Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **18**, No. 1, pp. 189-221

DOI: 10.4208/aamm.OA-2024-0123
February 2026

An Inverse Problem with the Final Overdetermination for the Mean Field Games System

Michael V. Klibanov[1], Jingzhi Li[2,*] and Hongyu Liu[3,*]

[1] *Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, NC 28223, USA*

[2] *Department of Mathematics & National Center for Applied Mathematics Shenzhen & SUSTech International Center for Mathematics, Southern University of Science and Technology, Shenzhen, Guangdong 518055, China*

[3] *Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong, China*

Received 7 June 2024; Accepted (in revised version) 25 July 2024

Abstract. The mean field games (MFG) theory has broad application in mathematical modeling of social phenomena. The Mean Field Games System (MFGS) is the key to the MFG theory. This is a system of two nonlinear parabolic partial differential equations with two opposite directions of time $t\in(0,T)$. The topic of Coefficient Inverse Problem (CIPs) for the MFGS is a newly emerging one. A CIP for the MFGS is studied. The input data are Dirichlet and Neumann boundary conditions either on a part of the lateral boundary (incomplete data) or on the whole lateral boundary (complete data). In addition to the initial conditions at $\{t=0\}$, terminal conditions at $\{t=T\}$ are given. The terminal conditions mean the final overdetermination. The necessity of assigning all these input data is explained. Hölder and Lipschitz stability estimates are obtained for the cases of incomplete and complete data respectively. These estimates imply uniqueness of the CIP.

AMS subject classifications: 35R30

Key words: The mean field games system, new Carleman estimates, Hölder and Lipschitz stability estimates, uniqueness.

1 Introduction

Social sciences play a significant role in the modern society. The Mean Field Games (MFG) theory studies the collective behavior of large populations of rational decision-makers (agents). This theory has a number of applications in the mathematical modeling

*Corresponding author.
Emails: mklibanv@uncc.edu (M. Klibanov), li.jz@sustech.edu.cn (J. Li), hongyliu@cityu.edu.hk (H. Liu)

of social phenomena. Some examples of these applications are, e.g., finance [1,40], sociology [2], fighting corruption [23,24], cyber security [24], etc.. Due to numerous applications of the MFG theory, it is important to study a variety of mathematical topics of this theory. In the current paper, we study a Coefficient Inverse Problem (CIP) for the Mean Field Games System (MFGS).

The MFG theory was first introduced in 2006-2007 in seminal works of Lasry and Lions [26–28] and of Huang, Caines and Malhamé [9,10]. The MFGS is the core of the MFG theory. This is a system of two coupled nonlinear parabolic Partial Differential Equations (PDEs) with two opposite directions of time. In the first equation time is running downwards. This is the Hamilton-Jacobi-Bellman equation (HJB). And in the second equation time is running upwards. This is the Fokker-Planck (FP) equation. Let $\Omega \subset \mathbb{R}^n, n \geq 1$ be a bounded domain with its boundary $\partial\Omega$ and let time $t \in (0,T)$. The state position of a representative agent is $x \in \Omega$. The HJB equation governs the value function $u(x,t)$ of each individual agent located at x at the moment of time t. The FP equation describes the evolution of the distribution of agents $m(x,t)$ over time $t \in (0,T)$.

CIPs for the MFGS is a newly emerging topic. Uniqueness theorems for CIPs for the MFGS in the case of the input data resulting from infinitely many measurements were obtained in [**?**,33,35–37]. We refer to [5,6] for numerical studies of CIPs for the MFGS. We consider the case of the input data resulting from a single measurement event [20,21, 34]. We also refer to [7,30,32,42] for some related studies. Previously a Hölder stability estimate was obtained in [21] for a CIP for the MFGS with a single measurement data. However, the statement of the CIP in [21] is significantly different from the one of this paper, see Subsection 3.1. In addition, Lipschitz stability estimate for a CIP with single measurement data was obtained in [12]. The MFGS of [12] does not contain the integral term, see Subsection 2.2 for this term.

Since the input data for CIPs are results of measurements, then they are given with errors. Hence, we are concerned with obtaining Hölder and Lipschitz stability estimates of the solution of our CIP with respect to the error in the input data. These stability estimates immediately imply uniqueness of our CIP.

In this paper we modify the framework, which was first proposed in [4], where the apparatus of Carleman estimates was introduced in the field of CIPs, also, see, e.g., [11, 13–17,41] and references cited therein for some follow up publications. Carleman estimates were introduced in the MFG theory in [18] and were used since then in [12,19,21] as well as in the current paper. The idea of our modification of the framework of [4] is outlined in Subsection 3.2.

It is natural to call the problem of this paper "CIP with the final overdetermination". Indeed, we assume that we know both initial and terminal conditions for both functions u and m as well as both Dirichlet and Neumann boundary conditions for these functions on either a part of the lateral boundary or on the whole lateral boundary. On the other hand, if a CIP for a single parabolic equation requires to find a coefficient of this equation, assuming that its solution is known at $\{t=0\}$ and at $\{t=T\}$, then such a CIP is called "CIP with the final overdetermination": we refer to [13, Section 9.1] for the Lipschitz

stability result for such a problem for the case of a single parabolic equation. However, the case of the MFGS with the final overdetermination was not considered previously. In addition, the technique of this paper is significantly different from the one of [13]. A CIP for one parabolic equation with the final overdetermination and the lateral boundary data was recently considered in [22]. A more detailed discussion of the statement of our CIP can be found in Subsections 3.1 and 3.2.

Remark 1.1. Traditionally minimal smoothness assumptions are of a secondary concern in the field of inverse problems, see, e.g., [38] and [39, Theorem 4.1]. Therefore, we are not concerned with these assumptions in the current paper.

All functions below are real valued ones. In Section 2, we first formulate the MFGS and outline four main difficulties of working with this system. Then we formulate our CIP. In Section 3, we first discuss our input data for our CIP. Next, we outline our idea of the above mentioned modification of the framework of [4]. We formulate our theorems in Section 4. Two theorems of this section about Hölder and Lipschitz stability are proven in Section 5. On the other hand, two theorems of Section 4 about Carleman estimates are proven in Appendix.

2 Statement of the coefficient inverse problem

2.1 Domains and spaces

First, we introduce some basic notations we use in this paper. For $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n$ denote $\overline{x}=(x_2,\cdots,x_n)$. To simplify the presentation, we consider only the case when our domain of interest $\Omega\subset\mathbb{R}^n$ is a rectangular prism. Let $A_i>0$, $i=1,\cdots,n$ and $T>0$ be some numbers. Also, let $\gamma\in(0,2A_1)$ be another number. Everywhere below domains and their boundaries are defined as:

$$\Omega=\{x:-A_i<x_i<A_i,\ i=1,\cdots,n\},\quad \Omega'=\{\overline{x}:-A_i<x_i<A_i,\ i=2,\cdots,n\}, \tag{2.1a}$$

$$\Gamma^-=\{x\in\partial\Omega:x_1=-A_1\},\quad \Gamma^+=\{x\in\partial\Omega:x_1=A_1\},\quad \Gamma_T^\pm=\Gamma^\pm\times(0,T), \tag{2.1b}$$

$$\partial_i^\pm\Omega=\{x\in\partial\Omega:x_i=\pm A_i\},\quad \partial_i^\pm\Omega_T=\partial_i^\pm\Omega\times(0,T),\quad i=2,\cdots,n, \tag{2.1c}$$

$$Q_T=\Omega\times(0,T),\quad S_T=\partial\Omega\times(0,T), \tag{2.1d}$$

$$\Omega_\gamma=\{x\in\Omega:x_1\in(-A_1+\gamma,A_1)\},\quad Q_{\gamma T}=\Omega_\gamma\times(0,T). \tag{2.1e}$$

We now introduce some spaces we will work with. Let $k\geq 1$ be an integers. Denote

$$C^{2k,k}(\overline{Q}_T)=\left\{u:\|u\|_{C^{2k,k}(\overline{Q}_T)}=\max_{\|\alpha\|+2m\leq 2k}\|D_x^\alpha\partial_t^m u\|_{C(\overline{Q}_T)}<\infty\right\}, \tag{2.2a}$$

$$H^{2k,k}(Q_T)=\left\{u:\|u\|^2_{H^{4,2}(Q_T)}=\sum_{\|\alpha\|+2m\leq 2k}\|D_x^\alpha\partial_t^m u\|^2_{L_2(Q_T)}<\infty\right\}, \tag{2.2b}$$

$$H^{2,1}(\partial_i^{\pm}\Omega_T)=\Bigg\{u:\|u\|^2_{H^{2,1}(\partial_i^{\pm}\Omega_T)}=\sum_{j=1,j\neq i}^{n}\|u_{x_j}\|^2_{L_2(\partial_i^{\pm}\Omega_T)}+\sum_{j,s=1,(j,s)\neq(i,i)}^{n}\|u_{x_jx_s}\|^2_{L_2(\partial_i^{\pm}\Omega_T)}$$
$$+\sum_{j=0}^{1}\|\partial_t^j u\|^2_{L_2(\partial_i^{\pm}\Omega_T)}<\infty\Bigg\}, \tag{2.2c}$$

$$H^{2,1}(S_T)=\Bigg\{u:\|u\|^2_{H^{2,1}(S_T)}=\sum_{i=1}^{n}\|u\|^2_{H^{2,1}(\partial_i^{\pm}\Omega_T)}<\infty\Bigg\}, \tag{2.2d}$$

$$H^{1,0}(\partial_i^{\pm}\Omega_T)=\Bigg\{u:\|u\|^2_{H^{1,0}(\partial_i^{\pm}\Omega_T)}=\sum_{j=1,j\neq i}^{n}\|u_{x_j}\|^2_{L_2(\partial_i^{\pm}\Omega_T)}+\|u\|^2_{L_2(\partial_i^{\pm}\Omega_T)}<\infty\Bigg\}, \tag{2.2e}$$

$$H^{1,0}(S_T)=\Bigg\{u:\|u\|^2_{H^{1,0}(S_T)}=\sum_{i=1}^{n}\|u\|^2_{H^{1,0}(\partial_i^{\pm}\Omega_T)}<\infty\Bigg\}. \tag{2.2f}$$

Spaces $H^{2,1}(\Gamma_T^{\pm})$ and $H^{1,0}(\Gamma_T^{\pm})$ are defined similarly with spaces $H^{2,1}(\partial_i^{\pm}\Omega_T)$ and $H^{1,0}(\partial_i^{\pm}\Omega_T)$ respectively.

2.2 The mean field games system

We consider a slightly simplified form of the MFGS of the second order [1,28]:

$$u_t(x,t)+\Delta u(x,t)-a(x)(\nabla u(x,t))^2/2+\int_{\Omega}K(x,y)m(y,t)dy$$
$$+s(x,t)m(x,t)=0, \qquad (x,t)\in Q_T, \tag{2.3a}$$
$$m_t(x,t)-\Delta m(x,t)-\mathrm{div}(a(x)m(x,t)\nabla u(x,t))=0, \qquad (x,t)\in Q_T, \tag{2.3b}$$

where $\nabla u=(u_{x_1},\cdots,u_{x_n})$, and conditions on functions $K(x,y)$, $a(x)$ and $s(x,t)$ are imposed later.

The term with the integral in (2.3) is called "global interaction term". This term has a deep applied meaning, which is explained in [18, page 634]. More precisely, $K(x,y)$ is the action on the agent occupying the point x by the agent occupying the point y. Hence, the integral term in (2.3) expresses the average action of all agents located at all points $y\in\Omega$ on the agent located at the point x.

2.3 Four main difficulties of working with MFGS (2.3)

We now outline four main difficulties of working with MFGS (2.3):

1. MFGS (2.3) is highly nonlinear. On the top of this is that any CIP for a PDE is nonlinear as well.

2. Two equations of system (2.3) have two opposite directions of time. Therefore, the classical theory of parabolic equations [25] does not work here.

3. The presence of the integral term in the first equation (2.3). Such terms are not present in all past works on CIPs for parabolic PDEs.

4. The presence of the Laplace operator Δu in the second equation (2.3), since this operator is involved in the principal part $(\partial_t+\Delta)u$ of the first equation (2.3).

Due to Items 1-4, the past theory of CIPs for a single parabolic PDE cannot be automatically applied to CIPs for MFGS (2.3). Rather, a significant additional effort is required for the latter.

2.4 The kernel $K(x,y)$ in (2.3)

Let $M>0$ be a number, $\delta(z)$, $z\in\mathbb{R}$ be the delta-function and $H(z)$ be the Heaviside function,

$$H(z)=\begin{cases} 1, & \text{if } z>0,\\ 0, & \text{if } z<0.\end{cases}$$

We cannot work with a general function $K(x,y)$. Hence, we assume below that the integral term in the first equation (2.3) has the following form:

$$\begin{aligned} &K(x,y)=b(x)\{\delta(x_1-y_1)K_1(\overline{x},\overline{y})+H(y_1-x_1)K_2(x,y)\}, \quad \text{with}\\ &\quad K_1\in C^4(\overline{\Omega'}\times\overline{\Omega'}), \quad \|K_1\|_{C^4(\overline{\Omega'}\times\overline{\Omega'})}<M, \quad K_2\in C^4(\overline{\Omega}\times\overline{\Omega}), \quad \|K_2\|_{C^4(\overline{\Omega}\times\overline{\Omega})}<M. \end{aligned} \tag{2.4}$$

Hence,

$$\begin{aligned} \int_\Omega K(x,y)m(y,t)dy=b(x)\Bigg[&\int_{\Omega'}K_1(\overline{x},\overline{y})m(x_1,\overline{y},t)d\overline{y}\\ &+\int_{x_1}^{A_1}\Big(\int_{\Omega'}K_2(x,y_1,\overline{y})m(y_1,\overline{y},t)d\overline{y}\Big)dy_1\Bigg]. \end{aligned} \tag{2.5}$$

Conditions imposed on the function $b(x)$ are specified later. A popular example of $K(x,y)$ is [31, Section 4.2]:

$$K(x,y)=b(x)\frac{1}{(2\pi)^n}\prod_{i=1}^n\frac{1}{\sigma_i}\exp\Big(-\frac{(x_i-y_i)^2}{2\sigma_i^2}\Big).$$

We recall that Gaussians approximate the $\delta-$ function in the sense of distributions, which justifies our choice of $\delta(x_1-y_1)K_1(\overline{x},\overline{y})$ in (2.4).

2.5 Coefficient inverse problem

It is hard to find a specific form of the kernel $K(x,y)$, see, e.g., [31, Section 4]. Hence, the recovery of at least a part of this kernel is of a significant interest, and this is what we do in the current paper. More precisely, we are interested in this paper in the recovery of

the coefficient $b(x)$ in (2.4). Following Remark 1.1, we are not concerned here with some extra smoothness conditions we impose below.

Coefficient Inverse Problem (CIP). Assume that functions $u,m\in C^{6,3}(\overline{Q}_T)$ satisfy equations (2.3), and let condition (2.4) holds. Let

$$u(x,0)=p(x),\quad m(x,0)=q(x),\qquad x\in\Omega, \tag{2.6a}$$
$$u(x,T)=F(x),\quad m(x,T)=G(x),\quad x\in\Omega, \tag{2.6b}$$
$$u|_{S_T}=f_0(x,t),\quad \partial_n u|_{S_T}=f_1(x,t), \tag{2.6c}$$
$$m|_{S_T}=g_0(x,t),\quad \partial_n u|_{S_T}=g_1(x,t). \tag{2.6d}$$

Determine the coefficient $b(x)$, assuming that the functions in the right hand sides of (2.6) are known.

