldots, J-1,\\ u_{n,0} = u. \end{cases}$$

Denote $D(u_{n,j}).h := \eta_{n,j}^h$ for $j \in 0, \ldots, J$. Then taking the direction derivatives for above equation yields

$$\begin{cases} \eta_{n,j+1}^h = \widehat{S}(\Delta t)\eta_{n,j}^h + \Delta t\widehat{S}(\Delta t)DF(u_{n,j}).\eta_{n,j}^h, & j \in 1, \ldots, J-1,\\ \eta_{n,0}^h = h. \end{cases}$$

Based on the form of semigroup $\{\widehat{S}(t) = e^{t(M-\sigma Id)}, t \geq 0\}$, we have the following recursion formula:

$$\begin{aligned} \eta_{n,J}^h &= e^{J\Delta t(M-\sigma Id)}\eta_{n,0}^h + \Delta t \sum_{j=0}^{J-1} e^{(J-j)\Delta t(M-\sigma Id)} DF(u_{n,j}).\eta_{n,j}^h \\ &= e^{\Delta T(M-\sigma Id)}h + \Delta t \sum_{j=0}^{J-1} e^{(J-j)\Delta t(M-\sigma Id)} DF(u_{n,j}).\eta_{n,j}^h. \end{aligned}$$

Applying the discrete Gronwall lemma yields the following inequality:

$$\sup_{1\leq n\leq N} \left\|\eta_{n,j}^h\right\|_{L_2(\Omega,\mathbb{H})} \leq C\|h\|_{L_2(\Omega,\mathbb{H})}. \tag{4.19}$$

Moreover, the derivative of $\mathscr{F}(t_{n-1}, t_n, u)$ can be written by

$$D\mathscr{F}(t_{n-1}, t_n, u).h = D(u_{n,J}).h = \eta_{n,J}^h,$$

where $J\Delta t = \Delta T$, that is, one gets

$$D\mathscr{F}(t_{n-1}, t_n, u).h = e^{\Delta T(M-\sigma Id)}h + \Delta t \sum_{j=0}^{J-1} e^{(J-j)\Delta t(M-\sigma Id)} DF(u_{n,j}).\eta_{n,j}^h. \tag{4.20}$$

On the other hand, since the stochastic exponential scheme is chosen as the coarse propagator $\mathscr{G}$, taking the direction derivative for u of formula (4.1) leads to

$$D\mathscr{G}(t_{n-1}, t_n, u).h = e^{\Delta T(M-\sigma Id)}h + \Delta T e^{\Delta T(M-\sigma Id)} DF(u).h. \tag{4.21}$$

Substituting formula (4.20) and (4.21) into formula (4.18), we obtain

$$\begin{aligned} &\|D\mathscr{R}(t_{n-1}, t_n, u).h\|_{L_2(\Omega,\mathbb{H})} \\ &= \|D\mathscr{F}(t_{n-1}, t_n, u).h - D\mathscr{G}(t_{n-1}, t_n, u).h\|_{L_2(\Omega,\mathbb{H})} \\ &= \left\|\Delta t \sum_{j=0}^{J-1} e^{(J-j)\Delta t(M-\sigma Id)} DF(u_{n,j}).\eta_{n,j}^h + \Delta T e^{\Delta T(M-\sigma Id)} DF(u).h\right\|_{L_2(\Omega,\mathbb{H})} \\ &\leq \left\|\Delta t \sum_{j=0}^{J-1} e^{(J-j)\Delta t(M-\sigma Id)} DF(u_{n,j}).\eta_{n,j}^h\right\|_{L_2(\Omega,\mathbb{H})} + \left\|\Delta T e^{\Delta T(M-\sigma Id)} DF(u).h\right\|_{L_2(\Omega,\mathbb{H})}. \end{aligned}$$

Utilizing the bounded derivatives condition of F, we get

$$\begin{aligned} &\|D\mathscr{R}(t_{n-1}, t_n, u).h\|_{L_2(\Omega,\mathbb{H})} \\ &\leq \Delta t \sum_{j=0}^{J-1} \left\|e^{(J-j)\Delta t(M-\sigma Id)}\right\|_{\mathscr{L}(\mathbb{H})} \left\|\eta_{n,j}^h\right\|_{L_2(\Omega,\mathbb{H})} \\ &\quad + C\Delta T \left\|e^{\Delta T(M-\sigma Id)}\right\|_{\mathscr{L}(\mathbb{H})} \|h\|_{L_2(\Omega,\mathbb{H})}. \end{aligned}$$

