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KAM and Geodesic Dynamics of Blackholes

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Abstract. In this paper we apply KAM theory and the Aubry-Mather theory for twist maps to the study of bound geodesic dynamics of a perturbed blackhole background. The general theories apply mainly to two observable phenomena: the photon shell (unstable bound spherical orbits) and the quasi-periodic oscillations (QPO). We prove that there is a gap structure in the photon shell that can be used to reveal information of the perturbation.

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Key words: KAM, photon shell, blackholes, Kerr.

1 Introduction

This paper is a companion paper of [31]. The main purpose is to study the stability of the geodesic dynamics of a blackhole background under perturbations. Schwarzschild and Kerr are among the important metrics. In this paper, we take the two metrics as examples to show how perturbative theory in Hamiltonian dynamics applies to yield interesting phenomena. Similar analysis also applies to many other solutions in general relativity. The geodesic dynamics of particles in either background is integrable and well-understood. However, when the metrics are perturbed slightly, the integrability is generically broken and chaotic motions occur. In reality, the blackholes modeled on either Schwarzschild and Kerr always undergo some perturbations, which makes a perturbative analysis necessary.

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The mathematical model that we use to study the geodesic dynamics of the blackholes is the following map called integrable twist map:

$$\phi_0\colon\ \mathbb{T}^1\times[0,1]\to\mathbb{T}^1\times[0,1],\quad (\theta,I)\mapsto(\theta+\nu(I),I), \tag{1.1}$$

where ν is assumed to be smooth and $\nu'(I)>c>0$ for some constant c and all $I\in[0,1]$. The dynamics of this example is as follows. Each I-circle is invariant and the dynamics on it is a rotation by $\nu(I)$. When $\nu(I)$ is rational, then each orbit on the I-circle is periodic, and when $\nu(I)$ is irrational, then each orbit is dense on the I-circle. When ϕ_0 is slightly perturbed within the class of smooth symplectic maps, we have the following picture for the dynamics. First, Moser's version of Kolmogorov-Arnold-Moser (KAM) theory implies that each I-circle with $\nu(I)$ a Diophantine number (see Definition A.1) is perturbed to a nearby smooth loop, called invariant circle, that remains invariant under the perturbed map and the dynamics on it can be conjugate to the original unperturbed rotation, provided the perturbation is sufficiently small in a smooth enough topology. For a fixed small perturbation of size ε, the set of remaining invariant circles in Moser's theorem has rotation numbers lying in a Cantor set (a nowhere dense closed set) of measure $1-\mathcal{O}(\sqrt{\varepsilon})$. Between two nearby invariant circles (not necessarily KAM curve), there is a gap region called Birkhoff instability region where the dynamics is very chaotic. There is an Aubry-Mather theory developed for general twist maps which gives the existence of some special orbits in the Birkhoff instability region, such as periodic orbits, heteroclinic orbits corresponding to rational rotation numbers and Cantor set like orbit corresponding to irrational rotation numbers. So Moser's theorem allows us to find a large measure set of the phase space where the motion is regular (quasiperiodic), while Aubry-Mather theory allows us to find some special orbits in the gaps. We give an outline of the two theories in Section 2.2 and more details in Appendix.

We first locate the part of phase space with bounded motions. For massless particles moving on null geodesics in Schwarzschild or Kerr background, these bounded motions are called bound photon orbits, or fundamental photon orbits in literature. Each such orbit is moving on a sphere with a fixed radius and is unstable under radial perturbations. This is an observable feature of the blackhole, which lies on the edges of the blackhole shadows [22,25]. If we also consider the motion of massive particles on timelike geodesics in Schwarzschild or Kerr background, again we have unstable bound spherical orbits, similar to the lightlike case. For simplicity, we use the term *photon shell* to call the set of bound spherical orbits that are unstable under radial perturbations, for both Schwarzschild and Kerr, and both lightlike or timelike geodesics. We show in this paper that both Moser's theorem and the twist map theory apply to the study of the dynamics on the photon shell in a perturbed