Thus, the functions in the right hand sides of first two lines of (2.6) are initial and terminal conditions. The right hand sides of the third and fourth lines of (2.6) are Dirichlet and Neumann boundary data, which are also called "lateral Cauchy data". We will consider the following two cases of the lateral Cauchy data:

1. Incomplete lateral Cauchy data. This is the case when in (2.6)

$$\text{functions } f_0,\, f_1,\, g_0,\, g_1 \text{ are known at } S_T\backslash\Gamma_T^- \text{ and unknown at } \Gamma_T^-. \tag{2.7}$$

   We obtain a Hölder stability estimate in this case.

2. Complete lateral Cauchy data. This is the case when in (2.6)

$$\text{functions } f_0,\, f_1,\, g_0,\, g_1 \text{ are known at the whole boundary } S_T. \tag{2.8}$$

   We obtain Lipschitz stability estimate in this case.

The input data (2.6) are generated by a single measurement event. As to these data, in the conventional case of the MFG theory, only functions $u(x,T)$ and $m(x,0)$ are given [1] as well as either Neumann or Dirichlet boundary condition for each of functions u,m. In the case of a practical mean field game process, other functions in (2.6) can be obtained via, e.g., polling of game participants at $t=0,T$ as well as at the boundary $\partial\Omega$, see, e.g., [5, p. 2].

3 Discussion

In this section we explain first why do we need the input data (2.6). Next, we briefly outline our idea of a modification of the framework of [4] in order to make it applicable to our CIP.

3.1 Discussion of the input data (2.6)

Let $t_0 \in (0,T)$ be a fixed moment of time. A Hölder stability estimate was obtained in [21] for a CIP for MFGS (2.3) in the case when the coefficient $a(x)$ is unknown, initial and terminal conditions in (2.6)

$$u(x,0),\ u(x,T),\ m(x,0),\ m(x,T) \tag{3.1}$$

are replaced with the assumption of the knowledge of functions $u(x,t_0)$ and $m(x,t_0)$, and also lateral Cauchy data in (2.6) are known in [21].

In the case of a single parabolic equation, uniqueness and stability results for CIPs with $x-$ dependent unknown coefficients were obtained only under the assumption that the solution of that equation is known at $t=t_0 \in (0,T)$ and the lateral Cauchy data are known as well, see, e.g., [11, 14–17, 41]. A similar statement is true for CIPs for MFGS (2.3) [12, 21]. These results were obtained using the framework of [4].

If, however, only the initial condition at $\{t=0\}$ and lateral Cauchy data are known, then a methodology of obtaining stability results for such CIPs does not exist yet even for the case of a single parabolic PDE. This explains our need of the knowledge in (2.6) of all four initial and terminal conditions (3.1) as well as the lateral Cauchy data.

On the other hand, if we would assume only the knowledge of functions in (3.1) as well as of only either Dirichlet or Neumann boundary condition for each of functions u,m, then we would not be able to consider the case of incomplete data (2.7). And even in the case of complete data we would likely need to impose some yet unknown additional conditions on operators in (2.3). For example, a similar CIP for a single parabolic equation with the data at $\{t=0\}$, $\{t=T\}$ and the Dirichlet boundary condition at the entire boundary is considereed in [13, Section 9.1]. An an additional condition in [13] was that the Dirichlet boundary value problem for the corresponding elliptic operator has at most one solution.

3.2 Our modification of the framework of [4]

The first step the framework of [4] transforms the considered CIP in an integral-differential equation, which does not contain the unknown coefficient. Integral terms in this equation are $t-$dependent Volterra integrals. Next, the application of a Carleman estimate to that equation leads to the desired result. This scheme works in the case of CIPs for hyperbolic and elliptic PDEs in the cases when the lateral Cauchy data are given, in addition to some initial conditions, see, e.g., [3, 14–16, 41] and [17, Chapter 3]. And in the case of CIPs for parabolic equations, this scheme works only if one replaces the initial data at $\{t=0\}$ with the data at $\{t=t_0 \in (0,T)\}$, see Subsection 3.1.

However, this framework does not work for our case when both initial data at $\{t=0\}$ and terminal data at $\{t=T\}$ are given in (2.6). More precisely, the straightforward application of the framework of [4] to our CIP leads to the presence of some parasitic integrals

over $\{t=0\}$ and $\{t=T\}$. These integrals appear when integrating the pointwise Carleman estimate over the time cylinder Q_T and applying the Gauss formula. The presence of these integrals does not allow us to obtain our desired Hölder and Lipschitz stability estimates and uniqueness theorem for our CIP.

Hence, we modify here the idea of [4]. More precisely, we arrange the above transformation in such a way that those parasitic integrals cancel each other. After our transformation, each of two transformed functions obtained from functions u and m in (2.3) attains the same values at $\{t=0\}$ and at $\{t=T\}$. Next, we apply to the resulting transformed system of integral differential equations two new Carleman estimates for operators $\partial_t+\Delta$ and $\partial_t-\Delta$. The new point of these estimates is that the Carleman Weight Function (CWF) in them is independent on t. Our CWF depends only on x_1: due to (2.4) and (2.5). On the other hand, in conventional Carleman estimates for parabolic equations with the lateral Cauchy data, CWFs always depend on both x and t, see, e.g., [17, Section 2.3], [29, Section 1 of Chapter 4] and [41].

4 Formulations of theorems

4.1 Carleman estimates

It is sufficient to prove Carleman estimates only for principal parts of Partial Differential Operators [17, Lemma 2.1.1]. There are two methods of proofs of Carleman estimates. The first method is presented in books [8, Sections 8.3 and 8.4], [13, Theorem 3.2.1], and it is based on symbols of Partial Differential Operators. This method is both elegant and short. However, it is based on the assumption of zero boundary conditions of involved functions. On the other hand, we work here with the non-zero boundary conditions, which play an important role in stability estimates for our CIP.

Therefore, primeraly due to our need to arrange the mutual cancellation of parasitic integrals over $\{t=0\}$ and $\{t=T\}$ (see Subsection 3.2), we need a painstaking analysis of boundary terms in our Carleman estimates. Thus, we use the second method. By this method, one first derives a pointwise Carleman estimate. Next, one integrates this estimate over the domain of interest. Boundary integrals occur due to the Gauss formula. In addition to our analysis of resulting integrals over $\{t=0\}$ and $\{t=T\}$, our derivation also allows us to analyze resulting boundary integrals over the lateral boundary S_T, which is important for our target stability estimates of $H^{2,1}(Q_{\gamma T})$ and $H^{2,1}(Q_T)$ norms of involved functions.

The derivation of any pointwise Carleman estimate is inevitably space consuming, see, e.g., [17, Section 2.3], [29, Section 1 of Chapter 4] and [41]. However, this is the price we pay for the incorporation of non-zero boundary conditions.

We remind that due to (2.4) and (2.5), our CWF depends only on the variable x_1. On the other hand, as stated in Subsection 3.2, it is critical for our CIP that CWF should be independent on t, which is unusual in Carleman estimates for parabolic operators.

Let $\nu>2$ and $\lambda>1$ be some large parameters, which we will choose later. Consider two functions ψ and $\varphi_{\lambda,\nu}$, where $\varphi_{\lambda,\nu}(x)$ is the CWF we work with. Thus,

$$\psi(x)=x_1+A_1+2, \quad \varphi_{\lambda,\nu}(x)=e^{2\lambda\psi^{\nu}}, \tag{4.1a}$$

$$\exp(2\lambda\cdot 2^{\nu})\le\varphi_{\lambda,\nu}(x)\le\exp[2\lambda(2A_1+2)^{\nu}] \quad \text{in } \Omega. \tag{4.1b}$$

Theorem 4.1 (Pointwise Carleman estimate for the operator $\partial_t-\Delta$)**.** *There exist sufficiently large numbers $\nu_0=\nu_0(A_1)>2$, $\lambda_0=\lambda_0(A_1)>1$ and a number $C=C(Q_T)>0$ depending only on the domain Ω such that for $\nu=\nu_0$, for all $\lambda\ge\lambda_0$ and for all functions $u\in C^{4,2}(\overline{Q}_T)$ the following pointwise Carleman estimate holds:*

$$\begin{aligned}(u_t-\Delta u)^2\varphi_{\lambda,\nu_0}\ge&\left(\frac{C}{\lambda}\right)\left(u_t^2+\sum_{i,j=1}^{n}u_{x_ix_j}^2\right)\varphi_{\lambda,\nu_0}+C\left[\lambda(\nabla u)^2+\lambda^3u^2\right]\varphi_{\lambda,\nu_0}\\&+\partial_tV+divU, \quad (x,t)\in Q_T,\end{aligned} \tag{4.2}$$

where U is an $n-$ D vector function. The function ∂_tV is:

$$\begin{aligned}\partial_tV=&\partial_t\left[(2\lambda/(2\lambda+1))\Big((u_{x_1}+\lambda\nu_0\psi^{\nu_0-1}u)^2+\sum_{i,j=2}^{n}u_{x_i}^2\Big)\psi^{-\nu_0+1}\varphi_{\lambda,\nu_0}\right]\\&+\partial_t\left[(2\lambda/(2\lambda+1))(-\lambda^2\nu_0^2\psi^{\nu_0-1}(1-2\psi^{-\nu_0}(\nu_0-1)/(\lambda\nu_0))u^2\varphi_{\lambda,\nu_0})\right]\\&+\partial_t((\lambda^2/(2\lambda+1))u^2\varphi_{\lambda,\nu_0}+(\nabla u)^2\varphi_{\lambda,\nu_0}/(2\lambda+1)).\end{aligned} \tag{4.3}$$

And the function $divU$ is:

$$\begin{aligned}divU=&\Big[(2\lambda/(2\lambda+1))(-2u_t(u_{x_1}+\lambda\nu\psi^{\nu_0-1}u)\varphi_{\lambda,\nu_0}\psi^{-\nu_0+1})\Big]_{x_1}\\&+\left[(2\lambda/(2\lambda+1))\Big(-2\lambda\nu_0(u_{x_1}+\lambda\nu_0\psi^{\nu_0-1}u)^2\varphi_{\lambda,\nu_0}+2\lambda\nu_0\sum_{i=2}^{n}u_{x_i}^2\varphi_{\lambda,\nu_0}\Big)\right]_{x_1}\\&+\left[(2\lambda/(2\lambda+1))(-2\lambda^3\nu_0^3\psi^{2\nu_0-2}(1-2\psi^{-\nu_0}(\nu_0-1)/(\lambda\nu_0))u^2\varphi_{\lambda,\nu_0})\right]_{x_1}\\&+\sum_{i=2}^{n}\Big[(2\lambda/(2\lambda+1))(-4\lambda\nu_0(u_{x_1}+\lambda\nu_0\psi^{\nu_0-1}u)u_{x_i}\varphi_{\lambda,\nu_0}-2u_tu_{x_i}\varphi_{\lambda,\nu_0}\psi^{-\nu_0+1})\Big]_{x_i}\\&+\Big[(2\lambda/(2\lambda+1))(-\lambda u_{x_1}u\varphi_{\lambda,\nu_0}+\lambda^2\nu_0\psi^{\nu_0-1}u^2\varphi_{\lambda,\nu_0})\Big]_{x_1}\\&+\sum_{i=2}^{n}[(2\lambda/(2\lambda+1))(-\lambda u_{x_i}u\varphi_{\lambda,\nu_0})]_{x_i}+\sum_{i=1}^{n}[(1/(2\lambda+1))(-2u_tu_{x_i}\varphi_{\lambda,\nu_0})]_{x_i}\\&+\sum_{i=2}^{n}[(1/(2\lambda+1))(-2u_{x_1x_i}u_{x_i}\varphi_{\lambda,\nu_0})]_{x_1}+\sum_{i=2}^{n}[(1/(2\lambda+1))(2u_{x_1x_1}u_{x_i}\varphi_{\lambda,\nu_0})]_{x_i}\\&+\sum_{i,j=2}^{n}\Big[(1/(2\lambda+1))(u_{x_jx_j}u_{x_i}\varphi_{\lambda,\nu_0}-u_{x_ix_j}u_{x_j}\varphi_{\lambda,\nu_0})\Big]_{x_i}.\end{aligned} \tag{4.4}$$

In particular, (4.3) leads to the following implications:

$$u(x,0)=u(x,T) \quad \to \quad V(x,0)=V(x,T) \quad \to \quad \int_{Q_T} \partial_t V dxdt=0. \tag{4.5}$$

Below $C=C(Q_T)>0$ denotes different constants depending only on the domain Ω. parameters.

Remark 4.1.

1. Formula (4.3) for the function V implies the key property, which we need: that parasitic integrals over $\{t=0\}$ and $\{t=T\}$, which occur when integrating (4.2) over Q_T, cancel each other, if $u(x,T)=u(x,0)$, and this is reflected in (4.5) and in the last line of (4.6) of Theorem 4.2. The necessity of (4.5) for our goal of obtaining stability estimates for our CIP is explained in Subsection 3.2.

2. Item 1 explains the reason of our need of a painstaking derivation of the precise formula (4.3) for $\partial_t V$ in the proof of Theorem 4.1. The reason of the derivation of precise formula (4.4) for $\mathrm{div} U$ is the necessity of the incorporation of estimates of boundary terms, especially those with u_t and $u_{x_i x_j}$, in the integral Carleman estimate (4.6) of Theorem 4.2.

Theorem 4.2 (Integral Carleman estimate for the operator $\partial_t-\Delta$)). *Let ν_0 and λ_0 be parameters chosen in Theorem* 4.1. *Then the following integral Carleman estimate holds:*

$$\begin{aligned}
&C\exp[3\cdot 2^{\nu_0}\lambda](\|u\|^2_{H^{2,1}(\Gamma_T^-)}+\|u_{x_1}\|^2_{H^{1,0}(\Gamma_T^-)})+C\exp[3\lambda(2A_1+2)^{\nu_0}](\|u\|^2_{H^{2,1}(S_T\setminus\Gamma_T^-)}\\
&\quad+\|\partial_n u\|^2_{H^{1,0}(S_T\setminus\Gamma_T^-)})+\int_{Q_T}(u_t-\Delta u)^2\varphi_{\lambda,\nu_0}dxdt\\
&\geq (C/\lambda)\int_{Q_T}\Big(u_t^2+\sum_{i,j=2}^n u_{ij}^2\Big)\varphi_{\lambda,\nu_0}dxdt+C\int_{Q_T}\Big(\lambda(\nabla u)^2+\lambda^3u^2\Big)\varphi_{\lambda,\nu_0}dxdt,\\
&\quad \forall u\in H^{4,2}(Q_T)\cap\{u:u(x,0)=u(x,T)\},\quad \forall\lambda\geq\lambda_0.
\end{aligned} \tag{4.6}$$

Since we have two parabolic operators in (2.3), whose principal parts are $\partial_t-\Delta$ and $\partial_t+\Delta$, then we need to formulate an analog of Carleman estimate (4.6) for the operator $\partial_t+\Delta$ as well. This is done in Theorem 4.3. We omit the proof of this theorem, since it is quite similar with the proofs of Theorems 4.1 and 4.2. As to the norms involved in (4.6), we refer to (2.2b)-(2.2f).

Theorem 4.3 (Integral Carleman estimates for the operator $\partial_t+\Delta$)**.** *Let ν_0 and λ_0 be two parameters chosen in Theorem* 4.1. *Then the direct analog of the Carleman estimate* (4.6) *holds true when $(u_t-\Delta u)^2$ is replaced with $(u_t+\Delta u)^2$.*

Remark 4.2. We prove Theorems 4.1 and 4.2 in Appendix. However, when carrying out other proofs below, we assume that Theorems 4.1 and 4.2 hold true.

