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The Generalized Riemann Problem for Two-Layer Shallow Water Equations with Two-Velocities

Shuzhi Liu[1] and Qinglong Zhang[2,*]

[1] *School of Statistics and Data Science, Ningbo University of Technology, Ningbo, Zhejiang 315211, China*
[2] *Corresponding author. School of Mathematics and Statistics, Ningbo University, Ningbo, Zhejiang 315211, China*

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Abstract. This paper proposes a direct Eulerian generalized Riemann problem (GRP) scheme for two-layer shallow water equations. The model takes into account the distinctions between different densities and velocities, and is obtained by taking the vertical averaging across the layer depth. The source terms generated from the mass and momentum exchange prevent us from solving the Riemann problem analytically. We consider an equivalent conservative two-layer model which describes the horizontal velocity with two degrees of freedom. The rarefaction wave and the shock wave are analytically resolved by using the Riemann invariants and Rankine-Hugoniot condition, respectively. Numerical simulations are also given on some typical problems in order to verify the good performance of the GRP method.

AMS subject classifications: 35L40, 76L05, 76M12, 76B70, 65M08

Key words: Generalized Riemann problem (GRP), two-layer shallow water equations, layer-wise discretization, Riemann invariants, Rankine-Hugoniot condition.

1 Introduction

The study on the multilayer shallow water equations has been of particular interest from both theoretical and numerical points of view, mainly due to its effectively in describing the laminar shallow water models. The two layers have different densities and velocities, and can be obtained by taking the vertical averaging of the layer depth. In this work, we are interested in the following shallow water equations in one-dimensional space with

*Corresponding author.
Emails: szliu@nbut.edu.cn (S. Liu), zhangqinglong@nbu.edu.cn (Q. Zhang)

the horizontal velocity described by two variables, which reads [2]

$$\begin{cases} h_t+(h\bar{u})_x=0, \\ (h\bar{u})_t+\left(h(\bar{u}^2+\hat{u}^2)+\frac{g}{2}h^2\right)_x=0, \\ \hat{u}_t+(\hat{u}\bar{u})_x=0, \end{cases} \tag{1.1}$$

where $h(x,t)$ denotes the depth of the water, $\bar{u}$ and $\hat{u}$ represent the average and the oriented standard deviation of the velocities, respectively. System (1.1) introduces two degrees of velocity and was derived in [2] from a general two-layer shallow water flow. For completeness and convenience to read, we post the bilayer shallow water model with flat topography as follows [3,33]

$$\begin{cases} (h_1)_t+(h_1u_1)_x=m_e, \\ (h_1u_1)_t+\left(h_1u_1^2+g\frac{h_1^2}{2}+gh_1h_2\right)_x=gh_2(h_1)_x+u_{in}m_e, \\ (h_2)_t+(h_2u_2)_x=-m_e, \\ (h_2u_2)_t+\left(h_2u_2^2+g\frac{h_2^2}{2}\right)_x=-gh_2(h_1)_x-u_{in}m_e. \end{cases} \tag{1.2}$$

where h_i and u_i $(i=1,2)$ represent the approximation of the layer depth and the horizontal velocity of the ith layer. m_e denotes the mass exchange from the second layer to the first layer, u_{in} is the interface velocity between two layers. Actually, (1.1) can be derived from (1.2) by assuming each layer has the same depth $h_1=h_2$, and by defining

$$u_{in}=\bar{u}=\frac{u_1+u_2}{2}, \quad \hat{u}=\frac{u_2-u_1}{2}, \tag{1.3}$$

one can find more details in [2] and references therein. System (1.1) also belongs to the well-known Benney equations [10]. Other systems describing shallow water equations and related problems can be found in [21,29,30,40].

As is well known, (1.2) is conditionally hyperbolic [3,4], which bring difficulties to solve the Riemann problem. Recently, a full set of conservation laws is studied in [5,32], the local well-posedness of the two-layer shallow water equations is obtained in [31]. Numerical treatments for the two-layer shallow water equations are also available. A relaxation approach system for (1.2) is proposed and investigated by [1]. More recently, a second order central-upwind scheme was derived in [23] which is proved to be well-balanced. A novel Q-scheme is applied to the two-layer shallow water equations in [15]. More works can be found in [11,13,21,23,43] and references cited therein. Compared to the original laminar model (1.1), (1.2) has a conservative type and the source terms generated from the mass exchange and momentum exchange are cancelled out by appropriate calculations. However, (1.1) is still interesting and quite useful when we are concerned about the saturate flows. Since (1.2) is conditionally hyperbolic and the coupling between