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Explicit, Monotone and Structure-Preserving Finite Difference Methods for Fisher-Kolmogorov-Petrovsky-Piscounov Equation and Allen-Cahn Equation

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Abstract. In this study, a class of explicit structure-preserving Du Fort-Frankel-type FDMs are firstly developed for Fisher-Kolmogorov-Petrovsky-Piscounov (Fisher-KPP) equation. They inherit some properties of the continuous problems, such as non-negativity, maximum principle and monotonicity. Besides, by using the discrete maximum principle, the error estimate in L^∞-norm is proven to be $\mathcal{O}(\tau+h_x^2+h_y^2+(\frac{\tau}{h_x})^2+(\frac{\tau}{h_y})^2)$ as some suitable conditions are satisfied. Here, τ, h_x and h_y are time step and spatial meshsizes in x- and y- directions, respectively. Then, as the current FDMs are used to solve Allen-Cahn equation, the obtained numerical solutions satisfy the discrete maximum principle and the discrete energy-dissipation law. Our methods are easy to be implemented because of explicitness. Finally, numerical results confirm theoretical findings and the efficiency of our methods.

AMS subject classifications: 65M06, 65M12

Key words: Fisher-KPP equation, Allen-Cahn equation, Du Fort-Frankel-type schemes, Structure-preserving FDMs, Maximum norm error estimate

1 Introduction

In this study, we are concerned with numerical solutions of the following two-dimensional (2D) initial boundary value problems (IBVPs)

$$u_t-\alpha\Delta u-bu(1-u^p)=0, \qquad (x,y)\in\Omega, \qquad 0<t\le T, \tag{1.1a}$$

$$u(x,y,0)=u_0(x,y), \qquad (x,y)\in\overline{\Omega}, \tag{1.1b}$$

$$u(x,y,t)=\varphi(x,y,t), \qquad (x,y)\in\partial\Omega, \qquad 0\le t\le T, \tag{1.1c}$$

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in which $\Omega=(a_1,b_1)\times(a_2,b_2)$, Δ denotes a Laplace operator, both α and b are positive constants. Eq. (1.1a) with $\alpha=p=1$, which is initially used to describe the propagation of an advantageous gene in a one-dimensional infinite medium (see [1,2]), has been investigated simultaneously and independently by Fisher [1] and Kolmogorov et al. [2] in 1937. Thus, Eq. (1.1a) with $\alpha=p=1$ is always named as Fisher equation or Fisher-Kolmogorov-Petrovsky-Piscounov (Fisher-KPP) equation. Up to now, the generalized Fisher-KPP equation (1.1a) has been extensively applied in various scientific and engineering fields, such as, epidermal wound healing [3], the nonlinear kinetics of nuclear reactors [4], diffusive phenomena in population genetics [5].

As they own strongly applied background, much attention has been paid on the mathematical and computational researches for them. For example, partly of exact solutions can be obtained using some analytical methods including tanh method etc. in [6–9]. Besides, theoretical findings, such as the existence and uniqueness, blow-up and the behavior of exact solutions, can be found in [2,10]. What is more, IBVPs (1.1a)–(1.1c) satisfy the following maximum principle, i.e., Proposition 1.1.

Proposition 1.1 (cf. [7,11–13]). *Suppose* $0\le u_0(x,y)\le 1$, $((x,y)\in\bar{\Omega})$ *and* $0\le\varphi(x,y,t)\le 1$, $((x,y)\in\partial\Omega\times[0,T])$. *Then the exact solution to the IBVP* (1.1a)–(1.1c) *satisfies* $0\le u(x,y,t)\le 1$, $((x,y,t)\in\bar{\Omega}\times[0,T])$.

Therefore, to avoid spurious solutions, numerical methods for IBVPs (1.1a)–(1.1c) should inherit this property. Recently, there has been a growing interest in the derivations and theoretical analyses of structure-preserving finite difference methods (SP-FDMs) for IBVPs (1.1a)–(1.1c), which can preserve the non-negativity or the maximum principle of continuous problems. For example, an explicit positivity-preserving FDM has been developed for IBVPs (1.1a)–(1.1c) in [11]. A two-level, nonlinear, positivity-preserving and maximum principle-satisfying FDM, which is first-order accurate in both time and space, has been derived in [12]. An explicit, positivity-preserving and maximum principle-satisfying FDM and an implicit, positivity-preserving and maximum principle-satisfying FDM have been studied for IBVPs (1.1a)–(1.1c) in [13], respectively. A two-level, explicit and non-negativity-preserving FDM and its Richardson extrapolation methods were suggested for IBVPs (1.1a)–(1.1c) in [14]. A linearly implicit and maximum-principle-satisfying FDM was derived for IBVPs (1.1a)–(1.1c) with $p=1$ in [15]. A positivity-preserving discontinuous Galerkin scheme was devised for IBVPs (1.1a)–(1.1c) in [16].

In (1.1a), taking $p=2$ yields an IBVPs of well-known Allen-Cahn equation as follows

$$u_t-\alpha\Delta u-b(u-u^3)=0,\qquad (x,y)\in\Omega,\qquad 0<t\le T, \tag{1.2a}$$

$$u(x,y,0)=u_0(x,y),\qquad (x,y)\in\overline{\Omega}, \tag{1.2b}$$

$$u(x,y,t)=\varphi(x,y,t),\qquad (x,y)\in\partial\Omega,\qquad 0\le t\le T, \tag{1.2c}$$

which has been firstly proposed by Allen and Cahn to describe the motion of anti-phase boundaries in crystalline solids in [17]. Nowadays it has been widely utilized in scientific and engineering fields, such as interfacial dynamics in material science [18,19], grain