4.2 Hölder and Lipschitz stability estimates

In the theory of Ill-Posed Problems, one often assumes that solution of such a problem belongs to an a priory chosen boundary set. Hence, let $M>0$ be the number of Subsection 2.4. Recalling Remark 1.1 and (2.2a), we introduce the following set of pairs of functions (u,m):

$$Y_1(M)=\Big\{(u,m)\in C^{6,3}(\overline{Q}_T):\|u\|_{C^{6,3}(\overline{Q}_T)},\ \|m\|_{C^{6,3}(\overline{Q}_T)}<M\Big\}. \tag{4.7}$$

Obviously,

$$\|u\|_{H^{6,3}(\overline{Q}_T)},\|m\|_{H^{6,3}(\overline{Q}_T)}\le CM,\quad \forall (u,m)\in Y_1(M). \tag{4.8}$$

We also assume that functions $a(x)$ and $s(x,t)$ in MFGS (2.3) satisfy the following conditions:

$$a\in C^3(\overline{\Omega}),\quad s\in C^{2,1}(\overline{Q}_T),\quad \|a\|_{C^3(\overline{\Omega})}<M,\quad \|s\|_{C^{2,1}(\overline{Q}_T)}<M. \tag{4.9}$$

In addition, let the unknown coefficient

$$b(x)\in Y_2(M)=\Big\{b:b\in C^4(\overline{\Omega}),\ \|b\|_{C^4(\overline{\Omega})}<M\Big\}. \tag{4.10}$$

Theorem 4.4 (Hölder stability for incomplete data, the case (2.7))**.** *Assume that there exists two vector functions $(u_i,m_i,b_i)\in Y_1(M)\times Y_2(M)$, $i=1,2$ satisfying the following analogs of conditions* (2.6)*:*

$$u_i(x,0)=p_i(x),\quad m_i(x,0)=q_i(x),\quad x\in\Omega,\qquad i=1,2, \tag{4.11a}$$
$$u_i(x,T)=F_i(x),\quad m_i(x,T)=G_i(x),\quad x\in\Omega,\qquad i=1,2, \tag{4.11b}$$
$$u_i|_{S_T}=f_{0,i}(x,t),\quad \partial_n u_i|_{S_T}=f_{1,i}(x,t),\qquad i=1,2, \tag{4.11c}$$
$$m_i|_{S_T}=g_{0,i}(x,t),\quad \partial_n m_i|_{S_T}=g_{1,i}(x,t),\qquad i=1,2. \tag{4.11d}$$

Assume that the lateral Cauchy data are incomplete as in the case (2.7)*, i.e., functions f_{0i}, f_{1i} in* (4.11) *are known for $(x,t)\in S_T\backslash\Gamma_T^-$ and are unknown for $(x,t)\in\Gamma_T^-$. Let $\delta\in(0,1)$ be a number characterizing the level of the error in the data* (4.11)*. More precisely, let*

$$\|p_1-p_2\|_{H^4(\Omega)}<\delta,\qquad \|q_1-q_2\|_{H^3(\Omega)}<\delta, \tag{4.12a}$$
$$\|F_1-F_2\|_{H^4(\Omega)}<\delta,\qquad \|G_1-G_2\|_{H^3(\Omega)}<\delta, \tag{4.12b}$$
$$\|\partial_t f_{0,1}-\partial_t f_{0,2}\|_{H^{2,1}(S_T\backslash\Gamma_T^-)}<\delta,\qquad \|\partial_t f_{1,1}-\partial_t f_{1,2}\|_{H^{1,0}(S_T\backslash\Gamma_T^-)}<\delta, \tag{4.12c}$$
$$\|\partial_t g_{0,1}-\partial_t g_{0,2}\|_{H^{2,1}(S_T\backslash\Gamma_T^-)}<\delta,\qquad \|\partial_t g_{1,1}-\partial_t g_{1,2}\|_{H^{1,0}(S_T\backslash\Gamma_T^-)}<\delta. \tag{4.12d}$$

Assume that condition (2.4) *holds. In addition, assume that there exists a number $c>0$ such that*

$$\min_{\overline{Q}_T}\Big\|\int_{\Omega'}K_1(\overline{x},\overline{y})m_2(x_1,\overline{y},t)d\overline{y}+\int_{x_1}^{A_1}\Big(\int_{\Omega'}K_2(x,y_1,\overline{y})m_2(y_1,\overline{y},t)d\overline{y}\Big)dy_1\Big\|\ge c. \tag{4.13}$$

Let $\gamma\in(0,2A_1)$ be the number in (2.1). Then there exist a sufficiently small number $\delta_0=\delta_0(M,c,\gamma,\Omega,T)\in(0,1)$ and a number $B=B(M,c,\gamma,\Omega,T)>0$, both numbers depending only on listed parameters, such that for all $\delta\in(0,\delta_0)$ the following Hölder stability estimates are valid with a certain number $\alpha\in(0,1)$:

$$\|\partial_t^j u_1-\partial_t^j u_2\|_{H^{2,1}(Q_{\gamma T})},\|\partial_t^j m_1-\partial_t^j m_2\|_{H^{2,1}(Q_{\gamma T})}\le B\delta^\alpha,\quad j=0,1, \tag{4.14a}$$

$$\|b_1-b_2\|_{L_2(\Omega_\gamma)}\le B\delta^\alpha. \tag{4.14b}$$

In particular, our CIP with the incomplete data as in (2.1) has at most one solution.

Remark 4.3. Below $B=B(M,c,\gamma,\Omega,T)>0$ and $C_1=C_1(M,c,\Omega,T)>0$ denote different numbers depending only on listed parameters.

Theorem 4.5 (Lipschitz stability for complete data, the case (2.8))**.** *Assume that there exists two vector functions* $(u_i,m_i,b_i)\in Y_1(M)\times Y_2(M)$, $i=1,2$ *satisfying conditions* (4.11)*. Assume that the lateral Cauchy data are complete as in* (2.8)*, i.e., functions* $f_{0i},f_{1i},g_{0i},g_{1i}$ *in* (4.11) *are known for all* $(x,t)\in S_T$*. In addition, let conditions* (2.4) *and* (4.13) *hold. Then the following Lipschitz stability estimates are valid:*

$$\begin{aligned}
&\|\partial_t^j u_1-\partial_t^j u_2\|_{H^{2,1}(Q_T)},\|\partial_t^j m_1-\partial_t^j m_2\|_{H^{2,1}(Q_T)}\\
&\le C_1(\|p_1-p_2\|_{H^4(\Omega)}+\|q_1-q_2\|_{H^4(\Omega)})+C_1(\|F_1-F_2\|_{H^4(\Omega)}+\|G_1-G_2\|_{H^4(\Omega)})\\
&\qquad+C_1(\|\partial_t f_{0,1}-\partial_t f_{0,2}\|_{H^{2,1}(S_{T^-})}+\|\partial_t f_{1,1}-\partial_t f_{1,2}\|_{H^{1,0}(S_T)})\\
&\qquad+C_1(\|\partial_t g_{0,1}-\partial_t g_{0,2}\|_{H^{2,1}(S_{T^-})}+\|\partial_t g_{1,1}-\partial_t g_{1,2}\|_{H^{1,0}(S_T)}),\quad j=0,1,\\
&\|b_1-b_2\|_{L_2(\Omega)}\le\|p_1-p_2\|_{H^4(\Omega)}+\|q_1-q_2\|_{H^4(\Omega)}+\|F_1-F_2\|_{H^4(\Omega)}+\|G_1-G_2\|_{H^4(\Omega)}\\
&\qquad\qquad+\|\partial_t f_{0,1}-\partial_t f_{0,2}\|_{H^{2,1}(S_{T^-})}+\|\partial_t f_{1,1}-\partial_t f_{1,2}\|_{H^{1,0}(S_T)}.
\end{aligned}$$

5 Proofs of Theorems 4.4 and 4.5

5.1 Proof of Theorem 4.4

First, we proceed with the transformation procedure as outlined in Subsection 3.2. Next, we apply to two resulting integral differential equations Carleman estimates of Theorems 4.2 and 4.3.

5.1.1 The transformation procedure

Consider the differences:

$$\widetilde{u}=u_1-u_2,\quad \widetilde{m}=m_1-m_2,\quad \widetilde{b}=b_1-b_2, \tag{5.1a}$$

$$\widetilde{p}=p_1-p_2,\quad \widetilde{q}=q_1-q_2,\quad \widetilde{F}=F_1-F_2,\quad \widetilde{G}=G_1-G_2, \tag{5.1b}$$

$$\widetilde{f}_0=f_{0,1}-f_{0,2},\quad \widetilde{f}_1=f_{1,1}-f_{1,2},\quad \widetilde{g}_0=g_{0,1}-g_{0,2},\quad \widetilde{g}_1=g_{1,1}-g_{1,2}. \tag{5.1c}$$

By (2.7), (4.11) and (5.1)

$$\widetilde{u}(x,0)=\widetilde{p}(x),\quad \widetilde{m}(x,0)=\widetilde{q}(x),\qquad x\in\Omega, \tag{5.2a}$$

$$\widetilde{u}(x,T)=\widetilde{F}(x),\quad \widetilde{m}(x,T)=\widetilde{G}(x),\quad x\in\Omega, \tag{5.2b}$$

$$\widetilde{u}|_{S_T\setminus\Gamma_T^-}=\widetilde{f}_0(x,t),\quad \partial_n\widetilde{u}|_{S_T\setminus\Gamma_T^-}=\widetilde{f}_1(x,t), \tag{5.2c}$$

$$\widetilde{m}|_{S_T\setminus\Gamma_T^-}=\widetilde{g}_0(x,t),\quad \partial_n\widetilde{m}|_{S_T\setminus\Gamma_T^-}=\widetilde{g}_1(x,t). \tag{5.2d}$$

Let y_1, z_1 and y_2, z_2 be two pairs of numbers. Denote $\widetilde{y}=y_1-y_2$, $\widetilde{z}=z_1-z_2$. Then

$$y_1z_1-y_2z_2=\widetilde{y}z_1+\widetilde{z}y_2. \tag{5.3}$$

Subtracting Eqs. (2.3) for (u_2,m_2,b_2) from the same equations for (u_1,m_1,b_1) and using (2.5), the first line of (5.1) as well as (5.3), we obtain two equations (5.4a) and (5.4b),

$$\begin{aligned}
&\widetilde{u}_t(x,t)+\Delta\widetilde{u}(x,t)-a(x)\nabla(u_1+u_2)(\nabla\widetilde{u}(x,t)/2+b_1(x)\int_{\Omega'}K_1(\overline{x},\overline{y})\widetilde{m}(x_1,\overline{y},t)d\overline{y}\\
&\quad+b_1(x)\int_{x_1}^{A_1}\Big(\int_{\Omega'}K_2(x,y_1,\overline{y})\widetilde{m}(y_1,\overline{y},t)d\overline{y}\Big)dy_1\\
&=-\widetilde{b}(x)\int_{\Omega'}K_1(\overline{x},\overline{y})m_2(x_1,\overline{y},t)d\overline{y}\\
&\qquad-\widetilde{b}(x)\int_{x_1}^{A_1}\Big(\int_{\Omega'}K_2(x,y_1,\overline{y})m_2(y_1,\overline{y},t)d\overline{y}\Big)dy_1,\quad (x,t)\in Q_T,
\end{aligned} \tag{5.4a}$$

$$\begin{aligned}
&\widetilde{m}_t(x,t)-\Delta\widetilde{m}(x,t)-\mathrm{div}(a(x)\widetilde{m}(x,t)\nabla u_2(x,t))-\mathrm{div}(a(x)m_1(x,t)\nabla\widetilde{u}(x,t))\\
&=0,\quad (x,t)\in Q_T.
\end{aligned} \tag{5.4b}$$

Divide both sides of Eq. (5.4a) by the function $R(x,t)$,

$$R(x,t)=-\int_{\Omega'}K_1(\overline{x},\overline{y})m_2(x_1,\overline{y},t)d\overline{y}-\int_{x_1}^{A_1}\Big(\int_{\Omega'}K_2(x,y_1,\overline{y})m_2(y_1,\overline{y},t)d\overline{y}\Big)dy_1. \tag{5.5}$$

By (4.13) and (5.5)

$$\frac{1}{\|R(x,t)\|}\geq\frac{1}{c}. \tag{5.6}$$

Denote

$$\overline{u}(x,t)=\frac{\widetilde{u}(x,t)}{R(x,t)}. \tag{5.7}$$

Then Eq. (5.4a) becomes:

$$\begin{aligned}
&\overline{u}_t+\Delta\overline{u}+P\nabla\overline{u}+Q\overline{u}+b_1(x)R^{-1}(x,t)\int_{\Omega'}K_1(\overline{x},\overline{y})\widetilde{m}(x_1,\overline{y},t)d\overline{y}\\
&\quad+b_1(x)R^{-1}(x,t)\int_{x_1}^{A_1}\Big(\int_{\Omega'}K_2(x,y_1,\overline{y})\widetilde{m}(y_1,\overline{y},t)d\overline{y}\Big)dy_1=-\widetilde{b}(x),
\end{aligned} \tag{5.8}$$

where (P,Q) is an $(n+1)-$ dimensional vector function with its $C^{2,1}(\overline{Q}_T)-$ components. Although it is easy to present the explicit formulas for its components, we are not doing this for brevity.

It follows from (5.2), (5.7) and (5.8) that the function $\overline{u}_t(x,t)$ attains the following values at $t=0,T$:

$$\begin{aligned}\overline{u}_t(x,0)=&-\Delta(R^{-1}(x,0)\widetilde{p}(x))-P(x,0)\nabla(R^{-1}(x,0)\widetilde{p}(x))\\&-Q(x,0)\nabla(R^{-1}(x,0)\widetilde{p}(x))-b_1(x)R^{-1}(x,0)\int_{\Omega'}K_1(\overline{x},\overline{y})\widetilde{q}(x_1,\overline{y})d\overline{y}\\&-b_1(x)R^{-1}(x,0)\int_{x_1}^{A_1}\Big(\int_{\Omega'}K_2(x,y_1,\overline{y})\widetilde{q}(y_1,\overline{y})d\overline{y}\Big)dy_1-\widetilde{b}(x)\\=&W_0(x)-\widetilde{b}(x),\end{aligned}\tag{5.9a}$$

$$\begin{aligned}\overline{u}_t(x,T)=&-\Delta(R^{-1}(x,T)\widetilde{F}(x))-P(x,T)\nabla(R^{-1}(x,0)\widetilde{F}(x))\\&-Q(x,T)\nabla(R^{-1}(x,T)\widetilde{F}(x))-b_1(x)R^{-1}(x,T)\int_{\Omega'}K_1(\overline{x},\overline{y})\widetilde{G}(x_1,\overline{y})d\overline{y}\\&-b_1(x)R^{-1}(x,T)\int_{x_1}^{A_1}\Big(\int_{\Omega'}K_2(x,y_1,\overline{y}\Big)\widetilde{G}(y_1,\overline{y})d\overline{y})dy_1-\widetilde{b}(x)\\=&W_T(x)-\widetilde{b}(x).\end{aligned}\tag{5.9b}$$

Next, (4.11), (5.2), (5.4b) and (5.7) imply similar formulas for $\widetilde{m}_t(x,0)$ and $\widetilde{m}_t(x,T)$,

$$\begin{aligned}\widetilde{m}_t(x,0)=&\Delta\widetilde{q}(x)+\mathrm{div}(a(x)\widetilde{q}(x)\nabla p_2(x))-\mathrm{div}(a(x)q_1(x)\nabla(\widetilde{p}(x)/R(x,0))\\=&Z_0(x),\end{aligned}\tag{5.10a}$$

$$\begin{aligned}\widetilde{m}_t(x,T)=&\Delta\widetilde{G}(x)+\mathrm{div}(a(x)\widetilde{G}(x)\nabla F_2(x))-\mathrm{div}(a(x)G_1(x)\nabla(\widetilde{F}(x)/R(x,T))\\=&Z_T(x).\end{aligned}\tag{5.10b}$$

Differentiate both Eqs. (5.4b) and (5.8) with respect to t. Then we obtain equations for the $t-$derivatives $\overline{v}(x,t)$ and $\overline{w}(x,t)$. An important property of these equations is that the function $\widetilde{b}(x)$ is not present in them, since this function is independent on t. Functions $\overline{w}(x,t)$ and $\overline{w}(x,t)$ are:

$$\overline{v}(x,t)=\overline{u}_t(x,t),\quad \overline{w}(x,t)=\widetilde{m}_t(x,t).\tag{5.11}$$

By the first line of (5.2) as well as by (5.7) and (5.11)

$$\overline{u}(x,t)=\int_0^t\overline{v}(x,\tau)d\tau+\frac{\widetilde{p}(x)}{R(x,0)},\quad \widetilde{m}(x,t)=\int_0^t\overline{w}(x,\tau)d\tau+\widetilde{q}(x).\tag{5.12}$$

Substitute (5.12) in equations for functions $\overline{v}(x,t)$ and $\overline{w}(x,t)$. Then introduce new functions

$$v(x,t)=\overline{v}(x,t)-\Big(W_0(x)\frac{t}{T}+W_T(x)\Big(1-\frac{t}{T}\Big)\Big),\tag{5.13a}$$

$$w(x,t)=\overline{w}(x,t)-\Big(Z_0(x)\frac{t}{T}+Z_T(x)\Big(1-\frac{t}{T}\Big)\Big),\tag{5.13b}$$

where functions $W_0(x)$, $W_T(x)$, $Z_0(x)$, $Z_T(x)$ are given in (5.9) and (5.10). It follows from (5.9), (5.10), (5.13a) and (5.13b) that

$$v(x,0)=v(x,T)=-\widetilde{b}(x), \tag{5.14a}$$
$$w(x,0)=w(x,T)=0. \tag{5.14b}$$

This finishes our transformation procedure outlined in Subsection 3.2. Indeed, comparing the last line of (4.6) with (5.14), we see that Carleman estimates of Theorems 4.2 and 4.3 can be applied to functions w and v respectively.