Using the contraction property of semigroup, we have

$$\begin{aligned} &\sup_{1\leq n\leq N} \|D\mathscr{R}(t_{n-1}, t_n, u).h\|_{L_2(\Omega,\mathbb{H})} \\ &\leq \Delta t \sum_{j=0}^{J-1} \sup_{1\leq n\leq N} \left\|\eta_{n,j}^h\right\|_{L_2(\Omega,\mathbb{H})} + C\Delta T\|h\|_{L_2(\Omega,\mathbb{H})}. \end{aligned}$$

Substituting the Gronwall inequality (4.19) into the above inequality leads to

$$\begin{aligned}
&\sup_{1\le n\le N} \|D\mathscr{R}(t_{n-1},t_n,u).h\|_{L_2(\Omega,\mathbb{H})}\\
&\le \Delta t JC\|h\|_{L_2(\Omega,\mathbb{H})} + C\Delta T\|h\|_{L_2(\Omega,\mathbb{H})}\\
&\le C\Delta T\|h\|_{L_2(\Omega,\mathbb{H})}.
\end{aligned}$$

In conclusion, it holds that

$$\begin{aligned}
&\sup_{1\le n\le N} \|\mathscr{R}(t_{n-1},t_n,u_2) - \mathscr{R}(t_{n-1},t_n,u_1)\|_{L_2(\Omega,\mathbb{H})}\\
&\le C\Delta T\|u_2-u_1\|_{L_2(\Omega,\mathbb{H})}, \quad \forall\, u_1,u_2\in\mathbb{H}. \qquad (4.22)
\end{aligned}$$

Substituting $u_{n-1}^{(k-1)}$ and u_{n-1}^{ref} into above formula derives lipschitz continuity property of the residual operator

$$\begin{aligned}
I_2 &= \big\|\mathscr{R}\big(t_{n-1},t_n,u_{n-1}^{(k-1)}\big) - \mathscr{R}\big(t_{n-1},t_n,u_{n-1}^{ref}\big)\big\|_{L_2(\Omega,\mathbb{H})}\\
&\le C\Delta T\big\|u_{n-1}^{(k-1)} - u_{n-1}^{ref}\big\|_{L_2(\Omega,\mathbb{H})}. \qquad (4.23)
\end{aligned}$$

Combining (4.17) and (4.23), we have

$$\begin{aligned}
\varepsilon_n^{(k)} &\le (1+C\Delta T)\big\|u_{n-1}^{(k)} - u_{n-1}^{ref}\big\|_{L_2(\Omega,\mathbb{H})} + C\Delta T\big\|u_{n-1}^{(k-1)} - u_{n-1}^{ref}\big\|_{L_2(\Omega,\mathbb{H})}\\
&= (1+C\Delta T)\varepsilon_{n-1}^{(k)} + C\Delta T\varepsilon_{n-1}^{(k-1)}.
\end{aligned}$$

According to Lemmas 4.1 and 4.2, it yields to

$$\begin{aligned}
\sup_{1\le n\le N}\varepsilon_n^{(k)} &\le C^k(1+C\Delta T)^{N-k-1}\binom{N-1}{k}\Delta T^k \sup_{1\le n\le N}\varepsilon_n^{(0)}\\
&\le \frac{C_k}{k!}\prod_{j=1}^{k}(N-j)\Delta T^k \sup_{1\le n\le N}\varepsilon_n^{(0)},
\end{aligned}$$

which leads to the final result

$$\sup_{1\le n\le N}\big\|u_n^{(k)} - u_n^{ref}\big\|_{L_2(\Omega,\mathbb{H})} \le \frac{C_k}{k!}\prod_{j=1}^{k}(N-j)\Delta T^k \sup_{1\le n\le N}\big\|u_n^{(0)} - u_n^{ref}\big\|_{L_2(\Omega,\mathbb{H})}.$$

The proof is complete. □

Remark 4.1. We can summarise Lipschitz continuity property of the residual operator $\mathscr{R}(t_{n-1},t_n,u)$: There exists $C\in(0,\infty)$ such that for $\Delta T\in(0,1]$ and $u_1,u_2\in\mathbb{H}$, we have

$$\sup_{1\le n\le N}\|\mathscr{R}(t_{n-1},t_n,u_2) - \mathscr{R}(t_{n-1},t_n,u_1)\|_{L_2(\Omega,\mathbb{H})} \le C\Delta T\|u_2-u_1\|_{L_2(\Omega,\mathbb{H})}.$$

Remark 4.2. When we fix the iteration number k, the convergence rate will be $\mathcal{O}(\Delta T)^k$.

