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Multidimensional simple wave and Monge-Ampère equation: theory and applications

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Abstract. This paper explores the properties and mathematical formulations of multidimensional simple waves, extending the well-established theory of one-dimensional simple waves to higher dimensions. The study focuses on the connection between simple waves and the Monge-Ampère equation, particularly in the context of gas dynamics and potential flows. Key aspects include the characterization of simple waves in unsteady and steady flows, the role of characteristic lines, and the application of Hodograph and Legendre transformations to derive solutions. The paper also addresses the challenges and open questions in extending simple wave theory to more complex systems, such as non-reducible systems, radiative heat transfer, and chemical reactions. The research highlights both theoretical advancements and practical applications, providing a foundation for future studies in this area.

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Key words: Simple wave; Monge-Ampère equation; Gas dynamics; Characteristic lines; Hodograph transformation.

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1 Introduction

Simple waves, as a fundamental model of wave phenomena, have significant applications across multiple fields. In physics, they are used to study classical wave behaviors such as sound and light waves [1], providing a theoretical foundation for understanding wave propagation, reflection, and interference. In engineering, simple wave models are widely applied in seismic wave analysis, underwater sonar detection, and structural vibration control, helping optimize technical solutions and improve system stability [2]. Additionally, fields like meteorology (e.g., atmospheric wave studies) and medical imaging (e.g., ultrasound technology) rely on simple wave principles [3]. The importance of studying simple waves lies in their role as a starting point for analyzing more complex wave phenomena. By using simplified mathematical models, they reveal essential wave characteristics, offering key theoretical support for interdisciplinary technological advancements. A deep understanding of simple waves not only advances fundamental science but also provides practical tools for solving real-world engineering challenges.

The foundational theories developed for one-dimensional simple waves provide a crucial framework for exploring more complex scenarios [4], including multidimensional flows and their connections to nonlinear partial differential equations such as the Monge-Ampère equation. We begin by revisiting the basic properties of one-dimensional simple waves, which serve as the building blocks for extending these concepts to higher dimensions.

1.1 Basic properties of one-dimensional simple waves

In one-dimensional unsteady isentropic flow or planar unsteady flow, for a reducible system of two unknown functions u,v with two variables x and y the system is given by:

$$A_1\frac{\partial u}{\partial x}+B_1\frac{\partial u}{\partial y}+C_1\frac{\partial v}{\partial x}+D_1\frac{\partial v}{\partial y}=0, \tag{1.1}$$

$$A_2\frac{\partial u}{\partial x}+B_2\frac{\partial u}{\partial y}+C_2\frac{\partial v}{\partial x}+D_2\frac{\partial v}{\partial y}=0, \tag{1.2}$$

where A_i,B_i,C_i,D_i are functions of u,v. For one-dimensional unsteady flow, simply replace x with t, y with x. If $J=\frac{\partial(u,v)}{\partial(x,y)}=0$, that is u,v are related. Functions $u=u(x,y)$, $v=v(x,y)$ map a region $(x,y)\in G$ to a family of curves. The solutions or flows of such equations are known as simple waves. These waves arise in many practical mechanical models, including planar detonations, two-dimensional steady flows, and one-dimensional isentropic flows, among others. This property has an intuitive physical interpretation.