Applications of Theorems 4.2 and 4.3. It is well known that Carleman estimates can work not only with equations but with inequalities as well. Hence, to simplify the presentation, we turn the above mentioned equations for functions v and w in two integral differential inequalities. The first inequality is:

$$\begin{aligned}\|v_t+\Delta v\| \le & C_1(\|\nabla v\|+\|v\|)+C_1\int_{\Omega'}\|w(x_1,\overline{y},t)\|d\overline{y}+C_1\int_{\Omega'}\Big(\int_0^t\|w(x_1,\overline{y},\tau)\|d\tau\Big)d\overline{y}\\ &+C_1\int_{x_1}^{A_1}\Big(\int_{\Omega'}\|w\|(y_1,\overline{y},t)d\overline{y}\Big)dy_1+C_1\int_{x_1}^{A_1}\Big[\int_{\Omega'}\Big(\int_0^t\|w\|(y_1,\overline{y},\tau)d\tau\Big)d\overline{y}\Big]dy_1\\ &+X_1(x,t), \quad (x,t)\in Q_T.\end{aligned} \tag{5.15}$$

The second inequality is:

$$\begin{aligned}\|w_t-\Delta w\| \le & C_1(\|\nabla w\|+\|w\|+\int_0^t(\|\nabla w\|+\|w\|)(x,\tau)d\tau)+C_1\Big(\|\nabla v\|+\int_0^t\|\nabla v\|(x,\tau)d\tau\Big)\\ &+C_1\Big(\|\Delta v\|+\int_0^t\|\Delta v\|(x,\tau)d\tau\Big)+X_2(x,t), \quad (x,t)\in Q_T.\end{aligned} \tag{5.16}$$

Note that the presence of integrals with respect to y in the third line of (5.15) is the indication of the third difficulty of working with MFGS (2.3) as in item 3 of Subsection 2.3. And the presence of the terms with $\|\Delta v\|$ in (5.16) is the indication of the fourth difficulty listed in item 4 of that subsection.

It easily follows from the second and third lines of (2.4), (4.7)-(4.10) and (5.5)-(5.13b) that functions X_1 and X_2 in (5.15) and (5.16) are such that

$$\begin{aligned}X_1,X_2\in L_2(Q_T), \quad & \|X_1\|^2_{L_2(Q_T)}+\|X_2\|^2_{L_2(Q_T)}\\ &\le C_1(\|\widetilde{p}\|^2_{H^4(\Omega)}+\|\widetilde{q}\|^2_{H^4(\Omega)}+\|\widetilde{F}\|^2_{H^4(\Omega)}+\|\widetilde{G}\|^2_{H^4(\Omega)}).\end{aligned} \tag{5.17}$$

As to the lateral Cauchy data for functions v and w, using boundary data in (5.1), (5.2) and the above transformation procedure combined with the considerations, which resulted in estimates (5.17), we obtain

$$v|_{S_T\setminus\Gamma_T^-}=m_0(x,t), \quad \partial_n v|_{S_T\setminus\Gamma_T^-}=m_1(x,t), \tag{5.18a}$$
$$w|_{S_T\setminus\Gamma_T^-}=z_0(x,t), \quad \partial_n w|_{S_T\setminus\Gamma_T^-}=z_1(x,t), \tag{5.18b}$$

where functions in the right hand sides of (5.18) can be estimated as:

$$\begin{aligned}&\|m_0\|^2_{H^{2,1}(S_T\setminus\Gamma_T^-)}+\|z_0\|^2_{H^{2,1}(S_T\setminus\Gamma_T^-)}+\|m_1\|^2_{H^{1,0}(S_T\setminus\Gamma_T^-)}+\|z_1\|^2_{H^{1,0}(S_T\setminus\Gamma_T^-)}\\ \le& C_1(\|\widetilde{f}_0\|^2_{H^{2,1}(S_T\setminus\Gamma_T^-)}+\|\widetilde{g}_0\|^2_{H^{2,1}(S_T\setminus\Gamma_T^-)})++C_1(\|\widetilde{f}_1\|^2_{H^{1,0}(S_T\setminus\Gamma_T^-)}+\|\widetilde{g}_1\|^2_{H^{1,0}(S_T\setminus\Gamma_T^-)})\\ &+C_1(\|\widetilde{p}\|^2_{H^4(\Omega)}+\|\widetilde{q}\|^2_{H^4(\Omega)}+\|\widetilde{F}\|^2_{H^4(\Omega)}+\|\widetilde{G}\|^2_{H^4(\Omega)}).\end{aligned} \tag{5.19}$$

Square both sides of each of inequalities (5.15) and (5.16), multiply by the CWF $\varphi_{\lambda,\nu_0}(x)$ in (4.1a) and integrate over the domain Q_T. Use Cauchy-Schwarz inequality, the right inequality (4.2) and (5.17). Also, note that since the function $\varphi_{\lambda,\nu_0}(x)$ depends only on x_1, then

$$\int_{Q_T}\Big(\int_0^t\|f\|(x,\tau)d\tau\Big)^2\varphi_{\lambda,\nu_0}(x)dxdt\le C_1\int_{Q_T}f^2(x,t)\varphi_{\lambda,\nu_0}(x)dxdt,\qquad \forall f\in L_2(Q_T), \tag{5.20a}$$

$$\int_{Q_T}\Big(\int_{\Omega'}\|f(x_1,\overline{y},t)\|d\overline{y}\Big)^2\varphi_{\lambda,\nu_0}(x)dxdt\le C_1\int_{Q_T}f^2(x,t)\varphi_{\lambda,\nu_0}(x)dxdt,\qquad \forall f\in L_2(Q_T). \tag{5.20b}$$

Hence, we obtain two inequalities. The first inequality is:

$$\begin{aligned}&\int_{Q_T}(v_t+\Delta v)^2\varphi_{\lambda,\nu_0}(x)dxdt\\ \le& C_1\int_{Q_T}((\nabla v)^2+v^2)\varphi_{\lambda,\nu_0}(x)dxdt+C_1\int_{Q_T}w^2\varphi_{\lambda,\nu_0}(x)dxdt\\ &+C_1\int_{Q_T}\Big(\int_{x_1}^{A_1}\int_{\Omega'}w^2(y_1,\overline{y},t)d\overline{y}dy_1\Big)\varphi_{\lambda,\nu_0}(x)dxdt\\ &+C_1\int_{Q_T}\Big[\int_{x_1}^{A_1}\int_{\Omega'}\Big(\int_0^t w^2(y_1,\overline{y},\tau)d\tau\Big)d\overline{y}dy_1\Big]\varphi_{\lambda,\nu_0}(x)dxdt\\ &+C_1(\|\widetilde{p}\|^2_{H^4(Q_T)}+\|\widetilde{q}\|^2_{H^4(Q_T)}+)\exp[2\lambda(2A_1+2)^{\nu_0}]\\ &+C_1(\|\widetilde{F}\|^2_{H^4(Q_T)}+\|\widetilde{G}\|^2_{H^4(Q_T)})\exp[2\lambda(2A_1+2)^{\nu_0}].\end{aligned} \tag{5.21}$$

We now estimate from the above the term in the third line of (5.21). When doing so, we recall that (4.1a) implies that the function $\varphi_{\lambda,\nu_0}(x)\equiv\varphi_{\lambda,\nu_0}(x_1)$ is increasing. Using (2.1), we obtain

$$\begin{aligned}&\int_{Q_T}\Big[\int_{x_1}^{A_1}\int_{\Omega'}\Big(\int_0^t w^2(y_1,\overline{y},\tau)d\tau\Big)d\overline{y}dy_1\Big]\varphi_{\lambda,\nu_0}(x)dxdt\\ \le& T\int_{0|}^T dt\int_{\Omega'}d\overline{x}\int_{\Omega'}d\overline{y}\int_{-A_1}^{A_1}\varphi_{\lambda,\nu_0}(x_1)\Big(\int_{x_1}^{A_1}w^2(y_1,\overline{y},t)dy_1\Big)dx_1\\ =& T\int_{0|}^T dt\int_{\Omega'}d\overline{x}\int_{\Omega'}d\overline{y}\int_{-A_1}^{A_1}\Big(\int_{-A_1}^{y_1}\varphi_{\lambda,\nu_0}(x_1)dx_1\Big)w^2(y_1,\overline{y},t)dy_1\\ \le&\int_{0|}^T dt\int_{\Omega'}d\overline{x}\int_{\Omega'}d\overline{y}\int_{-A_1}^{A_1}w^2(y_1,\overline{y},t)(y_1+A_1)\varphi_{\lambda,\nu_0}(y_1)dy_1\\ \le& C_1\int_{Q_T}w^2\varphi_{\lambda,\nu_0}(x)dxdt.\end{aligned}$$

Hence, (5.21) can be rewritten as:

$$\begin{aligned}
&\int_{Q_T}(v_t+\Delta v)^2\varphi_{\lambda,\nu_0}(x)dxdt\\
\le& C_1\int_{Q_T}((\nabla v)^2+v^2)\varphi_{\lambda,\nu_0}(x)dxdt+C_1\int_{Q_T}w^2\varphi_{\lambda,\nu_0}(x)dxdt\\
&+C_1(\|\widetilde{p}\|^2_{H^4(Q_T)}+\|\widetilde{q}\|^2_{H^4(Q_T)}+)\exp[2\lambda(2A_1+2)^{\nu_0}]\\
&+C_1(\|\widetilde{F}\|^2_{H^4(Q_T)}+\|\widetilde{G}\|^2_{H^4(Q_T)})\exp[2\lambda(2A_1+2)^{\nu_0}].
\end{aligned}\tag{5.22}$$

The second above mentioned second inequality is generated by (5.16) and the first estimate (5.20). This inequality is:

$$\begin{aligned}
&\int_{Q_T}(w_t-\Delta w)^2\varphi_{\lambda,\nu_0}(x)dxdt\\
\le& C_1\int_{Q_T}((\nabla w)^2+w^2)\varphi_{\lambda,\nu_0}(x)dxdt++C_1\int_{Q_T}((\nabla v)^2+v^2)\varphi_{\lambda,\nu_0}(x)dxdt\\
&+C_1\int_{Q_T}(\Delta v)^2\varphi_{\lambda,\nu_0}(x)dxdt+C_1(\|\widetilde{p}\|^2_{H^4(\Omega)}+\|\widetilde{q}\|^2_{H^4(\Omega)})\exp[2\lambda(2A_1+2)^{\nu_0}]\\
&+C_1(\|\widetilde{F}\|^2_{H^4(\Omega)}+\|\widetilde{G}\|^2_{H^4(\Omega)})\exp[2\lambda(2A_1+2)^{\nu_0}].
\end{aligned}\tag{5.23}$$

It follows from the last line of (4.6) and (5.14) that we can apply Carleman estimates of Theorems 4.3 and 4.2 to the left hand sides of (5.22) and (5.23) respectively. Let $\lambda_0>1$ be the parameter of Theorems 4.1-4.3. Hence, using (4.6), (5.18) and (5.19), we again obtain two estimates for all $\lambda\ge\lambda_0$. The first estimate is:

$$\begin{aligned}
&(1/\lambda)\int_{Q_T}\Big(v_t^2+\sum_{i,j=2}^n v_{ij}^2\Big)\varphi_{\lambda,\nu_0}dxdt+\int_{Q_T}(\lambda(\nabla v)^2+\lambda^3v^2)\varphi_{\lambda,\nu_0}dxdt\\
\le& C_1\int_{Q_T}((\nabla v)^2+v^2)\varphi_{\lambda,\nu_0}(x)dxdt+C_1\int_{Q_T}w^2\varphi_{\lambda,\nu_0}(x)dxdt+D,
\end{aligned}\tag{5.24}$$

where

$$\begin{aligned}
D=&C_1(\|\widetilde{p}\|^2_{H^4(\Omega)}+\|\widetilde{q}\|^2_{H^4(\Omega)})\exp[3\lambda(2A_1+2)^{\nu_0}]\\
&+C_1(\|\widetilde{F}\|^2_{H^4(\Omega)}+\|\widetilde{G}\|^2_{H^4(\Omega)})\exp[3\lambda(2A_1+2)^{\nu_0}]\\
&+C_1(\|\widetilde{f}_0\|^2_{H^{2,1}(S_T\setminus\Gamma_T^-)}+\|\widetilde{f}_1\|^2_{H^{1,0}(S_T\setminus\Gamma_T^-)})\exp[3\lambda(2A_1+2)^{\nu_0}]\\
&+C_1(\|\widetilde{g}_0\|^2_{H^{2,1}(S_T\setminus\Gamma_T^-)}+\|\widetilde{g}_1\|^2_{H^{2,1}(S_T\setminus\Gamma_T^-)})\exp[3\lambda(2A_1+2)^{\nu_0}]\\
&+C_1\exp[3\cdot 2^{\nu_0}\lambda](\|v\|^2_{H^{2,1}(\Gamma_T^-)}+\|v_{x_1}\|^2_{H^{1,0}(\Gamma_T^-)})\\
&+\backslash C_1\exp[3\cdot 2^{\nu_0}\lambda](\|w\|^2_{H^{2,1}(\Gamma_T^-)}+\|w_{x_1}\|^2_{H^{1,0}(\Gamma_T^-)}).
\end{aligned}\tag{5.25}$$

Here we make the estimate for D slightly stronger for a convenience of further derivations. Since C_1 denotes different numbers (Remark 4.3, then below D denotes different numbers with the same expression (5.25)).