Remark 4.3. The error between the reference solution u_n^{ref} by the fine propagator defined in (4.12) and the exact solution $u(t_n)$ defined in (3.2) do not affect the convergence rate of the parareal algorithm, due to

$$\sup_{1\le n\le N}\big\|u_n^{ref} - u(t_n)\big\|_{L_2(\Omega,\mathbb{H})} \le C(1+\|u_0\|_{L_2(\Omega,\mathbb{H})})\Delta t.$$

Therefore, it is sufficient to study the convergence order of the error between $u_n^{(k)}$ and u_n^{ref}.

Proposition 4.1 (Uniform Boundedness of Reference Solution u_n^{ref}, [11]). *There exists a constant $C \in (0,\infty)$ such that*

$$\sup_{t\in[0,T]} \left\|u_n^{ref}\right\|_{L_2(\Omega,\mathbb{H})} \leq C(1+\|u_0\|_{L_2(\Omega,\mathbb{H})}).$$

Proposition 4.2 (Uniform Boundedness of Parareal Algorithm Solution $u_n^{(k)}$). *There exists a constant $C \in (0,\infty)$ such that*

$$\sup_{t\in[0,T]} \left\|u_n^{(k)}\right\|_{L_2(\Omega,\mathbb{H})} \leq C(1+\|u_0\|_{L_2(\Omega,\mathbb{H})}).$$

5. Numerical Experiments

This section is devoted to investigating the convergence result with several parameters and the effect of the scale of noise on numerical solutions. Since the parareal algorithm in principle is a temporal algorithm, and the spatial discretization is not our focus in this article, we perform finite difference method to discretize spatially.

The mean-square error is used as

$$L_2 = \left(\varepsilon\left\|\vec{E}(t_n) - \vec{E}_n^{(k)}\right\|^2 + \mu\left\|\vec{H}(t_n) - \vec{H}_n^{(k)}\right\|^2\right)^{\frac{1}{2}}.$$

5.1. Convergence

5.1.1. One-dimensional transverse magnetic wave

We first consider the stochastic Maxwell equations with one-dimensional transverse magnetic wave driven by the standard Brownian motion

$$\begin{cases} \dfrac{\partial E_z}{\partial t} = \dfrac{1}{\epsilon}\dfrac{\partial H_y}{\partial x} - \sigma E_z + \lambda_1 \cdot \dot{W}, \\ \dfrac{\partial H_y}{\partial t} = \dfrac{1}{\mu}\dfrac{\partial E_z}{\partial x} - \sigma H_y + \lambda_2 \cdot \dot{W}, \end{cases}$$

by providing initial conditions

$$E_z(x,0) = \sin(x), \quad H_y(x,0) = -\sqrt{\frac{\epsilon}{\mu}}\sin(x)$$

for $t \in [0,1]$, $x \in [0,2\pi]$ and $\dot{W} = \dot{B}$.

Firstly, to ensure computational consistency and comparability, the parameters are normalized to $\varepsilon = 1$, $\mu = 1$ and $\lambda_1 = \lambda_2 = 1$. We apply the parareal algorithm to solve the numerical solution with the fine step-size $\Delta t = 2^{-8}$ and the coarse step-size $\Delta T = 2^{-6}$. The spatial mesh grid-size $\Delta x = \Delta y = 2\pi/100$. Fig. 5.1 demonstrates the evolution of the mean-square error $(\sup_{1\leq n\leq N} E\|u_n^{(k)} - u_n^{ref}\|^2)^{1/2}$ with the iteration number k. From the Fig. 5.1, we observe that the damping term speeds up the convergence of the numerical solutions and the error approaches 10^{-12} after $k = 11$ at least nearly, demonstrating the effective convergence of the proposed parareal algorithm.

Remark 5.1. From a numerical analysis point of view, the inclusion of damping coefficients usually accelerates the convergence of numerical solutions by suppressing oscillations and instability, resulting in a faster steady state or desired precision. However, too small damping may not be enough to accelerate the convergence rate and may even introduce instability.