The second estimate is:

$$
\begin{aligned}
&(1/\lambda)\int_{Q_T}\Big(w_t^2+\sum_{i,j=2}^n w_{ij}^2\Big)\varphi_{\lambda,\nu_0}dxdt+\int_{Q_T}(\lambda(\nabla w)^2+\lambda^3w^2)\varphi_{\lambda,\nu_0}dxdt\\
\le& C_1\int_{Q_T}((\nabla w)^2+w^2)\varphi_{\lambda,\nu_0}(x)dxdt+C_1\int_{Q_T}((\nabla v)^2+v^2)\varphi_{\lambda,\nu_0}(x)dxdt\\
&+C_1\int_{Q_T}(\Delta v)^2\varphi_{\lambda,\nu_0}(x)dxdt+D. \qquad (5.26)
\end{aligned}
$$

Choose a sufficiently large $\lambda_1=\lambda_1(M,c,\Omega,T)>\lambda_0$ such that for all $\lambda\ge\lambda_1$ and for all functions $h\in H^{1,0}(Q_T)$

$$
C_1\int_{Q_T}((\nabla h)^2+h^2)\varphi_{\lambda,\nu_0}(x)dxdt\le(1/2)\int_{Q_T}(\lambda(\nabla h)^2+\lambda^3h^2)\varphi_{\lambda,\nu_0}dxdt.
$$

The (5.24) implies

$$
\begin{aligned}
&(1/\lambda)\int_{Q_T}\Big(v_t^2+\sum_{i,j=2}^n v_{ij}^2\Big)\varphi_{\lambda,\nu_0}dxdt+\int_{Q_T}(\lambda(\nabla v)^2+\lambda^3v^2)\varphi_{\lambda,\nu_0}dxdt\\
\le& C_1\int_{Q_T}w^2\varphi_{\lambda,\nu_0}(x)dxdt+D,\quad \forall\lambda\ge\lambda_1. \qquad (5.27)
\end{aligned}
$$

Similarly, (5.26) implies

$$
\begin{aligned}
&(1/\lambda)\int_{Q_T}\Big(w_t^2+\sum_{i,j=2}^n w_{ij}^2\Big)\varphi_{\lambda,\nu_0}dxdt+\int_{Q_T}(\lambda(\nabla w)^2+\lambda^3w^2)\varphi_{\lambda,\nu_0}dxdt\\
\le& C_1\int_{Q_T}((\nabla v)^2+v^2)\varphi_{\lambda,\nu_0}(x)dxdt+C_1\int_{Q_T}(\Delta v)^2\varphi_{\lambda,\nu_0}(x)dxdt+D_2,\quad \forall\lambda\ge\lambda_1. \qquad (5.28)
\end{aligned}
$$

Divide (5.28) by λ^2 and sum up with (5.27). We obtain

$$
\begin{aligned}
&(1/\lambda)\int_{Q_T}\Big(v_t^2+\sum_{i,j=2}^n v_{ij}^2\Big)\varphi_{\lambda,\nu_0}dxdt+(1/\lambda^3)\int_{Q_T}\Big(w_t^2+\sum_{i,j=2}^n w_{ij}^2\Big)\varphi_{\lambda,\nu_0}dxdt\\
&+\int_{Q_T}(\lambda(\nabla v)^2+\lambda^3v^2)\varphi_{\lambda,\nu_0}dxdt+\int_{Q_T}\left[(1/\lambda)(\nabla w)^2+\lambda w^2\right]\varphi_{\lambda,\nu_0}dxdt\\
&+(C_1/\lambda^2)\int_{Q_T}(\Delta v)^2\varphi_{\lambda,\nu_0}(x)dxdt+D,\quad \forall\lambda\ge\lambda_1. \qquad (5.29)
\end{aligned}
$$

Since λ_1 is sufficiently large, then

$$
\frac{C_1}{\lambda^2}\int_{Q_T}(\Delta v)^2\varphi_{\lambda,\nu_0}(x)dxdt\le\frac{1}{2\lambda}\int_{Q_T}\Big(v_t^2+\sum_{i,j=2}^n v_{ij}^2\Big)\varphi_{\lambda,\nu_0}dxdt,\quad \forall\lambda\ge\lambda_1.
$$

Hence, (5.29) can be rewritten as:

$$\begin{aligned}&(1/\lambda)\int_{Q_T}\Big(v_t^2+\sum_{i,j=2}^n v_{ij}^2\Big)\varphi_{\lambda,\nu_0}dxdt+(1/\lambda^3)\int_{Q_T}\Big(w_t^2+\sum_{i,j=2}^n w_{ij}^2\Big)\varphi_{\lambda,\nu_0}dxdt\\&+\int_{Q_T}(\lambda(\nabla v)^2+\lambda^3v^2)\varphi_{\lambda,\nu_0}dxdt+\int_{Q_T}\left[(1/\lambda)(\nabla w)^2+\lambda(w)^2\right]\varphi_{\lambda,\nu_0}dxdt\\&+D,\quad\forall\lambda\geq\lambda_1.\end{aligned}\tag{5.30}$$

Using (4.12), (5.1) and (5.25), we obtain that, similarly with (5.19), the transformation procedure of Subsubsection 5.1.1 leads to the following estimate for D

$$\begin{aligned}D\leq&C_1\delta^2\exp[3\lambda(2A_1+2)^{\nu_0}]+C_1\exp[3\cdot2^{\nu_0}\lambda](\|v\|^2_{H^{2,1}(\Gamma_T^-)}+\|v_{x_1}\|^2_{H^{1,0}(\Gamma_T^-)})\\&+C_1\exp[3\cdot2^{\nu_0}\lambda](\|w\|^2_{H^{2,1}(\Gamma_T^-)}+\|w_{x_1}\|^2_{H^{1,0}(\Gamma_T^-)}).\end{aligned}\tag{5.31}$$

Next, using trace theorem and again similarly with (5.19), we obtain

$$\|v\|^2_{H^{2,1}(\Gamma_T^-)}+\|v_{x_1}\|^2_{H^{1,0}(\Gamma_T^-)}+\|w\|^2_{H^{2,1}(\Gamma_T^-)}+\|w_{x_1}\|^2_{H^{1,0}(\Gamma_T^-)}\leq C_1.$$

Hence, (5.31) imply that D can be estimated as:

$$D\leq C_1\exp[3\lambda(2A_1+2)^{\nu_0}]\delta^2+C_1\exp[3\cdot2^{\nu_0}\lambda]\tag{5.32}$$

and

$$\|v\|^2_{H^{2,1}(\Gamma_T^-)}+\|v_{x_1}\|^2_{H^{1,0}(\Gamma_T^-)}+\|w\|^2_{H^{2,1}(\Gamma_T^-)}+\|w_{x_1}\|^2_{H^{1,0}(\Gamma_T^-)}\leq C_1.$$

We now recall the domain $Q_{\gamma T}$ in (2.1), where $\gamma\in(0,2A_1)$ is an arbitrary but fixed number. By (4.1a) $\varphi_{\lambda,\nu_0}(x)\geq\exp(2\lambda(\gamma+2)^{\nu_0})$ in $Q_{\gamma T}$. Hence,

$$\exp(2\lambda(\gamma+2)^{\nu_0})\|f\|^2_{L_2(Q_{\gamma T})}\leq\int_{Q_T}f^2(x,t)\varphi_{\lambda,\nu_0}dxdt,\quad\forall f\in L_2(Q_T).\tag{5.33}$$

Hence, (5.30), (5.32) and (5.33) imply

$$\begin{aligned}&\|v\|_{H^{2,1}(Q_{\gamma T})}+\|w\|_{H^{2,1}(Q_{\gamma T})}\\&\leq C_1\left[\exp(1.5\lambda(2A_1+2)^{\nu_0})\delta+\exp(-1.5\lambda((\gamma+2)^{\nu_0}-2^{\nu_0}))\right],\quad\forall\lambda\geq\lambda_1.\end{aligned}\tag{5.34}$$

Choose now $\lambda=\lambda(\delta)$ such that (5.34)

$$\exp(1.5\lambda(2A_1+2)^{\nu_0})\delta=\exp(-1.5\lambda((\gamma+2)^{\nu_0}-2^{\nu_0})).$$

Hence,

$$1.5[(2A_1+2)^{\nu_0}+(\gamma+2)^{\nu_0}-2^{\nu_0}]\lambda=\ln(\delta^{-1}).\tag{5.35}$$

Recall that by Theorem 4.1, $\nu_0=\nu_0(A_1)$. Also, recall that $\lambda_1=\lambda_1(M,c,\gamma,\Omega,T)$. Hence, by (5.35)

$$\lambda(\delta)=\ln(\delta^{-1/d}),\quad d=1.5[(\gamma+2)^{\nu_0}-2^{\nu_0}+(2A_1+2)^{\nu_0}], \tag{5.36}$$

$$\text{with } \forall\delta\in(0,\delta_0),\quad \delta_0=\delta_0(M,c,\Omega,T):\ln(\delta_0^{-1/d})>\lambda_1. \tag{5.37}$$

Hence, by Remark 4.3, we should now replace $C_1=C_1(M,c,\Omega,T)>0$ with $B=B(M,c,\gamma,\Omega,T)>0$. Consider the number $\alpha\in(0,1)$,

$$\alpha(M,c,\gamma,\Omega,T)=\frac{1.5((\gamma+2)^{\nu_0}-2^{\nu_0})}{d}=\frac{(\gamma+2)^{\nu_0}-2^{\nu_0}}{(\gamma+2)^{\nu_0}-2^{\nu_0}+(2A_1+2)^{\nu_0}}. \tag{5.38}$$

It follows from (5.34)-(5.38) that the following Hölder stability estimate for functions v and w is valid:

$$\|v\|_{H^{2,1}(Q_{\gamma T})}+\|w\|_{H^{2,1}(Q_{\gamma T})}\le B\delta^{\alpha},\quad \forall\delta\in(0,\delta_0). \tag{5.39}$$

It follows from (5.14), (5.39) and trace theorem that estimate (4.14b) is valid, which is the second target estimate of this theorem.

To prove target estimates (4.14a) of this theorem, we recall again the transformation procedure of Subsubsection 5.1.1. Using (4.12), (5.1), (5.2), (5.5)-(5.13b) and triangle inequality, we obtain

$$\begin{aligned}
&\|v\|_{H^{2,1}(Q_{\gamma T})}+\|w\|_{H^{2,1}(Q_{\gamma T})}\\
\ge&\|\partial_t^j u_1-\partial_t^j u_2\|_{H^{2,1}(Q_{\gamma T})}+\|\partial_t^j m_1-\partial_t^j m_2\|_{H^{2,1}(Q_{\gamma T})}\\
&-(\|\widetilde{p}\|_{H^4(\Omega)}+\|\widetilde{q}\|_{H^4(\Omega)}+\|\widetilde{F}\|_{H^4(\Omega)}+\|\widetilde{G}\|_{H^4(\Omega)})\\
\ge&\|\partial_t^j u_1-\partial_t^j u_2\|_{H^{2,1}(Q_{\gamma T})}+\|\partial_t^j m_1-\partial_t^j m_2\|_{H^{2,1}(Q_{\gamma T})}-C_1\delta,\quad j=0,1.
\end{aligned} \tag{5.40}$$

Comparing this with (5.39) and using $\delta<\delta^{\alpha}$, $\forall\delta\in(0,\delta_0)$, we obtain (4.14a).

To prove uniqueness, we set $\delta=0$. Then (4.14a) and (4.14b) imply that

$$u_1(x,t)=u_2(x,t)=m_1(x,t)=m_2(x,t)=0\quad\text{in } Q_{\gamma T},\quad\text{and}\quad b_1(x)=b_2(x)\quad\text{in } \Omega_{\gamma}. \tag{5.41}$$

Since $\gamma\in(0,2A_1)$ is an arbitrary number, then, setting $\gamma\to 0$, we obtain that (5.41) holds for Q_T and Ω.

5.2 Proof of Theorem 4.5

The proof of this theorem can be carried out as an insignificant modification of the proof of Theorem 4.4. Indeed, since by (2.8) the lateral Cauchy data are known now at the entire boundary S_T, then we should not separate $S_T\backslash\Gamma_T^-$ from Γ_T^- in the above proof. In particular, first and second lines in the Carleman estimate (4.6) should be replaced with:

$$C\exp[3\lambda(2A_1+2)^{\nu_0}](\|u\|^2_{H^{2,1}(S_T)}+\|\partial_n u\|^2_{H^{1,0}(S_T^-)}). \tag{5.42}$$

Hence, D in (5.25) should be replaced with:

$$\begin{aligned}D=&C_1(\|\widetilde{p}\|^2_{H^4(\Omega)}+\|\widetilde{q}\|^2_{H^4(\Omega)}+)\exp[3\lambda(2A_1+2)^{\nu_0}]\\&+C_1(\|\widetilde{F}\|^2_{H^4(\Omega)}+\|\widetilde{G}\|^2_{H^4(\Omega)})\exp[3\lambda(2A_1+2)^{\nu_0}]\\&+C_1(\|\widetilde{f}_0\|^2_{H^{2,1}(S_T)}+\|\widetilde{f}_1\|^2_{H^{1,0}(S_T)})\exp[3\lambda(2A_1+2)^{\nu_0}]\\&+C_1(\|\widetilde{g}_0\|^2_{H^{2,1}(S_{T^-})}+\|\widetilde{g}_1\|^2_{H^{2,1}(S_{T^-})})\exp[3\lambda(2A_1+2)^{\nu_0}].\end{aligned} \tag{5.43}$$

Using (5.30), (5.42) and (5.43), we obtain

$$\begin{aligned}&(1/\lambda)\int_{Q_T}\Big(v_t^2+\sum_{i,j=2}^n v_{ij}^2\Big)\varphi_{\lambda,\nu_0}dxdt+(1/\lambda^3)\int_{Q_T}\Big(w_t^2+\sum_{i,j=2}^n w_{ij}^2\Big)\varphi_{\lambda,\nu_0}dxdt\\&+\int_{Q_T}(\lambda(\nabla v)^2+\lambda^3v^2)\varphi_{\lambda,\nu_0}dxdt+\int_{Q_T}\left[(1/\lambda)(\nabla w)^2+\lambda(w)^2\right]\varphi_{\lambda,\nu_0}dxdt\\&+C_1(\|\widetilde{p}\|^2_{H^4(\Omega)}+\|\widetilde{q}\|^2_{H^4(\Omega)})\exp[3\lambda(2A_1+2)^{\nu_0}]\\&+C_1(\|\widetilde{F}\|^2_{H^4(\Omega)}+\|\widetilde{G}\|^2_{H^4(\Omega)})\exp[3\lambda(2A_1+2)^{\nu_0}]\\&+C_1(\|\widetilde{f}_0\|^2_{H^{2,1}(S_T)}+\|\widetilde{f}_1\|^2_{H^{1,0}(S_T)})\exp[3\lambda(2A_1+2)^{\nu_0}]\\&+C_1(\|\widetilde{g}_0\|^2_{H^{2,1}(S_{T^-})}+\|\widetilde{g}_1\|^2_{H^{2,1}(S_{T^-})})\exp[3\lambda(2A_1+2)^{\nu_0}],\quad\forall\lambda\geq\lambda_1,\end{aligned} \tag{5.44}$$

Since by (4.2) $\varphi_{\lambda,\nu_0}(x)\geq\exp(2\lambda\cdot 2^{\nu})$, then we set in (5.44) $\lambda=\lambda_1$ divide by $\exp(2\lambda\cdot 2^{\nu})$ and then obtain similarly with (5.33) and (5.34):

$$\begin{aligned}&\|v\|_{H^{2,1}(Q_T)}+\|w\|_{H^{2,1}(Q_T)}\\\leq&C_1(\|\widetilde{p}\|_{H^4(\Omega)}+\|\widetilde{q}\|_{H^4(\Omega)}+\|\widetilde{F}\|_{H^4(\Omega)}+\|\widetilde{G}\|_{H^4(\Omega)})\\&+C_1(\|\widetilde{f}_0\|_{H^{2,1}(S_T)}+\|\widetilde{f}_1\|_{H^{1,0}(S_T)}+\|\widetilde{g}_0\|_{H^{2,1}(S_T)}+\|\widetilde{g}_1\|_{H^{1,0}(S_T)}).\end{aligned}$$

The rest of the proof is similar with the proof of Subsection 4.1, starting from (5.33).