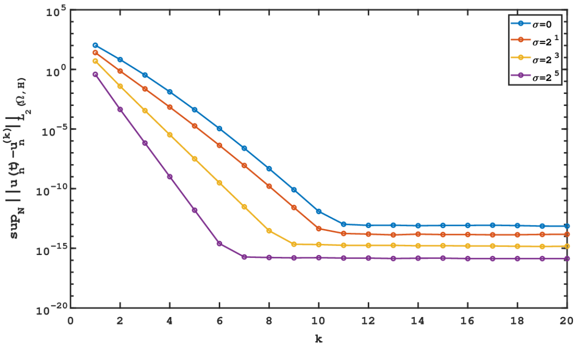

Fig. 5.1. Mean square error $(\sup_{1\leq n\leq N} E\|u_n^{(k)} - u_n^{ref}\|^2)^{1/2}$ of one-dimensional case with iteration number k for different values of $\sigma = 0, 2^1, 2^3, 2^5$.

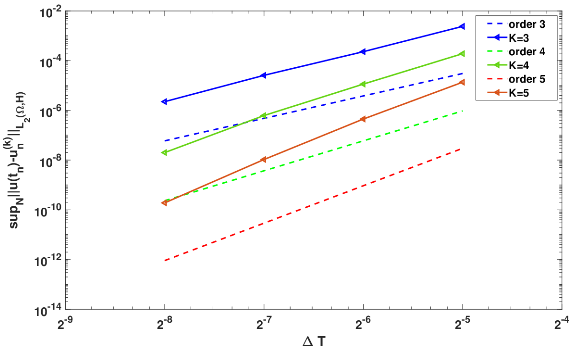

Fig. 5.2. Mean-square order of one-dimensional case with respect to $\Delta T = 2^{-i}, i = 5, 6, 7, 8$.

Subsequently, we choose the damping coefficient $\sigma = 2$ to calculate the convergence order of the proposed parareal algorithm. We compute the numerical solution with the fine step-size $\Delta t = 2^{-10}$ and the coarse step-size $\Delta T = 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}$. Fig. 5.2 reports the convergence order of the parareal algorithm with the iteration number $k = 3, 4, 5$. It is clearly shown that the mean-square convergence order always increases as the iteration number k increases.

5.1.2. Two-dimensional transverse magnetic waves

We consider the stochastic Maxwell equations with two-dimensional transverse magnetic polarization driven by trace class noise

$$\begin{cases} \dfrac{\partial E_z}{\partial t} = \dfrac{1}{\varepsilon}\left(\dfrac{\partial H_y}{\partial x} - \dfrac{\partial H_x}{\partial y}\right) - \sigma E_z + \lambda_1 \cdot \dot{W}, \\ \dfrac{\partial H_x}{\partial t} = -\dfrac{1}{\mu}\dfrac{\partial E_z}{\partial y} - \sigma H_x + \lambda_2 \cdot \dot{W}, \\ \dfrac{\partial H_y}{\partial t} = \dfrac{1}{\mu}\dfrac{\partial E_z}{\partial x} - \sigma H_y + \lambda_2 \cdot \dot{W}, \end{cases} \tag{5.1}$$

by providing initial conditions

$$E_z(x, y, 0) = \sin(3\pi x)\sin(4\pi y),$$

$$H_x(x, y, 0) = -0.8\cos(3\pi x)\sin(4\pi y),$$
$$H_y(x, y, 0) = -0.6\sin(3\pi x)\sin(4\pi y)$$

for $t \in [0, 1]$ and $(x, y) \in D = [0, 2\pi] \times [0, 2\pi]$. Following the formula (2.1), we choose

$$e_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right),$$

and $\lambda_n = n^{-(2r+1+\delta)}$ for some $\delta > 0$ and $r \geq 0$. In this case,

$$\mathrm{Tr}(Q) = \sum_{n=1}^{\infty}\lambda_n < \infty.$$

We construct the Wiener process as follows [8]:

$$W(t) = \sqrt{\frac{2}{a}}\sum_{n=1}^{\infty} n^{-(\frac{2r+1+\delta}{2})}\sin\left(\frac{n\pi x}{a}\right)\beta_n(t)$$

with $a = 2, r = 0.5$ and $\delta = 0.001$.

Firstly, the parameters are normalized to $\varepsilon = 1, \mu = 1$ and $\lambda_1 = \lambda_2 = 1$. We compute the fine numerical solution with step-size $\Delta t = 2^{-8}$ and the coarse numerical solution $\Delta T = 2^{-6}$. The spatial mesh grid-size is defined with $\Delta x = \Delta y = 2\pi/100$. Fig. 5.3 demonstrates the evolution of the mean-square error $(\sup_{1\leq n\leq N} E\|u_n^{(k)} - u_n^{ref}\|^2)^{1/2}$ with iteration number k. From the Fig. 5.3, we observe that the error decreases to approximately 10^{-11} after $k = 10$ nearly, thereby providing strong evidence for the convergence of the proposed algorithm.

Remark 5.2. In numerical simulation, the introduction of damping terms and the selection of parameters need to be careful to ensure the accuracy and physical authenticity of simulation results. Excessive damping may lead to excessive attenuation, thus affecting the accuracy of simulation results.

Secondly, in order to investigate the relationship between the convergence order and the iteration number, we choose the damping coefficient $\sigma = 2$ to calculate the convergence order of the proposed algorithm as taking the different iteration number k. We compute the numerical solution with the fine step-size $\Delta t = 2^{-8}$ and the coarse step-size $\Delta T = 2^{-3}, 2^{-4}, 2^{-5}, 2^{-6}$. Fig. 5.4 reports the convergence order of the proposed algorithm numerical error with the iteration number $k = 2, 3, 4$. Indeed, the numerical experiments reveal that the convergence order of the proposed algorithm increases as the iteration number k increases.

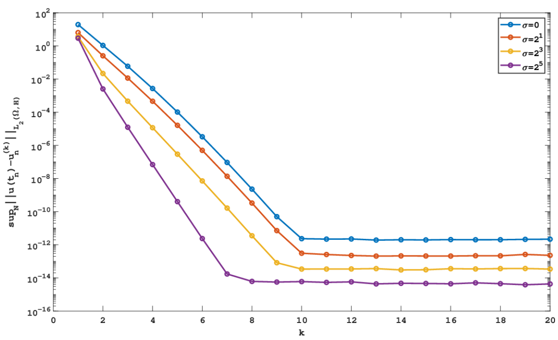

Fig. 5.3. Mean square error $(\sup_{1\leq n\leq N} E\|u_n^{(k)} - u_n^{ref}\|^2)^{1/2}$ of two-dimensional case with iteration number k for different values of $\sigma = 0, 2^1, 2^3, 2^5$.

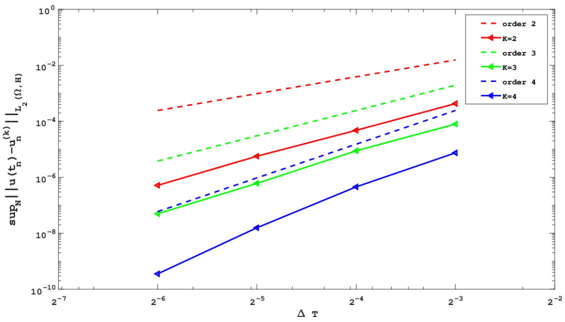

Fig. 5.4. Mean-square order of two-dimensional case with respect to $\Delta T = 2^{-i}, i = 3, 4, 5, 6$.

5.2. Impact of the scale of noise

We investigate the stochastic Maxwell equations with two-dimensional transverse magnetic polarization (5.1). To maintain consistency and facilitate comparison, the parameters are normalized to $\varepsilon = 1$, $\mu = 1$. For the numerical discretization, we take the fine step-size $\Delta t = 2^{-5}$, the coarse step-size $\Delta T = 2^{-3}$ and the spatial mesh grid-size $\Delta x = \Delta y = 1/50$. To elucidate the influence of noise scale on the numerical solution, we perform a series of numerical simulations with four scales of noise $\lambda_1 = \lambda_2 = 0, 2^1, 2^3, 2^5$, while maintaining a damping coefficient of $\sigma = 2^3$.

Fig. 5.5 presents the 10 contour plots of the numerical solution $E_z(x, y)$ with different scales of noise and Fig. 5.6 shows the electric field wave forms $E_z(x, y)$ with different scales of noise. Comparing with deterministic case (a) of Figs. 5.5 and 5.6, we can find that the oscillator of the wave forms (b)-(d) of Figs. 5.5 and 5.6 becomes more and more violent as the scale of the noise increases, i.e. from (a)-(d) of Figs. 5.5 and 5.6 it can be observed that the perturbation of the numerical solutions becomes more and more apparent as the scale of the noise increases.