Appendix

A Appendix: Proofs of Theorems 4.1 and 4.2

A.1 Proof of Theorem 4.1

Recall that in this theorem $u\in C^{4,2}(\overline{Q}_T)$. It is convenient not to fix ν in the major part of this proof. Rather, we assume that $\nu\geq\nu_0>1$ and set $\nu=\nu_0(A_1)$ only when being close to the end of the proof. The constant C is independent on ν. In this proof $\lambda\geq\lambda_0(A_1)$ and both parameters λ_0,ν_0 are sufficiently large. Furthermore since $\nu=\nu_0(A_1)$ in the end, then we assume that

$$\lambda\gg\nu. \tag{A.1}$$

Using (4.1a), change variables

$$v=ue^{\lambda\psi^{\nu}} \quad \to \quad u=ve^{-\lambda\psi^{\nu}}. \tag{A.2}$$

Hence,

$$u_t=v_te^{-\lambda\psi^{\nu}}, \quad u_{x_1}=(v_{x_1}-\lambda\nu\psi^{\nu-1}v)e^{-\lambda\psi^{\nu}}, \tag{A.3a}$$

$$u_{x_1x_1}=\Big\{v_{x_1x_1}-2\lambda\nu\psi^{\nu-1}v_{x_1}+\lambda^2\nu^2\psi^{2\nu-2}\left[1-2\psi^{-\nu}(\nu-1)/(\lambda\nu)\right]v\Big\}e^{-\lambda\psi^{\nu}}, \tag{A.3b}$$

$$u_{x_ix_i}=v_{x_ix_i}e^{-\lambda\psi^{\nu}}, \quad i,j=2,\cdots,n. \tag{A.3c}$$

By (A.2) and (A.3)

$$\begin{aligned}(u_t-\Delta u)^2\varphi_{\lambda,\nu}\psi^{-\nu+1}=\Big[&v_t-\Big(v_{x_1x_1}+\sum_{i=2}^{n}v_{x_ix_i}\Big)+2\lambda\nu\psi^{\nu-1}v_{x_1}\\ &-\lambda^2\nu^2\psi^{2\nu-2}\left[1-2\psi^{-\nu}(\nu-1)/(\lambda\nu)\right]v\Big]^2\psi^{-\nu+1}.\end{aligned} \tag{A.4}$$

Denote

$$z_1=v_t, \qquad z_2=-v_{x_1x_1}-\sum_{i,j=2}^{n}v_{x_ix_i}, \tag{A.5a}$$

$$z_3=2\lambda\nu\psi^{\nu-1}v_{x_1}, \qquad z_4=-\lambda^2\nu^2\psi^{2\nu-2}\left[1-2\psi^{-\nu}(\nu-1)/(\lambda\nu)\right]v. \tag{A.5b}$$

By (A.4) and (A.5)

$$\begin{aligned}&(u_t-\Delta u)^2\varphi_{\lambda,\nu}\psi^{-\nu+1}\left[(z_1+z_3)+z_2+z_4\right]^2\psi^{-\nu+1}\\ \geq&(z_1+z_3)^2\psi^{-\nu+1}+2z_1z_2\psi^{-\nu+1}+2z_1z_4\psi^{-\nu+1}+2z_2z_3\psi^{-\nu+1}+2z_3z_4\psi^{-\nu+1}.\end{aligned} \tag{A.6}$$

Step 1. Estimate from the below the term $2z_1z_2\psi^{-\nu+1}$ in (A.6).

We have:

$$\begin{aligned}2z_1z_2\psi^{-\nu+1}=&-2v_tv_{x_1x_1}\psi^{-\nu+1}-2\sum_{i,j=2}^{n}v_tv_{x_ix_i}\psi^{-\nu+1}\\ =&(-2v_tv_{x_1}\psi^{-\nu+1})_{x_1}+2v_{tx_1}v_{x_1}\psi^{-\nu+1}-2(\nu-1)\psi^{-\nu}v_tv_{x_1}\\ &+\sum_{i=2}^{n}(-2v_tv_{x_i}\psi^{-\nu+1})_{x_i}+\sum_{i=2}^{n}2v_{tx_i}v_{x_i}\psi^{-\nu+1}\\ =&-2(\nu-1)\psi^{-\nu}z_1v_{x_1}+(-2v_tv_{x_1}\psi^{-\nu+1})_{x_1}+\sum_{i=2}^{n}(-2v_tv_{x_i}\psi^{-\nu+1})_{x_i}\\ &+\partial_t\Big(v_{x_1}^2\psi^{-\nu+1}+\sum_{i,j=2}^{n}v_{x_i}^2\psi^{-\nu+1}\Big).\end{aligned}$$

Thus,

$$2z_1z_2\psi^{-\nu+1}=-2(\nu-1)\psi^{-\nu}z_1v_{x_1}+\partial_tV_1+\mathrm{div}U_1, \tag{A.7}$$

where

$$\partial_tV_1=\partial_t\left[\left((u_{x_1}+\lambda\nu\psi^{\nu-1}u)^2+\sum_{i,j=2}^{n}u_{x_i}^2\right)\varphi_{\lambda,\nu}\psi^{-\nu+1}\right], \tag{A.8a}$$

$$\mathrm{div}U_1=(-2u_t(u_{x_1}+\lambda\nu\psi^{\nu-1}u)\varphi_{\lambda,\nu}\psi^{-\nu+1})_{x_1}+\sum_{i=2}^{n}(-2u_tu_{x_i}\varphi_{\lambda,\nu}\psi^{-\nu+1})_{x_i}. \tag{A.8b}$$

Step 2. Using (A.5) and (A.7)-(A.8b), estimate from the below the term $(z_1+z_3)^2\psi^{-\nu+1}+2z_1z_2\psi^{-\nu+1}$ in (A.6).

We have:

$$\begin{aligned}(z_1+z_3)^2\psi^{-\nu+1}+2z_1z_2\psi^{-\nu+1}=&z_1^2\psi^{-\nu+1}+z_3^2\psi^{-\nu+1}+2z_1z_3\psi^{-\nu+1}\\&-2(\nu-1)\psi^{-\nu}z_1v_{x_1}-\partial_tV_1+\mathrm{div}U_1.\end{aligned} \tag{A.9}$$

By (A.5)

$$-2(\nu-1)\psi^{-\nu}z_1v_{x_1}=-\frac{(\nu-1)}{\lambda\nu}\psi^{-2\nu+1}z_1z_3.$$

Hence, (A.9) becomes

$$\begin{aligned}(z_1+z_3)^2\psi^{-\nu+1}+2z_1z_2\psi^{-\nu+1}=&\left[z_1^2+2z_1z_3(1-(\nu-1)/(\lambda\nu)\psi^{-\nu})+z_3^2\right]\psi^{-\nu+1}\\&+\partial_tV_1+\mathrm{div}U_1.\end{aligned} \tag{A.10}$$

Since $(1-(\nu-1)/(\lambda\nu)\psi^{-\nu})<1$ for sufficiently large λ, then

$$\left[z_1^2+2z_1z_3(1-(\nu-1)/(\lambda\nu)\psi^{-\nu})+z_3^2\right]\psi^{-\nu+1}\geq 0$$

as a quadratic polynomial with respect to z_1, z_3. Hence, (A.10) implies

$$(z_1+z_3)^2\psi^{-\nu+1}+2z_1z_2\psi^{-\nu+1}\geq\partial_tV_1+\mathrm{div}U_1. \tag{A.11}$$

Step 3. Using in (A.5), evaluate the term $2z_2z_3\psi^{-\nu+1}$ in (A.6).

We have:

$$\begin{aligned}2z_2z_3\psi^{-\nu+1}=&-4\lambda\nu v_{x_1}\left(v_{x_1x_1}+\sum_{i=2}^{n}v_{x_ix_i}\right)\\=&(-2\lambda\nu v_{x_1}^2)_{x_1}+\sum_{i=2}^{n}(-4\lambda\nu v_{x_1}v_{x_i})_{x_i}+\sum_{i=2}^{n}\left(4\lambda\nu v_{x_ix_1}v_{x_i}\right)\\=&(-2\lambda\nu v_{x_1}^2)_{x_1}+\sum_{i=2}^{n}(-4\lambda\nu v_{x_1}v_{x_i})_{x_i}+\left(2\lambda\nu\sum_{i=2}^{n}v_{x_i}^2\right)_{x_1}.\end{aligned}$$

Hence,

$$2z_2z_3\psi^{-\nu+1}=\mathrm{div}U_2, \tag{A.12a}$$

$$\begin{aligned}\mathrm{div}U_2=&\Big(-2\lambda\nu(u_{x_1}+\lambda\nu\psi^{\nu-1}u)^2\varphi_{\lambda,\nu}+2\lambda\nu\sum_{i=2}^{n}u_{x_i}^2\varphi_{\lambda,\nu}\Big)_{x_1}\\&+\sum_{i=2}^{n}(-4\lambda\nu(u_{x_1}+\lambda\nu\psi^{\nu-1}u)u_{x_i}\varphi_{\lambda,\nu})_{x_i}.\end{aligned} \tag{A.12b}$$

Step 4. Using in (A.5), estimate from the below the term $2z_1z_4\psi^{-\nu+1}+2z_3z_4\psi^{-\nu+1}$ in (A.6).

We have:

$$\begin{aligned}2z_1z_4\psi^{-\nu+1}+2z_3z_4\psi^{-\nu+1}=&-2\lambda^2\nu^2\psi^{\nu-1}(1-2\psi^{-\nu}(\nu-1)/(\lambda\nu))v_tv\\&-4\lambda^3\nu^3\psi^{2\nu-2}(1-2\psi^{-\nu}(\nu-1)/(\lambda\nu))v_{x_1}v\\\geq&\partial_t(-\lambda^2\nu^2\psi^{\nu-1}(1-2\psi^{-\nu}(\nu-1)/(\lambda\nu))v^2)\\&+\partial_{x_1}(-2\lambda^3\nu^3\psi^{2\nu-2}(1-2\psi^{-\nu}(\nu-1)/(\lambda\nu))v^2)\\&+2\lambda^3\nu^4\psi^{2\nu-3}(1-2\psi^{-\nu}(\nu-1)/(\lambda\nu))v^2.\end{aligned}$$

We have used here the inequality $(\nu-1)>\nu/2$, which is valid since $\nu>2$. Thus,

$$2z_1z_4\psi^{-\nu+1}+2z_3z_4\psi^{-\nu+1}\geq 2\lambda^3\nu^4\psi^{2\nu-3}\left[1-2\psi^{-\nu}(\nu-1)/(\lambda\nu)\right]v^2+\partial_tV_2+\mathrm{div}U_3, \tag{A.13a}$$

$$\partial_tV_2=\partial_t(-\lambda^2\nu^2\psi^{\nu-1}(1-2\psi^{-\nu}(\nu-1)/(\lambda\nu))u^2\varphi_{\lambda,\nu}), \tag{A.13b}$$

$$\mathrm{div}U_3=(-2\lambda^3\nu^3\psi^{2\nu-2}(1-2\psi^{-\nu}(\nu-1)/(\lambda\nu))u^2\varphi_{\lambda,\nu})_{x_1}. \tag{A.13c}$$

Step 5. Sum up (A.8a), (A.8b) and (A.11)-(A.13c).

Then, comparing with (A.4) and (A.5), we obtain

$$\begin{aligned}(u_t-\Delta u)^2\varphi_{\lambda,\nu}\psi^{-\nu+1}\geq&2\lambda^3\nu^4\psi^{2\nu-3}\left[1-2\psi^{-\nu}(\nu-1)/(\lambda\nu)\right]u^2\varphi_{\lambda,\nu}\\&+\partial_tV_3+\mathrm{div}U_4,\end{aligned} \tag{A.14a}$$

$$\begin{aligned}\partial_tV_3=\partial_tV_1+\partial_tV_2=&\partial_t\left[\Big((u_{x_1}+\lambda\nu\psi^{\nu-1}u)^2+\sum_{i,j=2}^{n}u_{x_i}^2\Big)\varphi_{\lambda,\nu}\psi^{-\nu+1}\right]\\&+\partial_t(-\lambda^2\nu^2\psi^{\nu-1}(1-2\psi^{-\nu}(\nu-1)/(\lambda\nu))u^2\varphi_{\lambda,\nu}),\end{aligned} \tag{A.14b}$$

$$\begin{aligned}\mathrm{div}U_4=&\mathrm{div}U_1+\mathrm{div}U_2+\mathrm{div}U_3\\=&(-2u_t(u_{x_1}+\lambda\nu\psi^{\nu-1}u)\varphi_{\lambda,\nu}\psi^{-\nu+1})_{x_1}\\&+\Big(-2\lambda\nu(u_{x_1}+\lambda\nu\psi^{\nu-1}u)^2\varphi_{\lambda,\nu}+2\lambda\nu\sum_{i=2}^{n}u_{x_i}^2\varphi_{\lambda,\nu}\Big)_{x_1}\\&+(-2\lambda^3\nu^3\psi^{2\nu-2}(1-2\psi^{-\nu}(\nu-1)/(\lambda\nu))u^2\varphi_{\lambda,\nu})_{x_1}\\&+\sum_{i=2}^{n}(-4\lambda\nu(u_{x_1}+\lambda\nu\psi^{\nu-1}u)u_{x_i}\varphi_{\lambda,\nu}-2u_tu_{x_i}\varphi_{\lambda,\nu}\psi^{-\nu+1})_{x_i}.\end{aligned} \tag{A.14c}$$

Step 6. Evaluate $(u_t-\Delta u)u\varphi_{\lambda,\nu}$.

We have:

$$\begin{aligned}(u_t-\Delta u)u\varphi_{\lambda,\nu}=&\partial_t((1/2)u^2\varphi_{\lambda,\nu})-u_{x_1x_1}u\varphi_{\lambda,\nu}-\sum_{i=2}^n u_{x_ix_i}u\varphi_{\lambda,\nu}\\ =&(-u_{x_1}u\varphi_{\lambda,\nu})_{x_1}+u_{x_1}^2\varphi_{\lambda,\nu}+2\lambda\nu\psi^{\nu-1}u_{x_1}u\varphi_{\lambda,\nu}\\ &+\sum_{i=2}^n(-u_{x_i}u\varphi_{\lambda,\nu})_{x_i}+\sum_{i=2}^n u_{x_i}^2\varphi_{\lambda,\nu}+\partial_t((1/2)u^2\varphi_{\lambda,\nu})\\ =&(\nabla u)^2\varphi_{\lambda,\nu}+(-u_{x_1}u\varphi_{\lambda,\nu}+\lambda\nu\psi^{\nu-1}u^2\varphi_{\lambda,\nu})_{x_1}\\ &-4\lambda^2\nu^2\psi^{2\nu-2}u^2\varphi_{\lambda,\nu}-\lambda\nu(\nu-1)\psi^{\nu-2}u^2\varphi_{\lambda,\nu}\\ &+\sum_{i=2}^n(-u_{x_i}u\varphi_{\lambda,\nu})_{x_i}+\partial_t((1/2)u^2\varphi_{\lambda,\nu}).\end{aligned}\tag{A.15}$$

Thus, since by (4.2)

$$\lambda^2\nu^2\psi^{2\nu-2}\gg\lambda\nu(\nu-1)\psi^{\nu-2}\quad\text{for}\quad\nu>2$$

and sufficiently large $\lambda>1$, then (A.15) implies that for these values of ν and λ

$$(u_t-\Delta u)u\varphi_{\lambda,\nu}\geq(\nabla u)^2\varphi_{\lambda,\nu}-C\lambda^2\nu^2\psi^{2\nu-2}u^2\varphi_{\lambda,\nu}+\partial_tV_4+\mathrm{div}U_5,\tag{A.16a}$$

$$\partial_tV_4=\partial_t\Big(\frac{1}{2}u^2\varphi_{\lambda,\nu}\Big),\tag{A.16b}$$

$$\mathrm{div}U_5=(-u_{x_1}u\varphi_{\lambda,\nu}+\lambda\nu\psi^{\nu-1}u^2\varphi_{\lambda,\nu})_{x_1}+\sum_{i=2}^n(-u_{x_i}u\varphi_{\lambda,\nu})_{x_i}.\tag{A.16c}$$

Step 7. Multiply (A.16a)-(A.16c) by λ and sum up with (A.14a)-(A.14c).

Use the inequality

$$2\lambda^3\nu^4\psi^{2\nu-3}\left[1-2\psi^{-\nu}(\nu-1)/(\lambda\nu)\right]>\lambda^3\nu^4\psi^{2\nu-3}>\lambda^3\nu^2\psi^{2\nu-2},\quad\forall\nu\geq\nu_0,$$

with a sufficiently large $\nu_0=\nu_0(A_1)>2$. We obtain for all sufficiently large $\lambda\geq\lambda_0=\lambda_0(A_1)>1$ and $\nu\geq\nu_0$:

$$\begin{aligned}&\lambda(u_t-\Delta u)u\varphi_{\lambda,\nu}+(u_t-\Delta u)^2\varphi_{\lambda,\nu}\psi^{-\nu+1}\\ &\geq+\lambda(\nabla u)^2\varphi_{\lambda,\nu}+C\lambda^3\nu^4\psi^{2\nu-2}u^2+\partial_tV_5+\mathrm{div}U_6,\end{aligned}\tag{A.17a}$$

$$\begin{aligned}\partial_tV_5=&\partial_tV_3+\partial_t(\lambda V_4)\\ =&\partial_t\left[((u_{x_1}+\lambda\nu\psi^{\nu-1}u)^2+\sum_{i,j=2}^n u_{x_i}^2)\varphi_{\lambda,\nu}\psi^{-\nu+1}\right]\\ &+\partial_t(-\lambda^2\nu^2\psi^{\nu-1}(1-2\psi^{-\nu}(\nu-1)/(\lambda\nu))u^2\varphi_{\lambda,\nu}+\partial_t((\lambda/2)u^2\varphi_{\lambda,\nu})),\end{aligned}\tag{A.17b}$$

$$\begin{aligned}\mathrm{div}U_6=&\mathrm{div}U_4+\mathrm{div}(\lambda U_5)\\ =&(-2u_t(u_{x_1}+\lambda\nu\psi^{\nu-1}u)\varphi_{\lambda,\nu}\psi^{-\nu+1})_{x_1}\end{aligned}$$

$$
\begin{aligned}
&+\Big(-2\lambda\nu(u_{x_1}+\lambda\nu\psi^{\nu-1}u)^2\varphi_{\lambda,\nu}+2\lambda\nu\sum_{i=2}^{n}u_{x_i}^2\varphi_{\lambda,\nu}\Big)_{x_1}\\
&+(-2\lambda^3\nu^3\psi^{2\nu-2}(1-2\psi^{-\nu}(\nu-1)/(\lambda\nu))u^2\varphi_{\lambda,\nu})_{x_1}\\
&+\sum_{i=2}^{n}(-4\lambda\nu(u_{x_1}+\lambda\nu\psi^{\nu-1}u)u_{x_i}\varphi_{\lambda,\nu}-2u_tu_{x_i}\varphi_{\lambda,\nu}\psi^{-\nu+1})_{x_i}\\
&+(-\lambda u_{x_1}u\varphi_{\lambda,\nu}+\lambda^2\nu\psi^{\nu-1}u^2\varphi_{\lambda,\nu})_{x_1}+\sum_{i=2}^{n}(-\lambda u_{x_i}u\varphi_{\lambda,\nu})_{x_i}. \qquad \text{(A.17c)}
\end{aligned}
$$

Applying Cauchy-Schwarz inequality to the left hand side of (A.17a) and also using (A.1), we obtain a pointwise Carleman estimate from the above for the lower order derivatives via $(u_t-\Delta u)^2\varphi_{\lambda,\nu}$,

$$
\begin{aligned}
(u_t-\Delta u)^2\varphi_{\lambda,\nu}\geq& C\lambda(\nabla u)^2\varphi_{\lambda,\nu}+C\lambda^3\nu^4\psi^{2\nu-2}u^2\\
&+\partial_tV_5+\mathrm{div}U_6,\quad \forall\lambda\geq\lambda_0=\lambda_0(A_1)>1,\quad \nu=\nu_0=\nu_0(A_1)>2. \qquad \text{(A.18)}
\end{aligned}
$$

We now need to incorporate estimates for derivatives $u_{x_ix_j}$ and u_t in a close analog of (A.18).

Step 8. Estimate again $(u_t-\Delta u)^2\varphi_{\lambda,\nu_0}$ from the below.

We now set $\nu=\nu_0$, see the beginning of the proof of Theorem 4.1. We have

$$
(u_t-\Delta u)^2\varphi_{\lambda,\nu_0}=u_t^2\varphi_{\lambda,\nu_0}-2u_t\Delta u\varphi_{\lambda,\nu_0}+(\Delta u)^2\varphi_{\lambda,\nu_0}. \qquad \text{(A.19)}
$$

We estimate separately from the below the second and the third terms in the righ hand side of (A.19).

Step 8.1. First, estimate from the below the term $u_t^2\varphi_{\lambda,\nu_0}-2u_t\Delta u\varphi_{\lambda,\nu_0}$ in (A.19). We have:

$$
\begin{aligned}
-2u_t\Delta u\varphi_{\lambda,\nu_0}=&-2u_tu_{x_1x_1}\varphi_{\lambda,\nu_0}+\sum_{i=2}^{n}(-2u_tu_{x_ix_i}\varphi_{\lambda,\nu_0})\\
=&(-2u_tu_{x_1}\varphi_{\lambda,\nu_0})_{x_1}+2u_{x_1t}u_{x_1}\varphi_{\lambda,\nu_0}+2\lambda\nu_0\psi^{\nu_0-1}u_tu_{x_1}\varphi_{\lambda,\nu_0}\\
&+\sum_{i=2}^{n}(-2u_tu_{x_i}\varphi_{\lambda,\nu_0})_{x_i}+\sum_{i=2}^{n}2u_{x_it}u_{x_i}\varphi_{\lambda,\nu_0}\\
=&\partial_t((\nabla u)^2\varphi_{\lambda,\nu_0})+\sum_{i=1}^{n}(-2u_tu_{x_i}\varphi_{\lambda,\nu_0})_{x_i}+2\lambda\nu_0\psi^{\nu_0-1}u_tu_{x_1}\varphi_{\lambda,\nu_0}. \qquad \text{(A.20)}
\end{aligned}
$$

By Young's inequality and (4.2)

$$
\begin{aligned}
2\lambda\nu_0\psi^{\nu_0-1}u_tu_{x_1}\varphi_{\lambda,\nu_0}\geq&-(1/2)u_t^2\varphi_{\lambda,\nu_0}-2\lambda^2\nu_0^2\psi^{2\nu_0-2}u_{x_1}^2\varphi_{\lambda,\nu_0}\\
\geq&-(1/2)u_t^2\varphi_{\lambda,\nu_0}-C\lambda^2u_{x_1}^2\varphi_{\lambda,\nu_0}. \qquad \text{(A.21)}
\end{aligned}
$$

Hence, using (A.19)-(A.21), we obtain

$$(u_t-\Delta u)^2\varphi_{\lambda,\nu_0}\geq(1/2)u_t^2\varphi_{\lambda,\nu_0}-C\lambda^2u_{x_1}^2\varphi_{\lambda,\nu_0}+(\Delta u)^2\varphi_{\lambda,\nu_0}+\mathrm{div}U_7, \tag{A.22a}$$

$$\partial_t V_6=\partial_t((\nabla u)^2\varphi_{\lambda,\nu_0}), \tag{A.22b}$$

$$\mathrm{div}U_7=\sum_{i=1}^n(-2u_tu_{x_i}\varphi_{\lambda,\nu_0})_{x_i}. \tag{A.22c}$$

Step 8.2. Second, estimate from the term $(\Delta u)^2\varphi_{\lambda,\nu_0}$ in (A.19). We have:

$$(\Delta u)^2\varphi_{\lambda,\nu_0}=u_{x_1x_1}^2\varphi_{\lambda,\nu_0}+\sum_{i=2}^n2u_{x_1x_1}u_{x_ix_i}\varphi_{\lambda,\nu_0}+\sum_{i,j=2}^n u_{x_ix_i}u_{x_jx_j}\varphi_{\lambda,\nu_0}. \tag{A.23}$$

Estimate the term:

$$\begin{aligned}\sum_{i=2}^n2u_{x_1x_1}u_{x_ix_i}\varphi_{\lambda,\nu_0}&=\sum_{i=2}^n(2u_{x_1x_1}u_{x_i}\varphi_{\lambda,\nu_0})_{x_i}-\sum_{i=2}^n(2u_{x_1x_1x_i}u_{x_i}\varphi_{\lambda,\nu_0})\\&=\sum_{i=2}^n(2u_{x_1x_1}u_{x_i}\varphi_{\lambda,\nu_0})_{x_i}+\sum_{i=2}^n(-2u_{x_1x_i}u_{x_i}\varphi_{\lambda,\nu_0})_{x_1}+2\sum_{i=2}^nu_{x_1x_i}^2\varphi_{\lambda,\nu_0}\\&\quad+\sum_{i=2}^n4\lambda\nu_0\psi^{\nu_0-1}u_{x_1x_i}u_{x_i}\varphi_{\lambda,\nu_0}.\end{aligned} \tag{A.24}$$

By Young's inequality and (4.2)

$$\sum_{i=2}^n4\lambda\nu_0\psi^{\nu_0-1}u_{x_1x_i}u_{x_i}\varphi_{\lambda,\nu_0}\geq-\sum_{i=2}^nu_{x_1x_i}^2\varphi_{\lambda,\nu_0}-C\lambda^2(\nabla u)^2\varphi_{\lambda,\nu_0}.$$

Hence, using (A.23) and (A.24), we obtain

$$(\Delta u)^2\varphi_{\lambda,\nu_0}\geq\sum_{i=1}^nu_{x_1x_i}^2\varphi_{\lambda,\nu_0}+\sum_{i,j=2}^n u_{x_ix_i}u_{x_jx_j}\varphi_{\lambda,\nu_0}-C\lambda^2(\nabla u)^2\varphi_{\lambda,\nu_0}+\mathrm{div}U_8, \tag{A.25a}$$

$$\mathrm{div}U_8=\sum_{i=2}^n(-2u_{x_1x_i}u_{x_i}\varphi_{\lambda,\nu_0})_{x_1}+\sum_{i=2}^n(2u_{x_1x_1}u_{x_i}\varphi_{\lambda,\nu_0})_{x_i}. \tag{A.25b}$$

We now estimate the following term in (A.25a):

$$\begin{aligned}\sum_{i,j=2}^n u_{x_ix_i}u_{x_jx_j}\varphi_{\lambda,\nu_0}&=\sum_{i,j=2}^n(u_{x_ix_i}u_{x_j}\varphi_{\lambda,\nu_0})_{x_j}-\sum_{i,j=2}^n u_{x_ix_ix_j}u_{x_j}\varphi_{\lambda,\nu_0}\\&=\sum_{i,j=2}^n(u_{x_ix_i}u_{x_j}\varphi_{\lambda,\nu_0})_{x_j}+\sum_{i,j=2}^n(-u_{x_ix_j}u_{x_j}\varphi_{\lambda,\nu_0})_{x_i}+\sum_{i,j=2}^nu_{x_ix_j}^2\varphi_{\lambda,\nu_0}.\end{aligned}$$

Combining this with (A.25a) and (A.25b), we obtain

$$(\Delta u)^2\varphi_{\lambda,\nu_0} \geq \sum_{i=1}^{n} u_{x_ix_j}^2\varphi_{\lambda,\nu_0} - C\lambda^2(\nabla u)^2\varphi_{\lambda,\nu_0} + \mathrm{div}U_9, \tag{A.26a}$$

$$\begin{aligned}\mathrm{div}U_9 = &\sum_{i=2}^{n}(-2u_{x_1x_i}u_{x_i}\varphi_{\lambda,\nu_0})_{x_1} + \sum_{i=2}^{n}(2u_{x_1x_1}u_{x_i}\varphi_{\lambda,\nu_0})_{x_i} \\ &+ \sum_{i,j=2}^{n}(u_{x_ix_i}u_{x_j}\varphi_{\lambda,\nu_0})_{x_j} + \sum_{i,j=2}^{n}(-u_{x_ix_j}u_{x_j}\varphi_{\lambda,\nu_0})_{x_i}.\end{aligned} \tag{A.26b}$$

Combining (A.26a) and (A.26b) with (A.22a)-(A.22c), we obtain

$$\begin{aligned}(u_t-\Delta u)^2\varphi_{\lambda,\nu_0} \geq &(1/2)u_t^2\varphi_{\lambda,\nu_0} + \sum_{i=1}^{n} u_{x_ix_j}^2\varphi_{\lambda,\nu_0} - C\lambda^2(\nabla u)^2\varphi_{\lambda,\nu_0} + \\ &+\partial_t((\nabla u)^2\varphi_{\lambda,\nu_0}) + \mathrm{div}U_{10},\end{aligned} \tag{A.27a}$$

$$\begin{aligned}\mathrm{div}U_{10} = &\mathrm{div}U_7 + \mathrm{div}U_9 \\ = &\sum_{i=1}^{n}(-2u_tu_{x_i}\varphi_{\lambda,\nu_0})_{x_i} + \sum_{i=2}^{n}(-2u_{x_1x_i}u_{x_i}\varphi_{\lambda,\nu_0})_{x_1} + \sum_{i=2}^{n}(2u_{x_1x_1}u_{x_i}\varphi_{\lambda,\nu_0})_{x_i} \\ &+ \sum_{i,j=2}^{n}(u_{x_ix_i}u_{x_j}\varphi_{\lambda,\nu_0})_{x_j} + \sum_{i,j=2}^{n}(-u_{x_ix_j}u_{x_j}\varphi_{\lambda,\nu_0})_{x_i}.\end{aligned} \tag{A.27b}$$

Step 9. Divide (A.27a) and (A.27b) by 2λ and sum up with (A.18), taking into account (A.17b) and (A.17c).

After suming up as indicated, we divide both parts of the resulting estimate by $(1+1/(2\lambda))$. Since we use the constant C, then this division will affect only terms under ∂_t and div signs. We obtain the target estimate (4.2) of this theorem, where

$$\partial_t V = \partial_t\Big(\frac{2\lambda V_5 + (\nabla u)^2\varphi_{\lambda,\nu_0}}{2\lambda+1}\Big), \quad \mathrm{div}U = \mathrm{div}\Big(\frac{2\lambda U_6 + U_{10}}{2\lambda+1}\Big). \tag{A.28}$$

Formulas (A.28) are equivalent with formulas (4.3), (4.4).

A.2 Proof of Theorem 4.2

It is convenient to assume first that $u \in C^{4,2}(\overline{Q}_T)$ since Theorem 4.1 is proven for these functions. Integrate (4.2) over Q_T. It follows from (4.3), (4.5) and the last line of (4.6) that integrals over $\{t=0\}$ and $\{t=T\}$ are mutually canceled. Therefore, by Gauss formula

$$\begin{aligned}&\int_{Q_T}(u_t-\Delta u)^2\varphi_{\lambda,\nu_0}dxdt \\ \geq &(C/\lambda)\int_{Q_T}\Big(u_t^2 + \sum_{i,j=1}^{n}u_{x_ix_j}^2\Big)\varphi_{\lambda,\nu_0}dxdt + C\int_{Q_T}\left[\lambda(\nabla u)^2 + \lambda^3u^2\right]\varphi_{\lambda,\nu_0}dxdt \\ &+\int_{S_T}U\cos(\mu,x)dS, \quad \forall\lambda\geq\lambda_0,\end{aligned} \tag{A.29}$$

where μ is the outward looking unit normal vector at $\partial\Omega$.