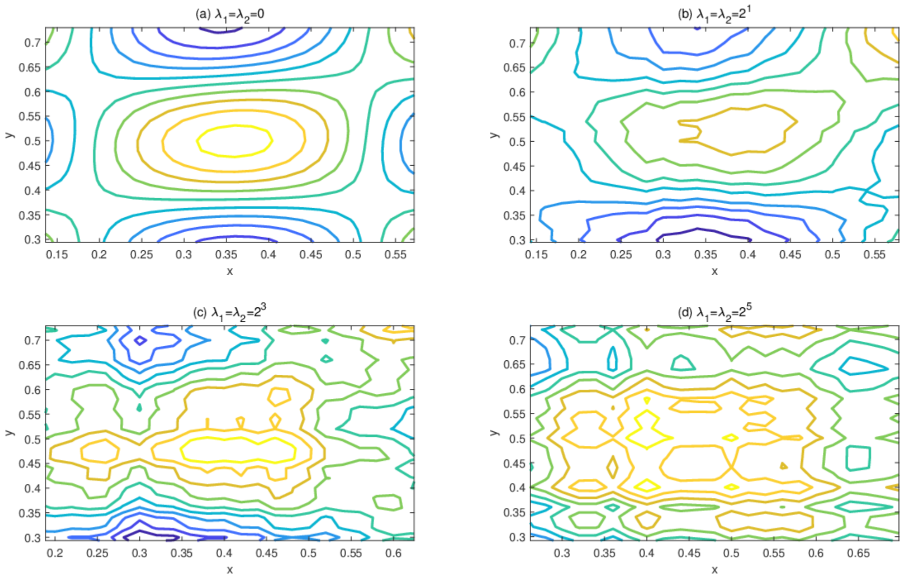

Fig. 5.5. 10 contour of $E_z(x, y)$ with different sizes of noise $\lambda_1 = \lambda_2 = 0, 2^1, 2^3, 2^5$ in the time $T = 1$.

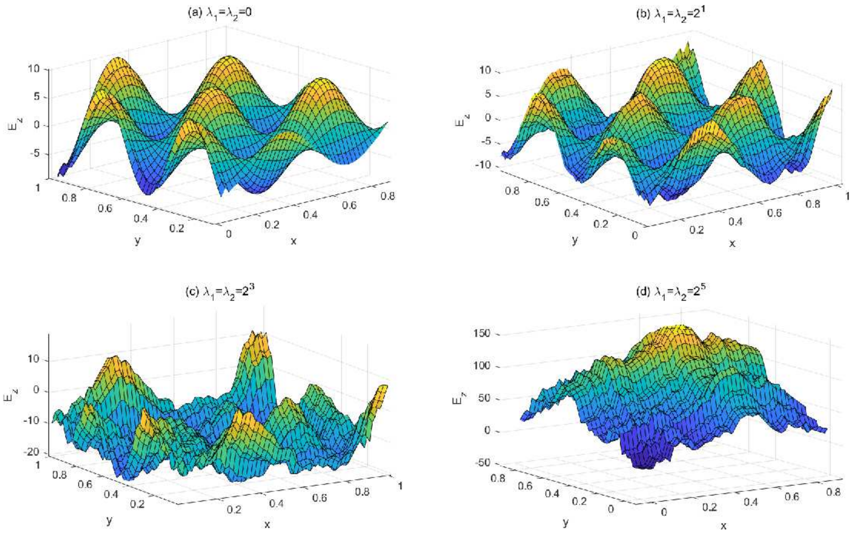

Fig. 5.6. $E_z(x,y)$ with different sizes of noise $\lambda_1=\lambda_2=0,2^1,2^3,2^5$ in the time $T=1$.

6. Conclusion

In this paper, we study the strong convergence analysis of the parareal algorithms for stochastic Maxwell equations with damping term driven by additive noise. Firstly the stochastic exponential scheme is chosen as the coarse propagator and the exact solution scheme is chosen as the fine propagator. And we propose our numerical schemes and establish the mean-square convergence estimate. Secondly, both the coarse propagator and the fine propagator choose the stochastic exponential scheme. Meanwhile, the error we considered in this section is the distance between the solution computed by the parareal algorithm and the reference solution generated by the fine propagator. It is shown that the convergence order of the proposed algorithms is linearly related to the iteration number k. At last, One- and two-dimensional numerical examples are performed to demonstrate convergence analysis with respect to damping coefficient and noise scale. One key idea from the proofs of two convergence results is that the residual operator in Theorem 4.2 is related to Lipschitz continuity properties, whereas Theorem 4.1 concerns the integrability of the exact solution. The future works will include the study for the parareal algorithms for the stochastic Maxwell equations driven by multiplicative noise and other choices of integrators as the coarse and fine propagators.

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