We now evaluate the term

$$\int_{S_T} U\cos(\mu,x)dS. \tag{A.30}$$

To do this, we use (2.2b)-(2.2f) and (4.4). First, consider the part Γ_T^+ of S_T. Obviously, $\mu=(1,0,\cdots,0)$ on Γ_T^+. Note that $dS=dx_2\cdots dx_n dt$ on Γ_T^+. By (4.4) and (A.30)

$$\begin{aligned}\int_{\Gamma_T^+} U\cos(\mu,x)dS=&\int_{\Gamma_T^+}\Big[(2\lambda/(2\lambda+1))(-2u_t(u_{x_1}+\lambda\nu_0\psi^{\nu_0-1}u)\varphi_{\lambda,\nu_0}\psi^{-\nu_0+1})\Big]dS\\&+\int_{\Gamma_T^+}\Big[(2\lambda/(2\lambda+1))(-2\lambda\nu_0(u_{x_1}+\lambda\nu_0\psi^{\nu_0-1}u)^2\varphi_{\lambda,\nu_0})\Big]dS\\&+\int_{\Gamma_T^+}\left[(2\lambda/(2\lambda+1))(2\lambda\nu_0\sum_{i=2}^n u_{x_i}^2\varphi_{\lambda,\nu_0})\right]dS\\&+\int_{\Gamma_T^+}\Big[(2\lambda/(2\lambda+1))(-2\lambda^3\nu_0^3\psi^{2\nu_0-2}(1-2\psi^{-\nu_0}(\nu_0-1)/(\lambda\nu_0))u^2\varphi_{\lambda,\nu_0})\Big]dS\\&+\int_{\Gamma_T^+}\Big[(2\lambda/(2\lambda+1))(-\lambda u_{x_1}u\varphi_{\lambda,\nu_0}+\lambda^2\nu_0\psi^{\nu_0-1}u^2\varphi_{\lambda,\nu_0})\Big]dS\\&+\int_{\Gamma_T^+}\sum_{i=2}^n\big[(1/(2\lambda+1))(-2u_{x_1x_i}u_{x_i}\varphi_{\lambda,\nu_0})\big]dS.\end{aligned}$$

Combining this equality with (4.2), we obtain

$$\begin{aligned}&\int_{\Gamma_T^+} U\cos(n,x)dS\\&\geq -C\exp[3\lambda(2A_1+2)^{\nu_0}](\|u\|^2_{H^{2,1}(\Gamma_T^+)}+\|u_{x_1}\|^2_{H^{1,0}(\Gamma_T^+)}).\end{aligned} \tag{A.31}$$

Second, consider the part Γ_T^+ of S_T. by (2.1) and (4.1a) $\varphi_{\lambda,\nu_0}=\exp(2^{\nu_0+1}\lambda)$ on Γ_T^-. Hence, we obtain similarly with (A.31)

$$\begin{aligned}&\int_{\Gamma_T^-} U\cos(\mu,x)dS\\&\geq -C\exp[3\cdot 2^{\nu_0}\lambda](\|u\|^2_{H^{2,1}(\Gamma_T^-)}+\|u_{x_1}\|^2_{H^{1,0}(\Gamma_T^-)}).\end{aligned} \tag{A.32}$$

We now evaluate the integral

$$\int_{\partial_i^+\Omega_T} U\cos(\mu,x)dS,\quad i=2,\cdots,n. \tag{A.33}$$

Obviously $\mu=(\delta_{i1},\cdots,\delta_{in})$, where

$$\delta_{ij}=\begin{cases}1, & \text{if } i=j,\\ 0, & \text{if } i\neq j.\end{cases}$$

Hence, by (4.4) and (A.33) for $i\in[2,n]$

$$\begin{aligned}\int_{\partial_i^+\Omega_T} U\cos(\mu,x)dS=&\int_{\partial_i^+\Omega_T}\Big[(2\lambda/(2\lambda+1))(-4\lambda\nu_0(u_{x_1}+\lambda\nu_0\psi^{\nu_0-1}u)u_{x_i}\varphi_{\lambda,\nu_0})\Big]dS\\&+\int_{\partial_i^+\Omega_T}\Big[(2\lambda/(2\lambda+1))(-2u_tu_{x_i})\psi^{-\nu_0+1}\varphi_{\lambda,\nu_0}\Big]dS\\&+\int_{\partial_i^+\Omega_T}[(2\lambda/(2\lambda+1))(-\lambda u_{x_i}u-2u_tu_{x_i})\varphi_{\lambda,\nu_0}]dS\\&+\int_{\partial_i^+\Omega_T}\Big[(1/(2\lambda+1))(u_{x_jx_j}u_{x_i}\varphi_{\lambda,\nu_0}-u_{x_ix_j}u_{x_j}\varphi_{\lambda,\nu_0})\Big]dS.\end{aligned}\tag{A.34}$$

Suppose that $i=j$ in the last line of (A.34). Then this term is:

$$\begin{aligned}&\int_{\partial_i^+\Omega_T}\Big[(1/(2\lambda+1))(u_{x_jx_j}u_{x_i}\varphi_{\lambda,\nu_0}-u_{x_ix_j}u_{x_j}\varphi_{\lambda,\nu_0})\Big]dS\\=&\int_{\partial_i^+\Omega_T}[(1/(2\lambda+1))(u_{x_ix_i}u_{x_i}\varphi_{\lambda,\nu_0}-u_{x_ix_i}u_{x_i}\varphi_{\lambda,\nu_0})]dS=0.\end{aligned}\tag{A.35}$$

Hence, the last line of (A.34) is not identically zero only if $i\neq j$. Hence, by (4.2)

$$\begin{aligned}&\int_{\partial_i^+\Omega_T}\Big[(1/(2\lambda+1))(u_{x_jx_j}u_{x_i}\varphi_{\lambda,\nu_0}-u_{x_ix_j}u_{x_j}\varphi_{\lambda,\nu_0})\Big]dS\\\geq&-C\exp[3\lambda(2A_1+2)^{\nu_0}](\|u\|^2_{H^{2,1}(\partial_i^+\Omega_T)}+\|\partial_nu\|^2_{H^{1,0}(\partial_i^+\Omega_T)}).\end{aligned}\tag{A.36}$$

Analysis of the sum of second, third and fourth lines of (A.34) shows that this sum can also be estimated from the below like in (A.36). Thus, combining the latter considerations with (A.30)-(A.36), we obtain

$$\int_{S_T} U\cos(\mu,x)dS\geq-C\exp[3\lambda(2A_1+2)^{\nu_0}](\|u\|^2_{H^{2,1}(S_T)}+\|\partial_nu\|^2_{H^{1,0}(S_T)}),$$

and also, more precisely, by (A.32),

$$\begin{aligned}\int_{S_T} U\cos(\mu,x)dS\geq&-C\exp[3\lambda(2A_1+2)^{\nu_0}](\|u\|^2_{H^{2,1}(S_T\setminus\Gamma_T^-)}+\|\partial_nu\|^2_{H^{1,0}(\Gamma_T^-)})\\&-C\exp[3\cdot2^{\nu_0}\lambda](\|u\|^2_{H^{2,1}(\Gamma_T^-)}+\|u_{x_1}\|^2_{H^{1,0}(\Gamma_T^-)}).\end{aligned}\tag{A.37}$$

Finally, density arguments ensure that one can replace $u\in C^{4,2}(\overline{Q}_T)$ with $u\in H^{4,2}(Q_T)$ in (A.29) and (A.37). The target estimate (4.6) of this theorem follows immediately.

Acknowledgements

The work of H. Liu was supported by the NSFC/RGC Joint Research Scheme, N_CityU101/21; ANR/RGC Joint Research Scheme, A_CityU203/19; and the Hong

Kong RGC General Research Funds (projects Nos. 11311122, 11304224 and 11300821). The work of J. Li was partially supported by the Shenzhen Sci-Tech Fund No. RCJC20200714114556020 and Guangdong Basic and Applied Research Fund No. 2023B1515250005.

References

[1] Y. Achdou, P. Cardaliaguet, F. Delarue, A. Porretta and F. Santambrogio, Mean Field Games, Cetraro, Italy 2019, Lecture Notes in Mathematics, C.I.M.E. Foundation Subseries, Volume 2281, Springer, 2019.

[2] D. Bauso, H. Tembine and T. Basar, *Opinion dynamics in social networks through mean-field games,* SIAM J. Control Optim., 54 (2016), pp. 3225–3257.

[3] M. Bellassoued and M. Yamamoto, Carleman Estimates and Applications to Inverse Problems for Hyperbolic Systems, Springer Monogr. Math., Springer, Tokyo, 2017.

[4] A. Bukhgeim and M. Klibanov, *Uniqueness in the large of a class of multidimensional inverse problems,* Soviet Mathematics Doklady, 17 (1981), pp. 244–247.

[5] Y. Chow, S. Fung, S. Liu, L. Nurbekyan and S. Osher, *A numerical algorithm for inverse problem from partial boundary measurement arising from mean field game problem,* Inverse Problems, 39 (2023), 014001.

[6] L. Ding, W. Li, S. Osher and W. Yin, *A mean field game inverse problem,* J. Sci. Comput., 92 (2022), 7.

[7] M. Ding, H. Liu and G. Zheng, *On inverse problems for several coupled PDE systems arising in mathematical biology,* J. Math. Biol., 87 (2023), Paper No. 86.

[8] L. Hörmander, Linear Partial Differential Operators, Springer Verlag, 1963.

[9] M. Huang, R. Malhamé and P. Caines, *Large population stochastic dynamic games: Closed-loop McKean–Vlasov systems and the Nash certainty equivalence principle,* Commun. Inf. Syst., 6 (2006), pp. 221–251.

[10] M. Huang, P. Caines and R. Malhamé, *Large-population cost-coupled LQG problems with nonuniform agents: individual-mass behavior and decentralized Nash equilibria,* IEEE Trans. Automat. Control, 52 (2007), pp. 1560–1571.

[11] O. Imanuvilov and M. Yamamoto, *Lipschitz stability in inverse parabolic problems by the Carleman estimate,* Inverse Probl., 14 (1998), pp. 1229–1245.

[12] O. Imanuvilov, H. Liu, and M. Yamamoto, *Lipschitz stability for determination of states and inverse source problem for the mean field game equations,* Inverse Problems and Imaging, published online, doi:10.3934/ipi.2023057, 2024.

[13] V. Isakov, Inverse Problems for Partial Differential Equations, Second Edition, Springer, New York, 2006.

[14] M. Klibanov, *Inverse problems in the 'large' and Carleman bounds,* Differential Equations, 20 (1984), pp. 755–760.

[15] M. Klibanov, *Inverse problems and Carleman estimates,* Inverse Probl., 8 (1992), pp. 575–596.

[16] M. Klibanov, *Carleman estimates for global uniqueness, stability and numerical methods for coefficient inverse problems,* J. Inverse Ill-Pose. P., 21 (2013), pp. 477–510.

[17] M. Klibanov and J. Li, Inverse Problems and Carleman Estimates: Global Uniqueness, Global Convergence and Experimental Data, De Gruyter, 2021.

[18] M. Klibanov and Y. Averboukh, *Lipschitz stability estimate and uniqueness in the retrospective analysis for the mean field games system via two Carleman estimates,* SIAM J. Math. Anal., 56

(2024), pp. 616–636.

[19] M. KLIBANOV, J. LI AND H. LIU, *Hölder stability and uniqueness for the mean field games system via Carleman estimates,* Stud. Appl. Math., 151 (2023), pp. 1447–1470.

[20] M. KLIBANOV, J. LI AND H. LIU, *On the mean field games system with lateral Cauchy data via Carleman estimates,* J. Inverse Ill-Pose. Probl. 32 (2024), pp. 277–295.

[21] M. KLIBANOV, *A coefficient inverse problem for the mean field games system,* Appl. Math. Optim., 88 (2023), 54.

[22] M. KLIBANOV, *Stability estimates for some parabolic inverse problems wit the final overdetermination via a new Carleman estimate,* Math. Methods Appl. Sci., 47 (2024), pp. 13258–13282.

[23] V. KOLOKOLTSOV AND O. MALAFEYEV, *Mean field game model of corruption,* Dyn. Games Appl., 7 (2017), pp. 34–47.

[24] V. KOLOKOLTSOV AND O. MALAFEYEV, Many Agent Games in Socio-Economic Systems: Corruption, Inspection, Coalition Building, Network Growth, Security, Springer Nature Switzerland AG, 2019.

[25] O. LADYZHENSKAYA, V. SOLONNIKOV AND N. URALCEVA, Linear and Quasilinear Equations of Parabolic Type, AMS, Providence, R.I., 1968.

[26] J. LASRY AND P. LIONS, *Jeux à champ moyen, I. Le cas stationnaire,* C. R. Math. Acad. Sci. Paris, 343 (2005), pp. 619–625.

[27] J. LASRY AND P. LIONS, *Jeux à champ moyen, II. Horizon fini et contrôle optimal,* C. R. Math. Acad. Sci. Paris, 343 (2006), pp. 679–684.

[28] J. LASRY AND P. LIONS, *Mean field games,* Japanese J. Math., 2 (2007), pp. 229–260.

[29] M. LAVRENTIEV, V. ROMANOV AND S. SHISHATSKII, Ill-Posed Problems of Mathematical Physics and Analysis, AMS, Providence: RI, 1986.

[30] Y. LI, H. LIU AND W. LO, *On inverse problems in predator-prey models,* J. Differential Equations, 397 (2024), pp. 349–376.

[31] S. LIU, M. JACOBS, W. LI, L. NURBEKYAN AND S. OSHER, *Computational methods for first order nonlocal mean field games with applications,* SIAM J. Numer. Anal., 59 (2021), pp. 2639–2668.

[32] H. LIU AND C. W. K. LO, *Determining a parabolic system by boundary observation of its nonnegative solutions with biological applications,* Inverse Probl., 40 (2024), Paper No. 025009.

[33] H. LIU, C. MOU AND S. ZHANG, *Inverse problems for mean field games,* Inverse Probl., 39 (2023), 085003.

[34] H. LIU AND M. YAMAMOTO, *Stability in determination of states for the mean field game equations,* Commun. Anal. Comput., 1 (2023), pp. 157–167.

[35] H. LIU AND C. W. K. LO, *Determining state space anomalies in mean field games,* Nonlinearity, 38(2) (2025), 025010.

[36] H. LIU AND S. ZHANG, *Simultaneously recovering running cost and Hamiltonian in mean field games system*.

[37] M. DING, H. LIU, AND G. ZHENG, *Determining a stationary mean field game system from full/partial boundary measurement,* SIAM J. Math. Anal., 57(1) (2025), pp. 661–681.

[38] R. NOVIKOV, *$\partial-$bar approach to approximate inverse scattering at fixed energy in three dimensions,* Int. Math. Res. Rep., 6 (2005), pp. 287–349.

[39] V. ROMANOV, Inverse Problems of Mathematical Physics, VNU Press, Utrecht, 1987.

[40] N. TRUSOV, *Numerical study of the stock market crises based on mean field games approach,* J. Inverse Ill-Posed Probl., 29 (2021), pp. 849–865.

[41] M. YAMAMOTO, *Carleman estimates for parabolic equations, Topical Review,* Inverse Problems, 25 (2009), 123013.

[42] S. YANG, Y. LIU, H. LIU AND C. WANG, *Numerical methods for semilinear fractional diffusion equations with time delay,* Adv. Appl. Math. Mech., 14 (2022), pp. 56–78.